Optimal Consumption and Portfolio Decisions with Stochastic Affine Interest Rate Model

HAO CHANG
Tianjin Polytechnic University
School of Science
Binshui West Road 399, Tianjin 300387;
Tianjin University
College of Management and Economics
Weijin Road 92, Tianjin 300072
CHINA
ch8683897@126.com

Abstract: This article is concerned with an investment and consumption problem with stochastic affine interest rate model, which includes the CIR model and the Vasicek model as special cases. The financial market is composed of three assets: one cash account, one stock and one zero-coupon bond. Moreover, the price dynamics of the stock and zero-coupon bond is affected by the dynamics of interest rate. Our objective is to seek an optimal consumption and portfolio decisions to maximize the expected discounted utility of intermediate consumption and terminal wealth in the finite horizon. By applying stochastic dynamic programming principle and variable change techniques, we obtain the explicit expressions of the optimal consumption and portfolio decisions in the power utility and logarithm utility cases. In order to analyze the impact of the parameters of interest rate on the optimal consumption and portfolio decisions, we provide a numerical example to illustrate our results.

Key–Words: Investment and consumption; stochastic affine interest rate model; stochastic dynamic programming principle; the closed-form solution;

1 Introduction

The problems of the optimal investment and optimal consumption have attracted more and more attentions of many researchers since the seminal work of Merton [1, 2]. In Merton's models, its goal was to choose the optimal consumption and portfolio decisions to maximize the total expected discounted utility of consumption in the infinite horizon. Stochastic optimal control theory was first adopted to obtain the closed-form solutions in the power utility and logarithm utility cases. These works had inspired literally hundreds of extensions and applications in recent years. For example, Fleming and Zariphopoulou [3], Vila and Zariphopoulou [4], and Yao and Zhang [5], studied the optimal consumption and portfolio decisions with borrowing constraints in different situations. Duffie et al. [6] and Sasha and Zariphopoulou [7], assumed that the financial market is incomplete and investigated the optimal consumption and portfolio decisions of the investors. Dumas and Luciano [8], Shreve and Soner [9], Liu and Loewenstein [10] and Dai et al. [11], explored the optimal consumption and portfolio decisions with transaction costs. These models enriched and extended the works of Merton, but in those papers interest rate and the volatility of the stock were almost supposed to be constants or bounded deterministic functions of the time.

However, interest rate is not always fixed in our real life. Recently, some scholars have studied some portfolio selection problems with stochastic interest rate and stochastic volatility. For instance, Fleming and Pang [12] investigated the optimal consumption and portfolio decisions with stochastic interest rate and proposed the sub-supersolution method to verify the existence of the optimal strategy. Munk and Sørensen [13] characterized the solution of the consumption and investment problem with power utility preference in a continuous-time dynamically complete market with stochastic changes in the opportunity set. Liu [14] explicitly solved the optimal consumption and portfolio decisions with stochastic market parameters in the several special cases when the asset returns are quadratic and the agent has a constant relative risk aversion coefficient. Fleming and Hernandez-Hernandez [15] and Chacko and Viceira [16] studied the optimal investment and optimal consumption strategies with Heston's stochastic volatility respectively, but they did not obtain the explicit expressions of the optimal portfolios. Noh and Kim [17] considered an optimal consumption problem with stochastic volatility and stochastic interest rate in an infinite time horizon, but they only verified the existence of the optimal trading policies. On the newest research results one can refer to the works of Chang et al. [18], Li et al. [19], Chang et al. [20], Chang and Rong [21], Guan and Liang [23], Chang and Lu [24] and so on. However, in those papers, Chang et al. [20] studied the optimal consumption and portfolio decisions under the constant elasticity of variance (CEV) model, which was the natural extension of geometric Brownian motion. Chang and Rong [21] investigated the optimal consumption and portfolio decisions with stochastic interest rate and stochastic volatility, in which interest rate was driven by the Cox-Ingersoll-Ross (CIR) model [22], while the volatility of the stock is governed by the Heston model. Guan and Liang [23] considered the inflation risk and interest rate risk in the reinsurance and investment problems, in which interest rate is suppose be driven by the Vasicek model [25]. As matter of fact, the Vasicek model and the CIR model are special cases of stochastic affine interest rate model, which is rarely studied in the portfolio selection problems.

In recent years, some scholars have paid more and more attentions to the portfolio selection problems with stochastic affine interest rate model. For example, Deelstra et al. [26] studied a defined contribution (DC) pension fund in the presence of a minimum guarantee, in which stochastic interest rate is affine. Gao [27] provided a Legendre transform-dual method to study a DC pension fund with affine interest rate. Chang et al. [28] considered stochastic liability process in an affine interest rate environment and investigated the optimal strategy under HARA utility. Guan and Liang [29] extended the model of Deelstra et al. [26] to the environments with affine interest rate and stochastic volatility. But as far as we know, the optimal consumption and portfolio decisions with stochastic affine interest rate model have not been reported in the existing literatures.

In this paper we investigate the optimal consumption and portfolio decisions with stochastic affine interest rate model. We assume that interest rate is described by an affine interest rate model and can affect the dynamics of stock price and zero-coupon bond. The objective of an investor is to maximize the expected discounted utility of intermediate consumption and terminal wealth in the finite horizon. Stochastic optimal control theory is applied to obtain the HJB equation for the value function, which is non-linear second-order partial differential equation. It is very hard to directly produce a closed-form solution of the HJB equation. In order to obtain the explicit expressions of the optimal investment and consumption strategies, we choose power utility and logarithm utility function as our analysis. In the power utility case, we repeatedly use variable change technique to change the HJB equation into (16), which is hard to solve directly. Inspired by the work of Liu [14], we can conjecture the structure of the solution of (16) and fit it successfully. The optimal consumption and portfolio decisions in the power utility case can be obtained and several special cases are also derived. In the logarithmic case, we can also use variable change approach to obtain the optimal consumption and portfolio decisions. A numerical example is given to illustrate our results. In summary, this article has three main contributions: (i) the optimal consumption and portfolio decisions with stochastic affine interest rate model is studied; (ii) inspired by the work of Liu [14], we provide the explicit solution to the equation (16); (iii) the closed-form solutions of the optimal consumption and portfolio decisions in the power utility and logarithm utility cases are achieved.

The remainder of this article is organized as follows. Section 2 describes problem formulation, including the financial market, the wealth process and the optimization criterion. In section 3, the HJB equation is obtained and the closed-form expressions of the optimal consumption and portfolio decisions are derived. Section 4 provides a numerical analysis. Section 5 concludes this article.

2 Problem Formulation

In this section, we describe the problem formulation, where financial market and wealth process and optimization criterion are all provided.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, where T denotes the fixed finite horizon of investment. The uncertainty involved by the financial market is described by two standard and independent Brownian motion $W_r(t)$ and $W_s(t)$, where $W_r(t)$ and $W_s(t)$ are defined on a complete probability space $(\Omega, \mathscr{F}_t, \{\mathscr{F}_t\}_{0\leq t\leq T}, P)$, where P is the real-world probability and the filtration $\{\mathscr{F}_t\}_{0\leq t\leq T}$ represents the information structure generated by Brownian motion $W_r(t)$ and $W_s(t)$.

2.1 The financial market

Assume that the financial market is composed of three assets, which can be traded continuously. The first asset is the risk-free asset (i.e. cash account), whose price at time t is denoted by $S_0(t)$, then $S_0(t)$

satisfies the following ordinary differential equation:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t)$ is interest rate of cash account.

In this article, suppose that the dynamics of $r(t)$ is described by the following stochastic differential equation(SDE):

$$\begin{aligned} &dr(t) = (a - br(t))dt - \sqrt{k_1 r(t) + k_2}dW_r(t), \\ &r(0) = r_0 > 0, \end{aligned} \tag{2}$$

with the coefficients $a, b, r(0), k_1$ and k_2 being positive constants.

Note that the dynamics recovers, as a special case, the Vasicek [25] (resp. Cox et al. [22]) dynamics, when k_1 (resp.k_2) is equal to zero. Under these dynamics, the term structure of interest rate is affine.

