On the Norms and Spreads of Fermat, Mersenne and Gaussian Fibonacci RFMLR Circulant Matrices

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Abstract: In this paper, we consider norms and spreads of RFMLR circulant matrices involving the Fermat, Mersenne sequences and Gaussian Fibonacci number, respectively. Firstly, we reviewed some properties of the Fermat, Mersenne sequences, Gaussian Fibonacci number and RFMLR circulant matrices. Furthermore, we give lower and upper bounds for the spectral norms and spread of these special matrices. Finally, we give several corollaries related to norms of Hadamard and Kronecker products of these matrices.

Key–Words: RFMLR circulant matrix, Norm, Spread, Fermat sequence, Mersenne sequence, Gaussian Fibonacci number

1 Introduction

Recently, there have been several papers on the norms of some special matrices [1, 2, 3, 4, 5, 6, 7, 8, 9, 11]. Akbulak [1] found upper and lower bounds for the spectral norms of Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{i-j} \equiv L_{i-j}$. Solak and Bozkurt [6] have found out upper and lower bounds for the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms $T_n = [\frac{1}{a+(i-j)b}]_{i,j=1}^{n}$, $H_n = [\frac{1}{a+(i+j)b}]_{i,j=1}^{n}$. Solak [8, 9] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ circulant matrices, where $a_{ij} \equiv F_{(\mathrm{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\mathrm{mod}(j-i,n))}$, then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms. In [3], the authors give upper and lower bounds for the spectral norms of matrices $A = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ and $B = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$, where $\{F_{k,n}\}_{n\in N}$ and $\{L_{k,n}\}_{n\in N}$ are k-Fibonacci and k-Lucas sequences respectively, and they also give the bounds for the spectral norms of Kronecker and Hadamard products of these matrices $A = Circ\Big(F_0^{(k,h)}, F_1^{(k,h)}, \cdots, F_{n-1}^{(k,h)}\Big)$ and $B = Circ\Big(L_0^{(k,h)}, L_1^{(k,h)}, \cdots, L_{n-1}^{(k,h)}\Big)$, where $F_n^{(k,h)}$ and $L_n^{(k,h)}$ are (k,h)-Fibonacci and (k,h)-Lucas numbers respectively[4].

Beginning with Mirsky [12] several author [13, 14, 15, 16, 17, 18, 19] have obtained bounds for the spread of a matrix.

Lately, some scholars gave the explicit determinant and inverse of the circulant and skew-circulant matrix involving famous numbers.

For any integer $m \geq 0$; let $F_m = 2^{2^m} + 1$ be the mth Fermat number. It is well known that F_m is prime for $m \leq 4$; but there is no other m for which F_m is known to prime. The Fermat and Mersenne sequences are defined by the following recurrence relations [20], respectively:

$$\mathbf{F}_{n+1} = 3\mathbf{F}_n - 2\mathbf{F}_{n-1} \quad (1)$$

$$\mathbf{M}_{n+1} = 3\mathbf{M}_n - 2\mathbf{M}_{n-1} \quad (2)$$

where $\mathbf{M}_0 = 0, \mathbf{M}_1 = 1, \mathbf{F}_0 = 2, \mathbf{F}_1 = 3$, for $n \geq 1$.

The Gaussian Fibonacci sequence [20, 21] is defined by the following recurrence relations:

$$G_{n+1} = G_n + G_{n-1}, \quad n \geq 1 \quad (3)$$

with the initial condition $G_0 = \mathrm{i}, G_1 = 1$. $G_n = F_n + \mathrm{i}F_{n-1}$, where F_n is the nth Fibonacci number, $\mathrm{i} = \sqrt{-1}$.

In [26], their Binet forms are given by

$$\mathbf{F}_n = 2^n + 1,$$
$$\mathbf{M}_n = 2^n - 1,$$
$$G_n = \frac{a^n - b^n + (a^{n-1} - b^{n-1})i}{a-b},$$

where a and b are the roots of the characteristic equation $x^2 - x - 1 = 0$.