The second asset is the stock, whose price is denoted by $S_1(t)$. The dynamics of $S_1(t)$ is governed by (referring to Deelstra et al. [26] and Gao [27], Chang et al. [28]):

$$\begin{aligned} &\frac{dS_1(t)}{S_1(t)} = r(t)dt + \sigma_1(dW_s(t) + \lambda_1 dt) \\ &+ \sigma_2\sqrt{k_1 r(t) + k_2}(dW_r(t) + \lambda_2\sqrt{k_1 r(t) + k_2}dt), \end{aligned} \tag{3}$$

with $S_1(0) = 1$ and λ_1, λ_2 (resp.σ_1, σ_2) being constants (resp. positive constants).

The third asset is a zero-coupon bond with maturity T, whose price at time T is denoted by $S_2(t,T)$. Then $S_2(t,T)$ evolves (see Deelstra et al. [26] and Gao [27], Chang et al. [28])

$$\begin{aligned} \frac{dS_2(t,T)}{S_2(t,T)} &= r(t)dt + \sigma_B(T-t, r(t)) \times (dW_r(t) \\ &\quad + \lambda_2\sqrt{k_1 r(t) + k_2}dt), \ S_2(T,T) = 1, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\sigma_B(T-t, r(t)) = h(T-t)\sqrt{k_1 r(t) + k_2}, \\ &h(t) = \frac{2(e^{mt} - 1)}{m - (b - k_1\lambda_2) + e^{mt}(m + b - k_1\lambda_2)}, \\ &m = \sqrt{(b - k_1\lambda_2)^2 + 2k_1}. \end{aligned}$$

2.2 The wealth process

Assume that the amount of money invested in the two risky assets (i.e. the stock and the zero-coupon bond) is denoted by $\pi_s(t)$ and $\pi_B(t)$ respectively. Letting $X(t)$ represents the wealth process at time t, then the amount invested in the risk-free asset is $\pi_0(t) = X(t) - \pi_s(t) - \pi_B(t)$. Suppose that the consumption rate is denoted by $C(t)$. Then the dynamics of $X(t)$ corresponding to $\pi(t) = (\pi_s(t), \pi_B(t))$ is given by

$$\begin{aligned} dX(t) = &\Big(X(t)r(t) + \pi_s\lambda_2\sigma_2(k_1 r(t) + k_2) + \pi_s\lambda_1\sigma_1 \\ &+\pi_B\lambda_2\sigma_B\sqrt{k_1 r(t) + k_2} - C(t)\Big)\, dt \\ &+ \left(\pi_s\sigma_2\sqrt{k_1 r(t) + k_2} + \pi_B\sigma_B\right) dW_r(t) \\ &+ \pi_s\sigma_1 dW_s(t), \quad X(0) = x_0 > 0, \end{aligned} \tag{5}$$

where interest rate $r(t)$ is a stochastic process and is driven by the SDE (2).

2.3 The optimization criterion

Definition 1 *(Admissible strategy) An consumption and investment strategy $(\pi(t), C(t))$ is said to be admissible if the following conditions are satisfied:*

(i) $(\pi(t), C(t))$ *is $\mathscr{F}_t$-progressively measurable, and* $\int_0^T \|\pi(t)\|^2 dt < \infty$, $\int_0^T C(t)dt < \infty$, $a.s.\ \forall T > 0$;

(ii) $E\left(\int_0^T (\pi_s\sigma_2\sqrt{k_1 r(t) + k_2} + \pi_B\sigma_B)^2 dt + \int_0^T (\pi_s\sigma_1)^2 dt\right) < \infty;$

(iii) The equation (5) has a unique solution on $[0,T]$ corresponding to any $(\pi(t), C(t))$.

Assume that the set of all the admissible consumption and investment strategies $(\pi(t), C(t))$ is denoted by $\Gamma = \{(\pi(t), C(t)) : \ 0 \le t \le T\}$. In this article, our goal is to maximize the following objective function:

$$\begin{aligned} \underset{(\pi(t),C(t))\in\Gamma}{Maximize}\ E\Bigg(\alpha\int_0^T e^{-\beta t}U_1(C(t))dt\ , \\ +(1-\alpha)e^{-\beta T}U_2(X(T))\Bigg) \end{aligned} \tag{6}$$

where β is the discount factor and the parameter α determines the relative importance of the intermediate consumption and the portfolio. When $\alpha = 0$, expected utility only depends on the terminal wealth and the problem is called a dynamic asset allocation problem. Utility function $U_1(\cdot) = U_2(\cdot) = U(\cdot)$ is strictly concave and satisfies the Inada conditions: $U'(+\infty) = 0$ and $U'(0) = +\infty$.

In this article, we choose power utility and logarithm utility for our analysis. Power utility is given by $U_1(x) = U_2(x) = \frac{x^\eta}{\eta}$, $\eta < 1$ and $\eta \neq 0$, where η is the risk aversion factor. Logarithm utility is defined as $U_1(x) = U_2(x) = \ln x$.

3 Optimal Consumption and Portfolio Decisions

In this section, we use the principle of stochastic dynamic programming to obtain the HJB equation and choose power utility and logarithm utility for our analysis. By applying variable change techniques we obtain the closed-form solutions of the optimal consumption and portfolio decisions.

We can define the value function $V(t,r,x)$ as

$$V(t,r,x) = \sup_{(\pi(t),C(t))\in\Gamma} E\left(\alpha\int_0^T e^{-\beta t}U_1(C(t))dt\right.$$
$$\left.+(1-\alpha)e^{-\beta T}U_2(X(T))\,|X(t)=x, r(t)=r\right),$$

with boundary condition

$$V(T,r,x) = (1-\alpha)\times e^{-\beta T}U_2(x).$$

For any $V(t,r,x)\in\mathbb{C}^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$, we define a variational operator:

$$\begin{aligned}\mathscr{A}^{\pi,C}V(t,r,x) &= V_t + \big(rx + \pi_s\lambda_1\sigma_1 + \pi_s\lambda_2\sigma_2\sigma_r^2\\ &+ \pi_B\lambda_2\sigma_B\sigma_r - C(t)\big)V_x\\ &+ \frac{1}{2}\left((\pi_s\sigma_1)^2 + (\pi_s\sigma_2\sigma_r + \pi_B\sigma_B)^2\right)V_{xx}\\ &+ (a-br)V_r + \frac{1}{2}\sigma_r^2V_{rr}\\ &- \sigma_r(\pi_s\sigma_2\sigma_r + \pi_B\sigma_B)V_{xr}\\ &+ \alpha e^{-\beta t}U_1(C(t)),\end{aligned}$$

where $\sigma_r = \sqrt{k_1r(t)+k_2}$ and H_t, H_x, H_{xx}, H_r, H_{rr}, H_{xr} denote partial derivatives of first-order and second-order with respect to the variables t, r, x. We use also similar notations for higher-order derivatives of other functions.

According to the principle of stochastic dynamic programming, we obtain the following HJB equation:

$$\sup_{(\pi(t),C(t))\in\Gamma}\left\{\mathscr{A}^{\pi,C}V(t,r,x)\right\} = 0. \tag{7}$$

The following theorem verifies that a solution to the HJB equation (7) is indeed the optimal solution to the problem (6).

Theorem 2 *If $H(t,r,x)\in\mathbb{C}^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ is a solution of the HJB equation (7), i.e. $H(t,r,x)$ satisfies the following variational equation:*

$$\sup_{(\pi(t),C(t))\in\Gamma}\left\{\mathscr{A}^{\pi,C}H(t,r,x)\right\} = 0,$$

$$H(T,r,x) = (1-\alpha)e^{-\beta T}U_2(x),$$

then we have $V(t,r,x)\le H(t,r,x)$ for an arbitrary admissible policy $(\pi(t),C(t))\in\Gamma$. Moreover, if there exists a $(\pi^*(t),C^*(t))\in\Gamma$ such that

$$(\pi^*(t),C^*(t))\in\arg\sup\left\{\mathscr{A}^{\pi,C}H(s,r(s),X(s))\right\},$$

then when $(\pi(t),C(t)) = (\pi^*(t),C^*(t))$, we have $H(t,r,x) = V(t,r,x)$, i.e. $(\pi^*(t),C^*(t))$ is indeed the optimal consumption and portfolio decisions of the problem (6).

Proof. See the Appendix. □

According to Theorem 1 and the first-order maximizing conditions, we get

$$\pi_s^*(t) = -\frac{\lambda_1}{\sigma_1}\cdot\frac{H_x}{H_{xx}}, \quad U_1'(C(t)) = \frac{H_x}{\alpha e^{-\beta t}}, \tag{8}$$

$$\pi_B^*(t) = \frac{\sigma_r(\lambda_1\sigma_2-\lambda_2\sigma_1)}{\sigma_1\sigma_B}\cdot\frac{H_x}{H_{xx}} + \frac{\sigma_r}{\sigma_B}\cdot\frac{H_{xr}}{H_{xx}}. \tag{9}$$

Putting (8) and (9) in (7), we obtain a non-linear second-order partial differential equation (PDE) for the value function $H(t,r,x)$:

$$\begin{aligned}&H_t + rxH_x + (a-br)H_r + \frac{1}{2}\sigma_r^2H_{rr}\\ &-\frac{1}{2}(\lambda_1^2+\lambda_2^2\sigma_r^2)\frac{H_x^2}{H_{xx}} - \frac{1}{2}\sigma_r^2\frac{H_{rx}^2}{H_{xx}}\\ &+\lambda_2\sigma_r^2\frac{H_xH_{xr}}{H_{xx}} - C^*(t)H_x\\ &+\alpha e^{-\beta t}U(C^*(t)) = 0.\end{aligned} \tag{10}$$

In the following subsection, we try our best to solve (10) in order to investigate the optimal consumption and portfolio decisions for power utility and logarithm utility respectively.

3.1 Power utility

Under power utility function, the value function $H(t,r,x)$ is conjectured to have the form

$$H(t,r,x) = f(t,r)e^{-\beta t}\frac{x^\eta}{\eta},$$

$$f(T,r) = 1-\alpha.$$

The partial derivatives for $H(t,r,x)$ are given by

$$\begin{aligned}&H_t = f_te^{-\beta t}\frac{x^\eta}{\eta} + f(-\beta)e^{-\beta t}\frac{x^\eta}{\eta},\\ &H_r = f_re^{-\beta t}\frac{x^\eta}{\eta}, \quad H_{rr} = f_{rr}e^{-\beta t}\frac{x^\eta}{\eta},\\ &H_x = fe^{-\beta t}x^{\eta-1}, \quad H_{rx} = f_re^{-\beta t}x^{\eta-1},\\ &H_{xx} = fe^{-\beta t}(\eta-1)x^{\eta-2}.\end{aligned} \tag{11}$$

Further, we get

$$\begin{aligned} &C^*(t) = (\frac{f}{\alpha})^{\frac{1}{\eta-1}}x, \quad \frac{H_x}{H_{xx}} = \frac{1}{\eta-1}x, \\ &\frac{H_{rx}}{H_{xx}} = \frac{1}{\eta-1} \cdot \frac{f_r}{f}x. \end{aligned} \tag{12}$$

Plugging (11) and (12) into (10) yields

$$\begin{aligned} &e^{-\beta t}\frac{x^\eta}{\eta}\left(f_t + \left(a - br + \frac{\eta\lambda_2\sigma_r^2}{\eta-1}\right)f_r\right. \\ &+ \left(\eta r - \beta - \frac{\eta\lambda_1^2}{2(\eta-1)} - \frac{\eta\lambda_2^2\sigma_r^2}{2(\eta-1)}\right)f \\ &+ \frac{1}{2}\sigma_r^2 f_{rr} - \frac{\eta}{2\eta-2}\sigma_r^2\frac{f_r^2}{f} \\ &\left.+(1-\eta)f^{\frac{\eta}{\eta-1}}\alpha^{\frac{1}{1-\eta}}\right) = 0. \end{aligned}$$

Eliminating the dependence on the variable x, we can obtain

$$\begin{aligned} &f_t + \left(\eta r - \beta - \frac{\eta\lambda_1^2}{2(\eta-1)} - \frac{\eta\lambda_2^2\sigma_r^2}{2(\eta-1)}\right)f \\ &+ \left(a - br + \frac{\eta\lambda_2\sigma_r^2}{\eta-1}\right)f_r + \frac{1}{2}\sigma_r^2 f_{rr} \\ &- \frac{\eta}{2\eta-2}\sigma_r^2\frac{f_r^2}{f} + (1-\eta)f^{\frac{\eta}{\eta-1}}\alpha^{\frac{1}{1-\eta}} = 0, \\ &f(T,r) = 1-\alpha. \end{aligned} \tag{13}$$

Assume that

$$\begin{aligned} &f(t,r) = g(t,r)^{1-\eta}, \\ &g(T,r) = (1-\alpha)^{\frac{1}{1-\eta}}. \end{aligned} \tag{14}$$

then we have

$$\begin{aligned} &f_t = (1-\eta)g^{-\eta}g_t, \quad f_r = (1-\eta)g^{-\eta}g_r, \\ &f_{rr} = -\eta(1-\eta)g^{-\eta-1}g_r^2 + (1-\eta)g^{-\eta}g_{rr}. \end{aligned} \tag{15}$$

Substituting (14) and (15) back into (13), we derive

$$\begin{aligned} &(1-\eta)g^{-\eta}\left(g_t + \left(\frac{\beta}{\eta-1} - \frac{\eta}{\eta-1}r\right.\right. \\ &\left.+\frac{\eta}{2(\eta-1)^2}\lambda_1^2 + \frac{\eta}{2(\eta-1)^2}\lambda_2^2\sigma_r^2\right)g \\ &+ \left(a - br + \frac{\eta}{\eta-1}\lambda_2\sigma_r^2\right)g_r \\ &\left.+\frac{1}{2}\sigma_r^2 g_{rr} + \alpha^{\frac{1}{1-\eta}}\right) = 0. \end{aligned}$$

Further, we get the equation

$$\begin{aligned} &g_t + \left(\frac{\beta}{\eta-1} - \frac{\eta}{\eta-1}r\right. \\ &\left.+\frac{\eta}{2(\eta-1)^2}\lambda_1^2 + \frac{\eta}{2(\eta-1)^2}\lambda_2^2\sigma_r^2\right)g \\ &+ \left(a - br + \frac{\eta}{\eta-1}\lambda_2\sigma_r^2\right)g_r + \frac{1}{2}\sigma_r^2 g_{rr} \\ &+ \alpha^{\frac{1}{1-\eta}} = 0, \quad g(T,r) = (1-\alpha)^{\frac{1}{1-\eta}}. \end{aligned} \tag{16}$$

The equation (16) is not easy to solve directly. Inspired by the work of Liu [14], we can transform (16) into another equation, which is easy to solve. Therefore, we need the following Lemma.