Lemma 1 *Let $\mathbf{F}_n$ be the n-th Fermat sequence and $\mathbf{M}_n$ be the n-th Mersenne sequence, G_n be the n-th Gaussian Fibonacci numbers, then we have*

$$1.\ \sum_{j=0}^{n-1}\mathbf{F}_j = \mathbf{F}_n + n - 2;$$

$$2.\ \sum_{j=0}^{n-1}\mathbf{F}_j\mathbf{F}_{j+1} = \frac{\mathbf{F}_n\mathbf{F}_{n+1}+6\mathbf{F}_n+3n-18}{3};$$

$$3.\ \sum_{j=0}^{n-1}\mathbf{F}_j^2 = \frac{4\mathbf{F}_{n-1}^2+4\mathbf{F}_{n-1}+3(n-5)}{3};$$

$$4.\ \sum_{j=0}^{n-1}\mathbf{M}_j = \mathbf{M}_n - n;$$

$$5.\ \sum_{j=0}^{n-1}\mathbf{M}_j\mathbf{M}_{j+1} = \frac{\mathbf{M}_n\mathbf{M}_{n+1}-6\mathbf{M}_n+3n}{3};$$

$$6.\ \sum_{j=0}^{n-1}\mathbf{M}_j^2 = \frac{4\mathbf{M}_{n-1}^2-4\mathbf{M}_{n-1}+3(n-1)}{3};$$

$$7.\ \sum_{j=0}^{n-1}G_j = G_{n+1} - G_1;$$

$$8.\ \sum_{j=0}^{n-1}G_jG_{j+1} = G_nG_{n+1} - G_nG_{n-1} + G_1(-1)^n + \tfrac{1}{2}G_0[1-(-1)^n];$$

$$9.\ \sum_{j=0}^{n-1}G_j^2 = G_{n-1}G_n - G_2.$$

Definition 2 *[22, 23] A row first-minus-last right (RFMLR) circulant matrix with the first row $(a_0, a_1, \cdots, a_{n-1})$, denoted by $\mathrm{RFMLRcircfr}(a_0, a_1, \cdots, a_{n-1})$, meaning a square matrix of the form*

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & m_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & m_2 & \ddots & \ddots & \vdots \\ a_2 & \vdots & \ddots & \ddots & a_1 \\ a_1 & a_2 - a_1 & \cdots & m_2 & m_1 \end{pmatrix} \tag{4}$$

where

$$m_1 = a_0 - a_{n-1},$$

and

$$m_2 = a_{n-1} - a_{n-2}.$$

It can be seen that the matrix with an arbitrary first row and the following rule for obtaining any other row from the previous one: Get the i+1st row by letting the first element of the ith row minus the last element of the ith row as the first element of the i+1st row, and then shifting the elements of the ith row, cyclically, one position to the right as the rest elements of the i+1st row.

Obviously, the RFMLR circulant matrix is determined by its first row, and RFMLR circulant matrix is a $x^n + x - 1$ circulant matrix [24] and is also a RFPrLR circulant matrix [25]. We define $\Theta_{(1,-1)}$ as the basic RFMLR circulant matrix, that is,

$$\begin{aligned}\Theta_{(1,-1)} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \\ &= \mathrm{RFMLRcircfr}(0, 1, 0, \cdots, 0)\end{aligned} \tag{5}$$

It is easily verified that $g(x) = x^n + x - 1$ has no repeated roots in its splitting field and $g(x) = x^n + x - 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(1,-1)}$. In addition, $\Theta_{(1,-1)}$ is nonderogatory and satisfies $\Theta_{(1,-1)}^j = \mathrm{RFMLRcircfr}(\underbrace{0, \cdots, 0}_{j}, 1, \underbrace{0, \cdots, 0}_{n-j-1})$ and $\Theta_{(1,-1)}^n = I_n - \Theta_{(1,-1)}$.

According to the structure of the powers of the basic RFMLR circulant matrix $\Theta_{(1,-1)}$, it is clear that

$$\begin{aligned} A &= \mathrm{RFMLRcircfr}(a_0, a_1 \cdots, a_{n-1}) \\ &= \sum_{i=0}^{n-1} a_i \Theta_{(1,-1)}^i \end{aligned} \tag{6}$$

Thus, A is a RFMLR circulant matrix if and only if $A = f(\Theta_{(1,-1)})$ for some polynomial $f(x)$. The polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ will be called the representer of the RFMLR circulant matrix A. Because of Definition 2 and Equation (6), it is clear that A is a RFMLR circulant matrix if and only if A commutes with $\Theta_{(1,-1)}$, that is, $A\Theta_{(1,-1)} = \Theta_{(1,-1)}A$.

In addition to the algebraic properties that can be easily derived from the representation (6), we mention that RFMLR circulant matrices have very nice structure. The product of two RFMLR circulant matrices is a RFMLR circulant matrix and A^{-1} is a RFMLR circulant matrix, too.

In this study, we define matrices of forms:let $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$, $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$ and $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$ be $n \times n$ matrices. Firstly, we give lower and upper bounds for the spectral norms of these matrices. Furthermore, we give some corollaries related to norms of Hadamard and Kronecker products of these matrices.