Lemma 3 *Defining the function $g(t,r)$, which is a solution of the equation (16), as the following form*

$$\begin{aligned} g(t,r) =& \alpha^{\frac{1}{1-\eta}}\int_t^T \hat{g}(s,r)ds \\ &+ (1-\alpha)^{\frac{1}{1-\eta}}\hat{g}(t,r), \end{aligned}$$

then we get

$$\begin{aligned} &\hat{g}_t + \left(\frac{\beta}{\eta-1} - \frac{\eta}{\eta-1}r\right. \\ &\left.+\frac{\eta}{2(\eta-1)^2}\lambda_1^2 + \frac{\eta}{2(\eta-1)^2}\lambda_2^2\sigma_r^2\right)\hat{g} \\ &+ \left(a - br + \frac{\eta}{\eta-1}\lambda_2\sigma_r^2\right)\hat{g}_r \\ &+ \frac{1}{2}\sigma_r^2\hat{g}_{rr} = 0, \quad \hat{g}(T,r) = 1. \end{aligned} \tag{17}$$

Proof. Defining a differential operator ∇ on any function $g(t,r)$ by

$$\begin{aligned} \nabla g(t,r) =& \left(\frac{\beta}{\eta-1} - \frac{\eta}{\eta-1}r\right. \\ &\left.+\frac{\eta}{2(\eta-1)^2}\lambda_1^2 + \frac{\eta}{2(\eta-1)^2}\lambda_2^2\sigma_r^2\right)g \\ &+ \left(a - br + \frac{\eta}{\eta-1}\lambda_2\sigma_r^2\right)g_r + \frac{1}{2}\sigma_r^2 g_{rr}, \end{aligned}$$

then we can rewrite (16) as

$$\begin{aligned} &\frac{\partial g(t,r)}{\partial t} + \nabla g(t,r) + \alpha^{\frac{1}{1-\eta}} = 0, \\ &g(T,r) = (1-\alpha)^{\frac{1}{1-\eta}}. \end{aligned} \tag{18}$$

On the other hand, we get

$$\frac{\partial g(t,r)}{\partial t} = -\alpha^{\frac{1}{1-\eta}}\hat{g}(t,r) + (1-\alpha)^{\frac{1}{1-\eta}} \cdot \frac{\partial \hat{g}(t,r)}{\partial t}$$

$$
\begin{aligned}
=&\alpha^{\frac{1}{1-\eta}}\left(\int_t^T \frac{\partial \hat{g}(s,r)}{\partial s}ds - \hat{g}(T,r)\right) \\
&+ (1-\alpha)^{\frac{1}{1-\eta}} \cdot \frac{\partial \hat{g}(t,r)}{\partial t},
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\nabla g(t,r) =&\alpha^{\frac{1}{1-\eta}} \int_t^T \nabla \hat{g}(s,r)ds \\
&+ (1-\alpha)^{\frac{1}{1-\eta}} \cdot \nabla \hat{g}(t,r).
\end{aligned} \tag{20}
$$

Putting (19) and (20) in the equation (18), we derive

$$
\begin{aligned}
&\alpha^{\frac{1}{1-\eta}}\left(\int_t^T \left(\frac{\partial \hat{g}(s,r)}{\partial s} + \nabla \hat{g}(s,r)\right)ds \right.\\
&\left. -\hat{g}(T,r) + 1\right) \\
&+ (1-\alpha)^{\frac{1}{1-\eta}} \cdot \left(\frac{\partial \hat{g}(t,r)}{\partial t} + \nabla \hat{g}(t,r)\right) = 0.
\end{aligned}
$$

So we have

$$
\frac{\partial \hat{g}(t,r)}{\partial t} + \nabla \hat{g}(t,r) = 0, \quad \hat{g}(T,r) = 1.
$$

Namely, (17) holds. It is obvious that we can verify (17). □

Lemma 4 *Assume that the solution of the equation (17) is of the form $\hat{g}(t,r) = e^{A(t)+B(t)r}$, with boundary conditions $A(T) = 0$ and $B(T) = 0$, then $A(t)$ and $B(t)$ are given by (26) and (24) respectively.*

Proof: Substituting $\hat{g}(t,r) = e^{A(t)+B(t)r}$ back into (17) yields

$$
\begin{aligned}
&e^{A(t)+B(t)r}\left[A'(t) - \frac{\beta}{1-\eta} + \frac{\eta}{2(\eta-1)}\lambda_1^2 + \frac{1}{2}k_2 B^2(t)\right. \\
&+ \frac{\eta}{2(\eta-1)^2}\lambda_2^2 k_2 + \left(a + \frac{\eta}{\eta-1}\lambda_2^2 k_2\right)B(t) \\
&+ r\left(B'(t) + \frac{1}{2}k_1 B^2(t) + \left(\frac{\eta}{\eta-1}\lambda_2^2 k_1 - b\right)B(t)\right. \\
&\left.\left. + \frac{\eta}{1-\eta} + \frac{\eta}{2(\eta-1)^2}\lambda_2^2 k_1\right)\right] = 0.
\end{aligned}
$$

Eliminating the dependence on r, we get

$$
\begin{aligned}
&B'(t) + \frac{1}{2}k_1 B^2(t) + \left(\frac{\eta}{\eta-1}\lambda_2^2 k_1 - b\right)B(t) \\
&+ \frac{\eta}{1-\eta} + \frac{\eta}{2(\eta-1)^2}\lambda_2^2 k_1 = 0,
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&A'(t) - \frac{\beta}{1-\eta} + \frac{\eta}{2(\eta-1)}\lambda_1^2 \\
&+ \frac{1}{2}k_2 B^2(t) + \frac{\eta}{2(\eta-1)^2}\lambda_2^2 k_2 \\
&+ \left(a + \frac{\eta}{\eta-1}\lambda_2^2 k_2\right)B(t) = 0.
\end{aligned} \tag{22}
$$

For the sake of simplicity, letting

$$
u = -\frac{1}{2}k_1, \quad v = b - \frac{\eta}{\eta-1}\lambda_2^2 k_1,
$$

$$
w = -\frac{\eta}{1-\eta} - \frac{\eta}{2(\eta-1)^2}\lambda_2^2 k_1,
$$

then the equation (21) can be rewritten as

$$
B'(t) = uB^2(t) + vB(t) + w, \quad B(T) = 0. \tag{23}
$$

Letting $v^2 - 4uw > 0$ and integrating (23) on the both sides from t to T, we get

$$
\begin{aligned}
&\int_t^T \left(\frac{1}{B(t)-m_1} - \frac{1}{B(t)-m_2}\right)dB(t) \\
&= u(m_1 - m_2)(T-t),
\end{aligned}
$$

where m_1 and m_2 are the two different roots of the following equation

$$
um^2 + vm + w = 0,
$$

namely,

$$
\begin{aligned}
m_1 = &-\frac{b(1-\eta) + \eta\lambda_2^2 k_1}{(\eta-1)k_1} \\
&+ \frac{\sqrt{(b(1-\eta)+\eta\lambda_2^2 k_1)^2 + k_1(2\eta^2 - 2\eta - \eta\lambda_2^2 k_1)}}{(\eta-1)k_1},
\end{aligned}
$$

$$
\begin{aligned}
m_2 = &-\frac{b(1-\eta) + \eta\lambda_2^2 k_1}{(\eta-1)k_1} \\
&- \frac{\sqrt{(b(1-\eta)+\eta\lambda_2^2 k_1)^2 + k_1(2\eta^2 - 2\eta - \eta\lambda_2^2 k_1)}}{(\eta-1)k_1}.
\end{aligned}
$$

So we have

$$
B(t) = \frac{m_1 m_2 (1 - e^{u(m_1-m_2)(T-t)})}{m_1 - m_2 e^{u(m_1-m_2)(T-t)}}. \tag{24}
$$

By calculating $(20) \times k_1 - (19) \times k_2$, we get

$$
\begin{aligned}
&k_1 A'(t) - k_2 B'(t) + (ak_1 + bk_2)B(t) \\
&-\frac{1}{1-\eta}(\beta k_1 - \frac{1}{2}\eta k_1 \lambda_1^2 + \eta k_2) = 0.
\end{aligned} \tag{25}
$$

If $k_1 \neq 0$, by integrating (25) on the both sides from t to T, we derive

$$
\begin{aligned}
A(t) = &\frac{k_2}{k_1}B(t) + \frac{ak_1 + bk_2}{k_1}\left(m_2(T-t)\right. \\
&\left.+\frac{1}{u}\left(\ln(m_1 - m_2) - \ln(m_1 - m_2 e^{u(m_1-m_2)(T-t)})\right)\right)
\end{aligned}
$$

$$-\frac{1}{(1-\eta)k_1}\left(\beta k_1 - \frac{1}{2}\eta k_1 \lambda_1^2 + \eta k_2\right)(T-t). \quad (26)$$

The proof of Lemma 4 is completed. □

Taking (15) into account, (12) can be rewritten as

$$\frac{H_x}{H_{xx}} = \frac{1}{\eta - 1}x, \quad \frac{H_{rx}}{H_{xx}} = -\frac{g_r}{g}x,$$

$$C^*(t) = g^{-1}\alpha^{\frac{1}{1-\eta}}x.$$

Finally, we can summarize the above conclusions in the following proposition 5.

Proposition 5 *When utility function is* $U_1(x) = U_2(x) = \frac{x^\eta}{\eta}$*,* $\eta < 1$ *and* $\eta \neq 0$*, the optimal consumption and portfolio decisions for the problem (6) are given by*

$$\begin{aligned}
\pi_s^*(t) =& \frac{1}{1-\eta}\cdot\frac{\lambda_1}{\sigma_1}X(t),\\
C^*(t) =& \alpha^{\frac{1}{1-\eta}}g^{-1}X(t),\\
\pi_B^*(t) =& \frac{1}{1-\eta}\cdot\frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)}{\sigma_1\sigma_B}X(t)\\
&- \frac{\sigma_r}{\sigma_B}\cdot\frac{g_r}{g}X(t),
\end{aligned}$$

where

$$\begin{aligned}
g = g(t,r) =& \alpha^{\frac{1}{1-\eta}}\int_t^T e^{A(s)+B(s)r(s)}ds\\
&+ (1-\alpha)^{\frac{1}{1-\eta}}e^{A(t)+B(t)r(t)},\\
g_r = \frac{\partial g(t,r)}{\partial r} =& \alpha^{\frac{1}{1-\eta}}\int_t^T B(s)e^{A(s)+B(s)r(s)}ds\\
&+ (1-\alpha)^{\frac{1}{1-\eta}}B(t)e^{A(t)+B(t)r(t)}.
\end{aligned}$$

Remark 6 *(i) The parameters* α *and* β *have no impact on* $\pi_s^*(t)$*, but have impact on* $\pi_B^*(t)$ *and* $C^*(t)$*. (ii) The parameters of affine interest rate, i,e, a,b,*k_1 *and* k_2*, have no impact on* $\pi_s^*(t)$*, but have impact on* $\pi_B^*(t)$ *and* $C^*(t)$*. (iii) The parameters* λ_2 *and* σ_2 *affect the dynamics of stock price, but have no influence on* $\pi_s^*(t)$*. (iv) The volatility parameters* σ_1*,*σ_2 *and* σ_B *have no effect on* $C^*(t)$*.*

In order to compare our results with those in the existing literatures, we discuss several special cases of Proposition 5.

Special case 1. In the Proposition 5, assume that $\eta = 0$, then we get

$$A(t) = \beta(t-T), \quad B(t) = 0,$$

$$\hat{g}(t,r) = e^{\beta(t-T)}.$$

According to (17), we derive

$$\begin{aligned}
g = g(t,r) =& \frac{\alpha}{\beta}(1 - e^{\beta(t-T)})\\
&+ (1-\alpha)e^{\beta(t-T)}.
\end{aligned}$$

Therefore, the results of Proposition 5 is reduced to

$$\pi_s^*(t) = \frac{\lambda_1}{\sigma_1}X(t),$$

$$\pi_B^*(t) = \frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)}{\sigma_1\sigma_B}X(t),$$

$$C^*(t) = \frac{\alpha}{\frac{\alpha}{\beta}(1-e^{\beta(t-T)}) + (1-\alpha)e^{\beta(t-T)}}X(t).$$

It is all well-known that power utility will be degenerated to logarithm utility function if $\eta = 0$. Moreover, we find that these results are in agreement with those of the following subsection 3.2.

Special case 2. If interest rate is a constant, i.e. $a = b = 0$ and $k_1 = k_2 = 0$, then the zero-coupon bond is degenerated to a risk-free asset. Therefore, the third asset will not be considered in this article. In addition, we derive

$$B(t) = \frac{\eta}{1-\eta}(T-t),$$

$$A(t) = \left(\frac{\eta}{2(\eta-1)}\lambda_1^2 - \frac{\beta}{1-\eta}\right)(T-t),$$

$$\begin{aligned}
g = g(t) =& \alpha^{\frac{1}{1-\eta}}\int_t^T e^{A(s)+B(s)r(s)}dt\\
&+ (1-\alpha)^{\frac{1}{1-\eta}}e^{A(t)+B(t)r(t)}.
\end{aligned}$$

Letting

$$\gamma = \frac{\eta}{2(\eta-1)}\lambda_1^2 - \frac{\beta}{1-\eta} + \frac{\eta}{1-\eta}r(t),$$

then we obtain

$$\begin{aligned}
g = g(t) =& \alpha^{\frac{1}{1-\eta}}\cdot\frac{1}{\gamma}(e^{\gamma(T-t)} - 1)\\
&+ (1-\alpha)^{\frac{1}{1-\eta}}e^{\gamma(T-t)}.
\end{aligned}$$

As a result, the optimal consumption and portfolio decisions with constant interest rate are given by

$$\pi_s^*(t) = \frac{1}{1-\eta}\cdot\frac{\lambda_1}{\sigma_1}X(t),$$

$$C^*(t) = \alpha^{\frac{1}{1-\eta}}g^{-1}(t)X(t).$$

Special case 3. If $\alpha = 0$ and $\beta = 0$, the objective function (6) is degenerated to maximize expected utility of terminal wealth and the model in this article is reduced to an asset allocation problem with affine interest rate. Finally, the optimal investment strategy is given by

$$\begin{aligned}\pi_s^*(t) =& \frac{1}{1-\eta} \cdot \frac{\lambda_1}{\sigma_1} X(t),\\ \pi_B^*(t) =& \frac{1}{1-\eta} \cdot \frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)}{\sigma_1\sigma_B} X(t)\\ &- \frac{\sigma_r}{\sigma_B} \cdot B(t)X(t),\end{aligned}$$

where $B(t)$ is still given by Lemma 4.

3.2 Logarithm utility

Under logarithm utility function, assume that the solution of (10) is of the form

$$H(t,r,x) = W(t,r)e^{-\beta t}\ln x + V(t,r),$$

$$W(T,r) = 1-\alpha, \quad V(T,r) = 0.$$

then the partial derivatives of $H(t,r,x)$ are

$$\begin{aligned} H_t &= W_t e^{-\beta t}\ln x + W(-\beta)e^{-\beta t}\ln x + V_t,\\ H_x &= We^{-\beta t}\frac{1}{x}, \quad H_{xx} = -We^{-\beta t}\frac{1}{x^2},\\ H_r &= W_r e^{-\beta t}\ln x + V_r, \quad H_{rx} = W_r e^{-\beta t}\frac{1}{x},\\ H_{rr} &= W_{rr}e^{-\beta t}\ln x + V_{rr}. \end{aligned} \tag{27}$$

Then we have

$$C^*(t) = \frac{\alpha}{W}x,$$

$$\frac{H_x}{H_{xx}} = -x, \quad \frac{H_{rx}}{H_{xx}} = -\frac{W_r}{W}x. \tag{28}$$

Putting (27) in the equation (10), we derive

$$\begin{aligned} &e^{-\beta t}\ln x\left(W_t - \beta W + (a-br)W_r + \frac{1}{2}\sigma_r^2 W_{rr} + \alpha\right)\\ &+ V_t + (a-br)V_r + \frac{1}{2}\sigma_r^2 V_{rr} - \alpha\\ &+ \left(r + \frac{1}{2}(\lambda_1^2 + \lambda_2^2\sigma_r^2)\right)We^{-\beta t} - \lambda_2\sigma_r^2 e^{-\beta t}W_r\\ &+ \frac{1}{2}\sigma_r^2 e^{-\beta t}\frac{W_r^2}{W} + \alpha e^{-\beta t}(\ln\alpha - \ln W) = 0. \end{aligned} \tag{29}$$