Definition 3 *[1] Let $A=(a_{ij})$ be an $n\times n$ matrix. The Euclidean (or Frobenius) norm, the spectral norm, the maximum column sum matrix norm, the maximum row sum matrix norm of the matrix A are, respectively,*

$$\|A\|_F=\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}},$$

$$\|A\|_2=\left(\max_{1\le i\le n}\lambda_i(A^*A)\right)^{\frac{1}{2}},$$

$$\|A\|_1=\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|,$$

$$\|A\|_\infty=\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|,$$

where A^* denotes the conjugate transpose of A.

The following inequality holds [10]:

$$\frac{1}{\sqrt{n}}\|A\|_F\le\|A\|_2\le\|A\|_F \tag{7}$$

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be $n\times n$ matrices. The Hadamard product of A and B is defined by $A\circ B=[a_{ij}b_{ij}]$. If $\|\cdot\|$ is any norm on $n\times m$ matrices, then

$$\|A\circ B\|\le\|A\|\cdot\|B\|.$$

Let A and B be arbitrary $n\times m$ matrices. Kronecker product of A and B is given to be [8]

$$A\otimes B=\begin{bmatrix} a_{11}B & \cdots & a_{1m}B\\ \vdots & & \vdots\\ a_{n1}B & \cdots & a_{nm}B\end{bmatrix}.$$

then

$$\|A\otimes B\|_F=\|A\|_F\|B\|_F.$$

Definition 4 *[13] Let $A=(a_{ij})$ be an $n\times n$ matrix with eigenvalues λ_i, $i=1,2,\dots,n$. The spread of A is defined as*

$$s(A)=\max_{i,j}|\lambda_i-\lambda_j|.$$

An upper bound for the spread due to Mirsky [12] states that

$$s(A)\le\sqrt{2\|A\|_F^2-\frac{2}{n}|\mathrm{tr}A|^2} \tag{8}$$

where $\|A\|_F$ denotes the Frobenius norm of A and $\mathrm{tr}A$ is the trace of A.

2 On the Norms and Spreads of RFMLR Circulant Matrices with the Fermat Sequence

Theorem 5 *Let $A=\mathrm{RFMLRcircfr}(\mathbf{F}_0,\mathbf{F}_1,\cdots,\mathbf{F}_{n-1})$, where $\{\mathbf{F}_j\}_{0\le j\le n-1}$ denote the Fermat sequence given by (1). For $n\ge1$, then three kinds of norms of A are given by*

$$\|A\|_1=\|A\|_\infty=\mathbf{F}_n+\mathbf{F}_{n-1}+n-4,$$

and

$$\|A\|_F=\sqrt{\frac{1}{18}(\alpha_1+\alpha_2+\alpha_3)},$$

where

$$\alpha_1=24(2n-1)(\mathbf{F}_{n-1}^2+\mathbf{F}_{n-1})-32(\mathbf{F}_{n-2}^2+\mathbf{F}_{n-2}),$$
$$\alpha_2=2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n+6\mathbf{F}_{n-1}),$$
$$\alpha_3=6(20-12n)\mathbf{F}_{n-1}-3(9n^2-33n+136).$$

Proof: By Definition 3 and Lemma 1, we have

$$\begin{aligned}\|A\|_1=\|A\|_\infty&=\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\\&=\sum_{i=0}^{n-1}\mathbf{F}_i+\mathbf{F}_{n-1}-\mathbf{F}_0\\&=\mathbf{F}_n+\mathbf{F}_{n-1}+n-4.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}\|A\|_F^2&=n\sum_{j=0}^{n-1}\mathbf{F}_j^2+\sum_{j=1}^{n-1}j\mathbf{F}_j^2-2\sum_{j=1}^{n-2}j\mathbf{F}_j\mathbf{F}_{j+1}\\&\quad-2(n-1)\mathbf{F}_0\mathbf{F}_{n-1}\\&=n\sum_{j=0}^{n-1}\mathbf{F}_j^2+\sum_{k=1}^{n-1}\sum_{j=n-k}^{n-1}\mathbf{F}_j^2-2\sum_{k=1}^{n-2}\sum_{j=n-k-1}^{n-2}\mathbf{F}_j\mathbf{F}_{j+1}\\&\quad-2(n-1)\mathbf{F}_0\mathbf{F}_{n-1}\\&=n\sum_{j=0}^{n-1}\mathbf{F}_j^2+\sum_{k=1}^{n-1}\left(\sum_{j=0}^{n-1}\mathbf{F}_j^2-\sum_{j=0}^{n-k-1}\mathbf{F}_j^2\right)\\&\quad-2\sum_{k=1}^{n-2}\left(\sum_{j=0}^{n-2}\mathbf{F}_j\mathbf{F}_{j+1}-\sum_{j=0}^{n-k-2}\mathbf{F}_j\mathbf{F}_{j+1}\right)\\&\quad-2(n-1)\mathbf{F}_0\mathbf{F}_{n-1}\\&=\tfrac{1}{18}(\alpha_1+\alpha_2+\alpha_3).\end{aligned}$$