Eliminating the dependence on x, (29) can be split into the following two equations:

$$\begin{aligned} &W_t - \beta W + (a-br)W_r + \frac{1}{2}\sigma_r^2 W_{rr} + \alpha = 0,\\ &W(T,r) = 1-\alpha; \end{aligned} \tag{30}$$

$$\begin{aligned} &V_t + (a-br)V_r + \frac{1}{2}\sigma_r^2 V_{rr} - \alpha\\ &+ \left(r + \frac{1}{2}(\lambda_1^2 + \lambda_2^2\sigma_r^2)\right)We^{-\beta t}\\ &- \lambda_2\sigma_r^2 e^{-\beta t}W_r + \frac{1}{2}\sigma_r^2 e^{-\beta t}\frac{W_r^2}{W}\\ &+ \alpha e^{-\beta t}(\ln\alpha - \ln W) = 0, \quad V(T,r) = 0. \end{aligned} \tag{31}$$

Lemma 7 *Assume that the solution of (30) is of the form $W(t,r) = D(t) + E(t)r$, with boundary conditions given by $D(T) = 1-\alpha$ and $E(T) = 0$, then*

$$D(t) = (1-\alpha)e^{-\beta(T-t)} - \frac{\alpha}{\beta}[e^{-\beta(T-t)} - 1],$$

$$E(t) = 0.$$

Proof. Introducing $W(t,r) = D(t) + E(t)r$ into (30), we get

$$D'(t) - \beta D(t) + aE(t) + \alpha$$

$$+ r(E'(t) - (\beta+b)E(t)) = 0.$$

Eliminating the dependence on r, we obtain two equations:

$$E'(t) - (\beta+b)E(t) = 0,$$
$$E(T) = 0;$$
$$D'(t) - \beta D(t) + aE(t) + \alpha = 0,$$
$$D(T) = 1-\alpha;$$

Solving the above two equations, we complete the proof. □

Lemma 8 *Suppose that (31) is of the solution $V(t,r) = F(t) + G(t)r$, with boundary conditions $F(T) = 0$ and $G(T) = 0$, then $F(t)$ and $G(t)$ are determined by (36) and (35) respectively.*

Proof. Putting $V(t,r) = F(t) + G(t)r$ in the equation (31), we obtain

$$\begin{aligned} &F'(t) + aG(t) + \frac{1}{2}(\lambda_1^2 + k_2\lambda_2^2)D(t)e^{-\beta t}\\ &- \alpha + \alpha e^{-\beta t}(\ln\alpha - \ln D(t))\\ &+ r\left(G'(t) - bG(t) + (\frac{1}{2}k_1\lambda_2^2 + 1)D(t)e^{-\beta t}\right) = 0. \end{aligned} \tag{32}$$

Comparing the coefficients on the both sides of (32), we can decompose (32) into two equations:

$$\begin{aligned} &G'(t) - bG(t) + (\frac{1}{2}k_1\lambda_2^2 + 1)D(t)e^{-\beta t} = 0,\\ &G(T) = 0; \end{aligned} \tag{33}$$

$$F'(t) + aG(t) + \frac{1}{2}(\lambda_1^2 + k_2\lambda_2^2)D(t)e^{-\beta t}$$
$$-\alpha + \alpha e^{-\beta t}(\ln\alpha - \ln D(t)) = 0, \quad F(T) = 0. \tag{34}$$

Solving the equation (33) and (34), we get

$$G(t) = \frac{1}{b}(\frac{1}{2}k_1\lambda_2^2+1)(1-\alpha-\frac{\alpha}{\beta})e^{-\beta T}(1-e^{-b(T-t)})$$
$$+\frac{1}{\beta+b}(\frac{1}{2}k_1\lambda_2^2+1)\frac{\alpha}{\beta}e^{-\beta T}(e^{\beta(T-t)} - e^{-b(T-t)}), \tag{35}$$
$$F(t) = \int_t^T aG(t)dt + \int_t^T \frac{1}{2}(\lambda_1^2 + k_2\lambda_2^2)D(t)e^{-\beta t}dt$$
$$+\int_t^T (\alpha\ln\alpha e^{-\beta t} - \alpha)dt - \int_t^T \alpha e^{-\beta t}\ln D(t)dt. \tag{36}$$

Therefore, Lemma 8 is completed. □

Taking Lemma 7 into account, (28) is reduced to

$$C^*(t) = \frac{\alpha}{D(t)}x,$$

$$\frac{H_x}{H_{xx}} = -x, \quad \frac{H_{rx}}{H_{xx}} = -\frac{W_r}{W}x = 0.$$

Therefore, we can obtain the optimal trading policy of the problem (6) in the logarithm utility case.

Proposition 9 *If utility function is given by $U_1(x) = U_2(x) = \ln x$, the optimal investment and consumption strategies for the problem (6) are*

$$\pi_s^*(t) = \frac{\lambda_1}{\sigma_1}X(t),$$

$$\pi_B^*(t) = \frac{\sigma_r(\lambda_2\sigma_1 - \lambda_1\sigma_2)}{\sigma_1\sigma_B}X(t),$$

$$C^*(t) = \frac{\alpha}{(1-\alpha)e^{-\beta(T-t)} - \frac{\alpha}{\beta}[e^{-\beta(T-t)} - 1]}X(t).$$

Remark 10 *(i) The parameters α and β have no impact on $\pi_s^*(t)$ and $\pi_B^*(t)$, but have impact on $C^*(t)$. (ii) The parameters of affine interest rate model, i,e, a,b,k_1 and k_2, have no impact on $\pi_s^*(t)$, $\pi_B^*(t)$ and $C^*(t)$. It means that the optimal investment and consumption strategies under affine interest rate model agrees with those under constant interest rate model when utility function is logarithmic. (iii) The parameters λ_2 and σ_2 affect the dynamics of stock price, but have no influence on $\pi_s^*(t)$ and $C^*(t)$. (iv) The volatility parameters σ_1,σ_2 and σ_B have no effect on $C^*(t)$.*

4 Numerical Analysis

In this section, we take power utility for example, and provide a numerical example to illustrate the effect of market parameters on the optimal investment and consumption strategy. Throughout numerical analysis, unless otherwise stated, the basic market parameters are given by $a = 0.018712$, $b = 0.2339$, $k_1 = 0.00729316$, $k_2 = 0$, $r(0) = 0.05$, $\sigma_1 = 0.2$, $\lambda_1 = 0.2$, $\sigma_2 = 0.02$, $\lambda_2 = 1$.

The set of parameters representing the financial market is consistent with the numerical analysis presented by Deelstra et al. [26] and Gao [27]. For the sake of calculating convenience, we assume that $\alpha = 0.6$, $\beta = 0.05$, $\eta = -1$, $t = 0$, $T = 1$, $x_0 = 100$. According to the results of Proposition 1, we can draw some graphs in the following Figure 1 and Figure 8. Notice that in the Figure 1–Figure 4, the dashed line represents the amount $\pi_s^*(t)$ invested in the stock and the orange line represents the amount $\pi_B^*(t)$ invested in the zero-coupon bond, while the amount $\pi_0^*(t)$ invested in the cash account is denoted by the thick line.

4.1 The effect on the optimal investment strategy

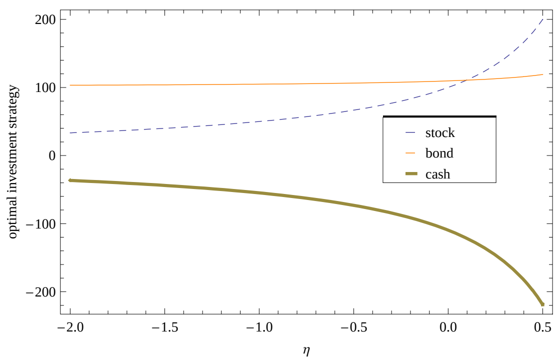

Figure 1: The impact of η on $\pi_s^*(t)$ and $\pi_B^*(t)$.

We can summarize some conclusions from the Figure 1-Figure 4.