Thus

$$\|A\|_F=\sqrt{\frac{1}{18}(\alpha_1+\alpha_2+\alpha_3)} \tag{9}$$

where

$$\alpha_1=24(2n-1)(\mathbf{F}_{n-1}^2+\mathbf{F}_{n-1})-32(\mathbf{F}_{n-2}^2+\mathbf{F}_{n-2}),$$
$$\alpha_2=2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n+6\mathbf{F}_{n-1}),$$
$$\alpha_3=6(20-12n)\mathbf{F}_{n-1}-3(9n^2-33n+136).$$

□

Theorem 6 *Let* $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$*, where* $\{\mathbf{F}_j\}_{0\le j\le n-1}$ *denote the Fermat sequence given by (1), then*

$$\sqrt{\frac{1}{18n}(\alpha_1+\alpha_2+\alpha_3)} \le \|A\|_2,$$

where

$$\alpha_1 = 24(2n-1)(\mathbf{F}_{n-1}^2+\mathbf{F}_{n-1})-32(\mathbf{F}_{n-2}^2+\mathbf{F}_{n-2}),$$

$$\alpha_2 = 2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n+6\mathbf{F}_{n-1}),$$

$$\alpha_3 = 6(20-12n)\mathbf{F}_{n-1}-3(9n^2-33n+136),$$

and

$$\|A\|_2 \le 2(\mathbf{F}_n+n-2),$$

where $\|\cdot\|_2$ *is the spectral norm.*

Proof: The matrix A is of the form

$$A = \begin{pmatrix} \mathbf{F}_0 & \mathbf{F}_1 & \cdots & \mathbf{F}_{n-2} & \mathbf{F}_{n-1} \\ \mathbf{F}_{n-1} & t_1 & \mathbf{F}_1 & \cdots & \mathbf{F}_{n-2} \\ \vdots & t_2 & \ddots & \ddots & \vdots \\ \mathbf{F}_2 & \vdots & \ddots & \ddots & \mathbf{F}_1 \\ \mathbf{F}_1 & \mathbf{F}_2-\mathbf{F}_1 & \cdots & t_2 & t_1 \end{pmatrix} \quad (10)$$

where

$$t_1 = \mathbf{F}_0 - \mathbf{F}_{n-1},$$

and

$$t_2 = \mathbf{F}_{n-1} - \mathbf{F}_{n-2}.$$

We know that $\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2 \le \|A\|_F$ from equivalent norms, where $\|\cdot\|_F$ is the Frobenius norm. By Equation (9), we can get

$$\frac{1}{\sqrt{n}}\|A\|_F = \sqrt{\frac{1}{18n}(\alpha_1+\alpha_2+\alpha_3)},$$

so

$$\sqrt{\frac{1}{18n}(\alpha_1+\alpha_2+\alpha_3)} \le \|A\|_2,$$

where

$$\alpha_1 = 24(2n-1)(\mathbf{F}_{n-1}^2+\mathbf{F}_{n-1})-32(\mathbf{F}_{n-2}^2+\mathbf{F}_{n-2}),$$

$$\alpha_2 = 2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n+6\mathbf{F}_{n-1}),$$

$$\alpha_3 = 6(20-12n)\mathbf{F}_{n-1}-3(9n^2-33n+136).$$

On the other hand, supposed that

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

then

$$A = \sum_{j=0}^{n-1}\mathbf{F}_j Q_1^j - \sum_{j=1}^{n-2}\mathbf{F}_{n-j-1}Q_2^j - \mathbf{F}_{n-1}Q_3.$$

We can get

$$\begin{aligned} \|A\|_2 &= \left\|\sum_{j=0}^{n-1}\mathbf{F}_j Q_1^j - \sum_{j=1}^{n-2}\mathbf{F}_{n-j-1}Q_2^j - F_{n-1}Q_3\right\|_2 \\ &\le \sum_{j=0}^{n-1}\mathbf{F}_j\|Q_1\|_2^j + \sum_{j=1}^{n-2}\mathbf{F}_{n-j-1}\|Q_2\|_2^j \\ &\quad +\mathbf{F}_{n-1}\|Q_3\|_2 \end{aligned} \quad (11)$$