(a1) $\pi_s^*(t)$ and $\pi_B^*(t)$ increase with respect to the parameter η respectively, while $\pi_0^*(t)$ just decreases with respect to η. This coincides with the economic implication of the parameter η. In the power utility theory, risk aversion coefficient measuring the risk preference of an investor is the value of $1-\eta$. Therefore, the bigger the value of η, the smaller the risk aversion coefficient $1-\eta$. It will make the investor invest more money in the stock and zero-coupon bond, while invest less money in the cash account.

(a2) $\pi_s^*(t)$ is almost fixed and $\pi_B^*(t)$ is increasing in the parameter b, while $\pi_0^*(t)$ is decreasing in b.

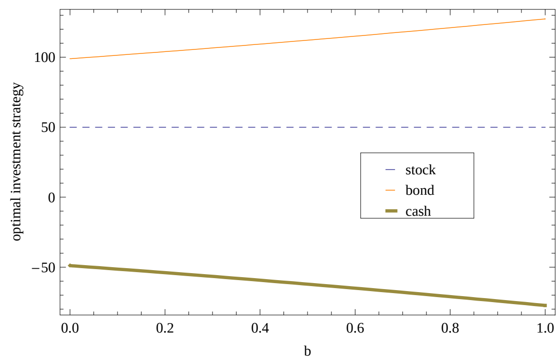

Figure 2: The impact of b on $\pi_s^*(t)$ and $\pi_B^*(t)$.

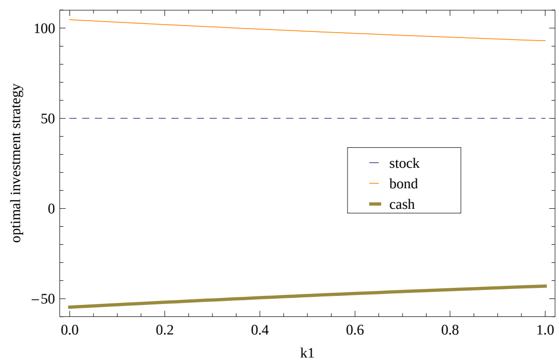

Figure 3: The impact of k_1 on $\pi_s^*(t)$ and $\pi_B^*(t)$.

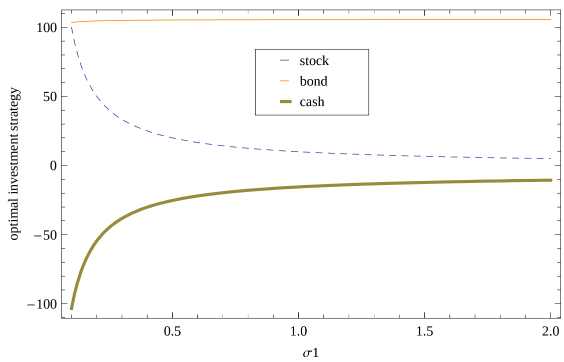

Figure 4: The impact of σ_1 on $\pi_s^*(t)$ and $\pi_B^*(t)$.

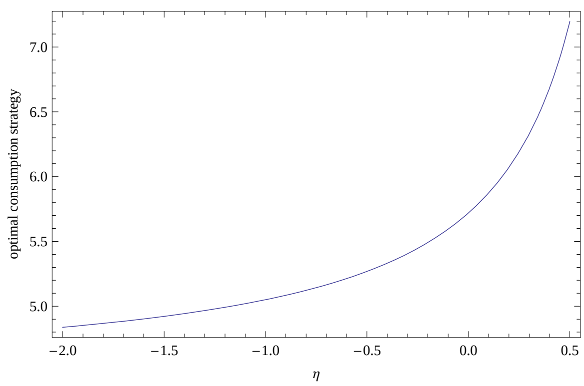

Figure 5: The impact of η on $C^*(t)$.

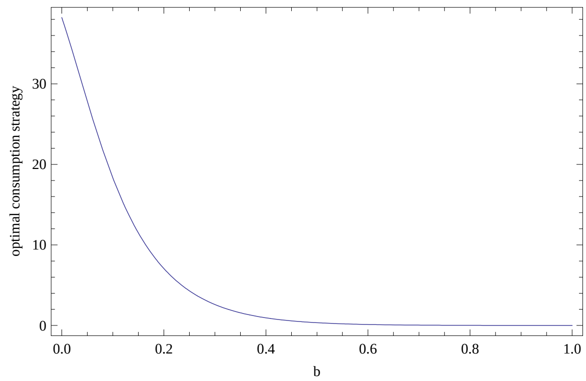

Figure 6: The impact of b on $C^*(t)$.

From the economic implication of b, when the value of b becomes larger, the expected value of interest rate $r(t)$ will become smaller. It leads to that the volatility risk of interest rate will become smaller. Therefore, the larger the value of b, the more the amount invested in the zero-coupon bond, while the less the amount invested in the cash account.

(a3) $\pi_B^*(t)$ decreases with respect to the parameter k_1, while $\pi_0^*(t)$ increases in k_1 and $\pi_s^*(t)$ is almost fixed. Meantime, $\pi_s^*(t)$ is not affected by the parameter k_1. This remark is intuitive, because the parameter k_1 measures the volatility of interest rate. The larger the value of k_1, the more the risk generated by interest rate, and hence the less an investor wishes to invest in the zero-coupon. This is the reason that the investor invests more money in the cash account.

(a4) $\pi_s^*(t)$decreases as the value of σ_1 becomes bigger, while $\pi_0^*(t)$ increases. In addition, the parameter σ_1 has no influence on $\pi_B^*(t)$. Form the economic implication of σ_1, the parameter σ_1 represents the volatility risk of the stock, the bigger the value of σ_1, the more the volatility risk of the stock, and it leads to that the more aggressive the investor is. Hence, the investor wishes to decrease the amount invested in the stock, while invest more money in the cash account.

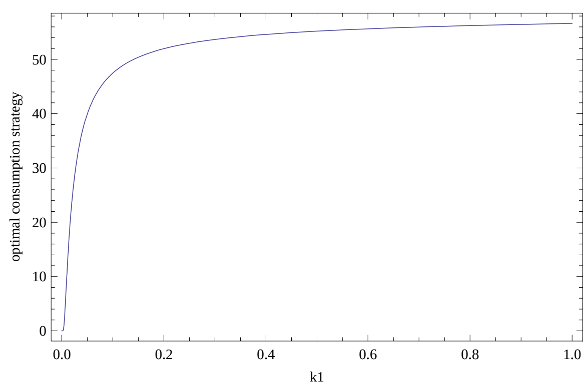

Figure 7: The impact of k_1 on $C^*(t)$.

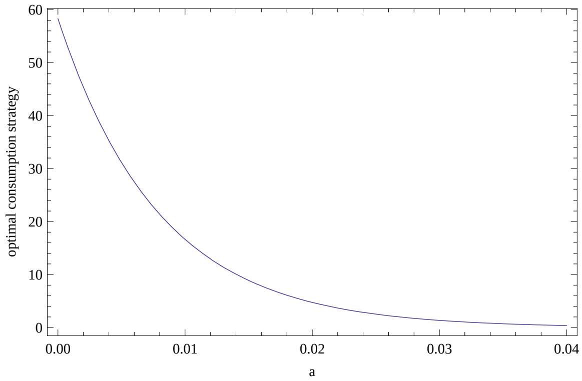

Figure 8: The impact of a on $C^*(t)$.

4.2 The effect on the optimal consumption strategy

We can illustrate how market parameter has an effect on the optimal consumption strategy $C^*(t)$ from the Figure 5–Figure 8.

(b1) $C^*(t)$ increases with respect to the parameter η. As matter of fact, when the value of η becomes large, the amount invested in the stock and zero-coupon bond is increasing, which results in the wealth of the investor will become large. Therefore, the investor has more money to consume.

(b2) $C^*(t)$is decreasing in the parameter b. It can be seen from (a2) that when the value of b is increasing the amount invested in the zero-coupon bond is increasing, which implies that the wealth of the investor is increasing. However, contrary to the intuitive, the amount that an investor should consume is decreasing when the value of b becomes bigger. We should keep this point in mind in the practice of investment.

(b3) $C^*(t)$will increase as the parameter k_1 becomes larger. This implies that the bigger the value of k_1, the more the risk of interest rate, and meantime the more the investor can consume.

(b4) $C^*(t)$ decreases with respect to the parameter a. It shows that when the value of a becomes bigger, the expected value of interest rate will also become much larger, and the investor should consume less money.

5 Conclusions

In this article, we consider consumption behavior of the individuals on the basis of the works of Deelstra et al. [26] and Gao [27], Chang et al. [28] and study the optimal consumption and portfolio decisions with stochastic affine interest rate model. The financial market is composed of three assets: one cash account, one stock and one zero-coupon bond. The explicit expressions of the optimal consumption and portfolio decisions are successfully obtained in the power utility and logarithm utility cases. Finally, we provide a numerical example to analyze the effect of market parameters on the optimal consumption and portfolio decisions and display some economic implications.

In future research on the consumption and investment problems, it would be very interesting to extend our financial market to the cases of the more sophisticated environments, such as the investment and consumption problems with interest rate risk and inflation risk. On the other hand, we also consider an investment and consumption problem in the regime-switching and affine interest rate framework. Nevertheless, we leave these points to future research.

Appendix:

The Proof of Theorem 1. Letting $Q=[0,\infty)\times[0,\infty)$, we take a sequence of bounded open sets Q_i with $Q_i\subset Q_{i+1}\subset Q$, $i=1,2,\cdots$, and $Q=\bigcup_{i=1}^{\infty}Q_i$. For $(r,x)\in Q_1$, denote the exit time of $(r(t),X(t))$ from Q_i by τ_i. when $i\to\infty$, we have $\tau_i\wedge T\to T$.

(i) Consider an arbitrary admissible policy $(\pi(t),C(t))$.

Applying $It\hat{o}'s$ formula to $H(t,r,x)$ on $[t,T]$, we obtain

$$\begin{aligned}&H(T,r(T),X(T))=H(t,r,x)\\&+\int_t^T\mathscr{A}^{\pi,C}H(s,r(s),X(s))ds\end{aligned}$$

$$\begin{aligned}&+\int_t^T\pi_s\sigma_1H_x(s,r(s),X(s))dW_s(s)\\&+\int_t^T\left(\pi_s\sigma_2\sqrt{k_1r(t)+k_2}+\pi_B\sigma_B\right)\\&\times H_x(s,r(s),X(s))dW_r(s)\\&-\int_t^T\sqrt{k_1r(t)+k_2}H_r(s,r(s),X(s))dW_r(s).\end{aligned}$$

Considering $\sup\limits_{(\pi(t),C(t))\in\Gamma}\left\{\mathscr{A}^{\pi,C}H(t,r,x)\right\}=0$, which implies the variational inequality

$\mathscr{A}^{\pi,C}H(t,r,x) \leq 0$, we have

$$\begin{aligned}&H(T,r(T),X(T)) \leq H(t,r,x)\\&+\int_t^T \pi_s\sigma_1 H_x(s,r(s),X(s))dW_s(s)\\&+\int_t^T \left(\pi_s\sigma_2\sqrt{k_1r(t)+k_2}+\pi_B\sigma_B\right)\\&\times H_x(s,r(s),X(s))dW_r(s)\\&-\int_t^T \sqrt{k_1r(t)+k_2}H_r(s,r(s),X(s))dW_r(s).\end{aligned}$$

For the last three terms on the right hand of the above inequality are square-integrable martingales with zero expectation. Hence, we have

$$\begin{aligned}&E\left(H(T,r(T),X(T))\,|X(t)=x,r(t)=r\right)\\&\leq H(t,r,x).\end{aligned}$$

Further, taking the supermum, we get

$$\begin{aligned}\sup_{(\pi(t),C(t))\in\Gamma} &E\left(H(T,r(T),X(T))\,|X(t)=x,r(t)=r\right)\\&\leq H(t,r,x).\end{aligned}$$

and it results in

$$V(t,r,x) \leq H(t,r,x).$$

(ii) $E\left(H(\tau_i\wedge T, r(\tau_i\wedge T), X(\tau_i\wedge T))\right) < \infty$ for a specific strategy $(\pi^*(t),C^*(t))$.

Applying $It\hat{o}'s$ formula on $[0,\tau_i\wedge T]$ once again, we have

$$\begin{aligned}&H((\tau_i\wedge T, r(\tau_i\wedge T), X(\tau_i\wedge T)) = H(0,r_0,x_0)\\&+\int_0^{\tau_i\wedge T} \mathscr{A}^{\pi^*,C^*}H(s,r(s),X(s))ds\\&+\int_0^{\tau_i\wedge T} \pi_s\sigma_1 H_x(s,r(s),X(s))dW_s(s)\\&+\int_0^{\tau_i\wedge T} \left(\pi_s\sigma_2\sqrt{k_1r(t)+k_2}+\pi_B\sigma_B\right)\\&\times H_x(s,r(s),X(s))dW_r(s)\\&-\int_0^{\tau_i\wedge T} \sqrt{k_1r(t)+k_2}H_r(s,r(s),X(s))dW_r(s).\end{aligned}$$

For a specific strategy $(\pi^*(t),C^*(t))$ satisfies (10), i.e. $\mathscr{A}^{\pi^*,C^*}H(s,r(s),X(s))=0$, and the last three terms are also square-integrable martingales. Hence, taking the expectation on both sides on the above equation, we obtain

$$\begin{aligned}&E\left(H((\tau_i\wedge T, r(\tau_i\wedge T), X(\tau_i\wedge T))\right)\\&= H(0,r_0,x_0) < \infty.\end{aligned}$$

(iii) $V(t,r,x) = H(t,r,x)$ for a specific strategy $(\pi^*(t),C^*(t))$.

Applying $It\hat{o}'s$ formula on $[t,\tau_i\wedge T]$ once again, similarly, we have

$$\begin{aligned}&H((\tau_i\wedge T, r(\tau_i\wedge T), X(\tau_i\wedge T)) = H(t,r,x)\\&+\int_t^{\tau_i\wedge T} \mathscr{A}^{\pi^*,C^*}H(s,r(s),X(s))ds\\&+\int_t^{\tau_i\wedge T} \pi_s\sigma_1 H_x(s,r(s),X(s))dW_s(s)\\&+\int_t^{\tau_i\wedge T} \left(\pi_s\sigma_2\sqrt{k_1r(t)+k_2}+\pi_B\sigma_B\right)\\&\times H_x(s,r(s),X(s))dW_r(s)\\&-\int_t^{\tau_i\wedge T} \sqrt{k_1r(t)+k_2}H_r(s,r(s),X(s))dW_r(s).\end{aligned}$$

Taking the expectation yields

$$\begin{aligned}&H(t,r,x) = E(H((\tau_i\wedge T, r(\tau_i\wedge T),\\&X(\tau_i\wedge T))\,|r(t)=t,X(t)=x).\end{aligned}$$

Taking the limitation, we get

$$\begin{aligned}&H(t,r,x) = \lim_{i\to\infty} E(H(\tau_i\wedge T, r(\tau_i\wedge T),\\&X(\tau_i\wedge T))\,|X(t)=x,r(t)=r).\end{aligned}$$

As a result, we derive

$$\begin{aligned}&V(t,r,x) = \sup_{(\pi(t),C(t))\in\Gamma} E(\alpha\int_0^T e^{-\beta t}U_1(C(t))dt\\&+(1-\alpha)e^{-\beta T}U_2(X(T))\,|X(t)=x,r(t)=r)\\&=\lim_{i\to\infty} E(H(\tau_i\wedge T, r(\tau_i\wedge T),\\&\quad X(\tau_i\wedge T))\,|X(t)=x,r(t)=r)\\&=H(t,r,x).\end{aligned}$$

Therefore, it implies that $(\pi^*(t),C^*(t))$ is indeed the optimal investment-consumption strategy for the problem (6). □

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