Since

$$Q_1^H Q_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$Q_2^H Q_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and

$$Q_3^H Q_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we can get

$$\|Q_1\|_2 = \|Q_2\|_2 = \|Q_3\|_2 = 1.$$

So the other result is obtained as follows

$$\begin{aligned}\|A\|_2 \leq \sum_{j=1}^{n-1} \mathbf{F}_j \|Q_1\|_2^j + \sum_{j=1}^{n-2} \mathbf{F}_{n-j-1} \|Q_2\|_2^j \\ +\mathbf{F}_{n-1}\|Q_3\|_2 \\ = 2\sum_{j=0}^{n-1} \mathbf{F}_j = 2(\mathbf{F}_n + n - 2) \qquad (12)\end{aligned}$$

Thus the proof is completed. □

Theorem 7 *Let* $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$*, where* $\{\mathbf{F}_j\}_{0\leq j\leq n-1}$ *denote the Fermat sequence given by* (1)*, then*

$$s(A) \leq \sqrt{\frac{1}{9}(\alpha_1+\alpha_2+\alpha_3) - \frac{2}{n}[(n-1)\mathbf{F}_{n-1} - 2n]^2},$$

where

$$\begin{aligned}\alpha_1 &= 24(2n-1)(\mathbf{F}_{n-1}^2 + \mathbf{F}_{n-1}) - 32(\mathbf{F}_{n-2}^2 + \mathbf{F}_{n-2}),\\ \alpha_2 &= 2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n + 6\mathbf{F}_{n-1}),\\ \alpha_3 &= 6(20-12n)\mathbf{F}_{n-1} - 3(9n^2 - 33n + 136).\end{aligned}$$

Proof: The trace of A, $\mathrm{tr}A = n\mathbf{F}_0 - (n-1)\mathbf{F}_{n-1}$. By Theorem 6 and by Equation (8), we have

$$s(A) \leq \sqrt{\frac{1}{9}(\alpha_1+\alpha_2+\alpha_3) - \frac{2}{n}[(n-1)\mathbf{F}_{n-1} - 2n]^2},$$

where

$$\begin{aligned}\alpha_1 &= 24(2n-1)(\mathbf{F}_{n-1}^2 + \mathbf{F}_{n-1}) - 32(\mathbf{F}_{n-2}^2 + \mathbf{F}_{n-2}),\\ \alpha_2 &= 2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n + 6\mathbf{F}_{n-1}),\\ \alpha_3 &= 6(20-12n)\mathbf{F}_{n-1} - 3(9n^2 - 33n + 136).\end{aligned}$$

Thus the proof is completed. □

3 On the Norms and Spreads of RFMLR Circulant Matrices with the Mersenne Sequence

Theorem 8 *Let* $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$*, where* $\{\mathbf{M}_j\}_{0\leq j\leq n-1}$ *denote Mersenne sequence given by (2), then three kinds of norms of* B *are given by*

$$\|B\|_1 = \|B\|_\infty = \mathbf{M}_n + \mathbf{M}_{n-1} - n,$$

and

$$\|B\|_F = \sqrt{\frac{1}{18}(\beta_1+\beta_2+\beta_3)},$$

where

$$\begin{aligned}\beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2 - \mathbf{M}_{n-1})\\ &+2(8-3n)(3\mathbf{M}_{n-1}^2 - 2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1} - 32(\mathbf{M}_{n-2}^2 - \mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2 + 21n + 16).\end{aligned}$$

Proof: By Definition 3 and Lemma 1, we have

$$\begin{aligned}\|B\|_1 = \|B\|_\infty &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|\\ &= \sum_{i=0}^{n-1} \mathbf{M}_i + \mathbf{M}_{n-1} - \mathbf{M}_0\\ &= \mathbf{M}_n + \mathbf{M}_{n-1} - n.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}\|B\|_F^2 &= n\sum_{j=0}^{n-1}\mathbf{M}_j^2 + \sum_{j=1}^{n-1} j\mathbf{M}_j^2 - 2\sum_{j=1}^{n-2} j\mathbf{M}_j\mathbf{M}_{j+1}\\ &= n\sum_{j=0}^{n-1}\mathbf{M}_j^2 + \sum_{k=1}^{n-1}\sum_{j=n-k}^{n-1}\mathbf{M}_j^2 -\\ &\quad 2\sum_{k=1}^{n-2}\sum_{j=n-k-1}^{n-2}\mathbf{M}_j\mathbf{M}_{j+1}\\ &= n\sum_{j=0}^{n-1}\mathbf{M}_j^2 + \sum_{k=1}^{n-1}\left(\sum_{j=0}^{n-1}\mathbf{M}_j^2 - \sum_{j=0}^{n-k-1}\mathbf{M}_j^2\right)\\ &\quad -2\sum_{k=1}^{n-2}\left(\sum_{j=0}^{n-2}\mathbf{M}_j\mathbf{M}_{j+1} - \sum_{j=0}^{n-k-2}\mathbf{M}_j\mathbf{M}_{j+1}\right)\\ &= \tfrac{1}{18}(\beta_1+\beta_2+\beta_3).\end{aligned}$$

Thus

$$\|B\|_F = \sqrt{\frac{1}{18}(\beta_1+\beta_2+\beta_3)} \qquad (13)$$

where

$$\begin{aligned}\beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2 - \mathbf{M}_{n-1})\\ &+2(8-3n)(3\mathbf{M}_{n-1}^2 - 2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1} - 32(\mathbf{M}_{n-2}^2 - \mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2 + 21n + 16).\end{aligned}$$

□

Theorem 9 *Let $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$, where $\{\mathbf{M}_j\}_{0\le j\le n-1}$ denote Mersenne sequence given by (2), then*

$$\sqrt{\frac{1}{18n}(\beta_1+\beta_2+\beta_3)} \le \|B\|_2,$$

where

$$\begin{aligned}\beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2-\mathbf{M}_{n-1})\\ &\quad +2(8-3n)(3\mathbf{M}_{n-1}^2-2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1}-32(\mathbf{M}_{n-2}^2-\mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2+21n+16),\end{aligned}$$

and

$$\|B\|_2 \le 2(\mathbf{M}_n - n),$$

where $\|\cdot\|_2$ is the spectral norm.

Proof: The matrix B is of the form

$$B = \begin{pmatrix} \mathbf{M}_0 & \mathbf{M}_1 & \cdots & \mathbf{M}_{n-2} & \mathbf{M}_{n-1}\\ \mathbf{M}_{n-1} & d_1 & \mathbf{M}_1 & \cdots & \mathbf{M}_{n-2}\\ \vdots & d_2 & \ddots & \ddots & \vdots\\ \mathbf{M}_2 & \vdots & \ddots & \ddots & \mathbf{M}_1\\ \mathbf{M}_1 & d_3 & \cdots & d_2 & d_1 \end{pmatrix} \quad (14)$$

where

$$\begin{aligned}d_1 &= \mathbf{M}_0-\mathbf{M}_{n-1},\\ d_2 &= \mathbf{M}_{n-1}-\mathbf{M}_{n-2},\\ d_3 &= \mathbf{M}_2-\mathbf{M}_1.\end{aligned}$$

We know that $\frac{1}{\sqrt{n}}\|B\|_F \le \|B\|_2 \le \|B\|_F$ from equivalent norms, where $\|\cdot\|_F$ is the Frobenius norm. By Equation(13), we can get

$$\frac{1}{\sqrt{n}}\|B\|_F = \sqrt{\frac{1}{18n}(\beta_1+\beta_2+\beta_3)},$$

so

$$\sqrt{\frac{1}{18n}(\beta_1+\beta_2+\beta_3)} \le \|B\|_2,$$

where

$$\begin{aligned}\beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2-\mathbf{M}_{n-1})\\ &\quad +2(8-3n)(3\mathbf{M}_{n-1}^2-2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1}-32(\mathbf{M}_{n-2}^2-\mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2+21n+16).\end{aligned}$$

On the other hand, according to Q_1, Q_2 and Q_3 defined in Theorem 6, then we can get

$$B = \sum_{j=0}^{n-1}\mathbf{M}_j Q_1^j - \sum_{j=1}^{n-2}\mathbf{M}_{n-j-1}Q_2^j - \mathbf{M}_{n-1}Q_3.$$

Because

$$\|Q_1\|_2 = \|Q_2\|_2 = \|Q_3\|_2 = 1,$$

so the other result is obtained as follows

$$\begin{aligned}\|B\|_2 &\le \sum_{j=1}^{n-1}\mathbf{M}_j\|Q_1\|_2^j + \sum_{j=1}^{n-2}\mathbf{M}_{n-j-1}\|Q_2\|_2^j\\ &\quad +\mathbf{M}_{n-1}\|Q_3\|_2\\ &= 2\sum_{j=0}^{n-1}\mathbf{M}_j = 2(\mathbf{M}_n-n) \qquad (15)\end{aligned}$$

Thus the proof is completed. □

Theorem 10 *Let $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$, where $\{\mathbf{M}_j\}_{0\le j\le n-1}$ denote Mersenne sequence given by (2), then*

$$s(B) \le \sqrt{\frac{1}{9}(\beta_1+\beta_2+\beta_3) - \frac{2}{n}(n-1)^2\mathbf{M}_{n-1}^2},$$

where

$$\begin{aligned}\beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2-\mathbf{M}_{n-1})\\ &\quad +2(8-3n)(3\mathbf{M}_{n-1}^2-2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1}-32(\mathbf{M}_{n-2}^2-\mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2+21n+16).\end{aligned}$$

Proof: The trace of B, $\mathrm{tr}B = (1-n)\mathbf{M}_{n-1}$. By Theorem 9 and by Equation (8), we have

$$s(B) \le \sqrt{\frac{1}{9}(\beta_1+\beta_2+\beta_3) - \frac{2}{n}(n-1)^2\mathbf{M}_{n-1}^2},$$

where

$$\begin{aligned}\beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2-\mathbf{M}_{n-1})\\ &\quad +2(8-3n)(3\mathbf{M}_{n-1}^2-2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1}-32(\mathbf{M}_{n-2}^2-\mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2+21n+16).\end{aligned}$$

Thus the proof is completed. □

4 On the Norms and Spreads of RFMLR Circulant Matrices with the Gaussian Fibonacci Number

Theorem 11 *Let $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, where $\{G_j\}_{0\le j\le n-1}$ denote Gaussian Fibonacci number given by (3), then three kinds of norms of C are given by*

$$\|C\|_1 = \|C\|_\infty = 2G_{n-1} - G_n - G_2,$$

and

$$\|C\|_F = \sqrt{2(2\gamma_1+\gamma_3-\gamma_4)+(2n-7)\gamma_2+\gamma_5},$$

where

$$\gamma_1 = G_nG_{n-1},$$
$$\gamma_2 = G_nG_{n-2},$$
$$\gamma_3 = G_{n-2}G_{n-3},$$
$$\gamma_4 = (n-1)G_0G_{n-1},$$
$$\gamma_5 = [(4-n)(-1)^n-3]G_0 +[2(n-3)(-1)^n-1]G_1-nG_2.$$

Proof: By Definition 3 and Lemma 1, we have

$$\begin{aligned}\|C\|_1 = \|C\|_\infty &= \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\\ &= \sum_{i=0}^{n-1}G_i+G_{n-1}-G_0\\ &= 2G_{n+1}-G_n-G_2.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}\|C\|_F^2 &= n\sum_{j=0}^{n-1}G_j^2+\sum_{j=1}^{n-1}jG_j^2-2\sum_{j=1}^{n-2}jG_jG_{j+1}\\ &\quad -2(n-1)G_0G_{n-1}\\ &= n\sum_{j=0}^{n-1}G_j^2+\sum_{k=1}^{n-1}\sum_{j=n-k}^{n-1}G_j^2\\ &\quad -2\sum_{k=1}^{n-2}\sum_{j=n-k-1}^{n-2}G_jG_{j+1}\\ &\quad -2(n-1)G_0G_{n-1}\\ &= n\sum_{j=0}^{n-1}G_j^2+\sum_{k=1}^{n-1}\left(\sum_{j=0}^{n-1}G_j^2-\sum_{j=0}^{n-k-1}G_j^2\right)\\ &\quad -2\sum_{k=1}^{n-2}\left(\sum_{j=0}^{n-2}G_jG_{j+1}-\sum_{j=0}^{n-k-2}G_jG_{j+1}\right)\\ &\quad -2(n-1)G_0G_{n-1}\\ &= 4G_nG_{n-1}+(2n-7)G_nG_{n-2}+2G_{n-2}G_{n-3}\\ &\quad -2(n-1)G_0G_{n-1}+[(4-n)(-1)^n-3]G_0\\ &\quad +[2(n-3)(-1)^n-1]G_1-nG_2,\end{aligned}$$

thus

$$\|C\|_F = \sqrt{2(2\gamma_1+\gamma_3-\gamma_4)+(2n-7)\gamma_2+\gamma_5} \tag{16}$$

where

$$\gamma_1 = G_nG_{n-1},$$
$$\gamma_2 = G_nG_{n-2},$$
$$\gamma_3 = G_{n-2}G_{n-3},$$
$$\gamma_4 = (n-1)G_0G_{n-1},$$
$$\gamma_5 = [(4-n)(-1)^n-3] +G_0[2(n-3)(-1)^n-1]G_1-nG_2.$$

□

Theorem 12 *Let the $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, where $\{G_j\}_{0\le j\le n-1}$ denote Gaussian Fibonacci number given by (3), then*

$$\sqrt{\frac{1}{n}\left[2(2\gamma_1+\gamma_3-\gamma_4)+(2n-7)\gamma_2+\gamma_5\right]} \le \|C\|_2,$$

where

$$\gamma_1 = G_nG_{n-1},$$
$$\gamma_2 = G_nG_{n-2},$$
$$\gamma_3 = G_{n-2}G_{n-3},$$
$$\gamma_4 = (n-1)G_0G_{n-1},$$
$$\gamma_5 = [(4-n)(-1)^n-3]G_0+[2(n-3)(-1)^n-1]G_1-nG_2,$$

and

$$\|C\|_2 \le 2(G_{n+1}-G_n),$$

where $\|\cdot\|_2$ is the spectral norm.

Proof: The matrix C is of the form

$$\begin{pmatrix} G_0 & G_1 & \cdots & G_{n-2} & G_{n-1}\\ G_{n-1} & f_1 & G_1 & \cdots & G_{n-2}\\ \vdots & f_2 & \ddots & \ddots & \vdots\\ G_2 & \vdots & \ddots & \ddots & G_1\\ G_1 & G_2-G_1 & \cdots & f_2 & f_1 \end{pmatrix} \tag{17}$$

where

$$f_1 = G_0-G_{n-1},$$

and

$$f_2 = G_{n-1}-G_{n-2}.$$

We know that $\frac{1}{\sqrt{n}}\|C\|_F \le \|C\|_2 \le \|C\|_F$ from equivalent norms. By Equation (16) we can get

$$\frac{1}{\sqrt{n}}\|C\|_F = \sqrt{\frac{1}{n}\left[2(2\gamma_1+\gamma_3-\gamma_4)+(2n-7)\gamma_2+\gamma_5\right]},$$

so

$$\sqrt{\frac{1}{n}\left[2(2\gamma_1+\gamma_3-\gamma_4)+(2n-7)\gamma_2+\gamma_5\right]} \le \|C\|_2$$

where

$$\begin{aligned}\gamma_1 &= G_n G_{n-1},\\ \gamma_2 &= G_n G_{n-2},\\ \gamma_3 &= G_{n-2} G_{n-3},\\ \gamma_4 &= (n-1)G_0 G_{n-1},\\ \gamma_5 &= [(4-n)(-1)^n - 3]G_0\\ &+[2(n-3)(-1)^n - 1]G_1 - nG_2.\end{aligned}$$

On the other hand, according to Q_1, Q_2 and Q_3 defied in Theorem 6, we can get

$$C = \sum_{j=0}^{n-1} G_j Q_1^j - \sum_{j=1}^{n-2} G_{n-j-1} Q_2^j - G_{n-1} Q_3.$$

Because

$$\|Q_1\|_2 = \|Q_2\|_2 = \|Q_3\|_2 = 1,$$

the other result can be obtained as follows

$$\begin{aligned}\|C\|_2 \leq & \sum_{j=0}^{n-1} G_j \|Q_1\|_2^j + \sum_{j=1}^{n-2} G_{n-j-1} \|Q_2\|_2^j\\ & +G_{n-1}\|Q_3\|_2\\ &= 2\sum_{j=0}^{n-1} G_j = 2(G_{n+1} - G_n) \qquad (18)\end{aligned}$$

Thus, the proof is completed. □

Theorem 13 *Let* $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, *where* $\{G_i\}_{0\leq j\leq n-1}$ *denote Gaussian Fibinacci number given by (3), then*

$$s(C) \leq \sqrt{4\Delta_1(n) - 2\Delta_2(n) - nG_2 - 2n\Delta_3(n)},$$

where

$$\Delta_1(n) = 2G_n G_{n-1} + G_{n-2} G_{n-3} - (n-1)G_0 G_{n-1},$$

$$\Delta_2(n) = [(4-n)(-1)^n - 3]G_0 + [2(n-3)(-1)^n - 1]G_1,$$
$$\Delta_3(n) = [(n-1)G_{n-1} - nG_0]^2.$$

Proof: The trace of C, $\mathrm{tr}C = nG_0 - (n-1)G_{n-1}$. By Theorem 12 and by Equation (8), we have

$$s(C) \leq \sqrt{4\Delta_1(n) - 2\Delta_2(n) - nG_2 - 2n\Delta_3(n)},$$

where

$$\Delta_1(n) = 2G_n G_{n-1} + G_{n-2} G_{n-3} - (n-1)G_0 G_{n-1},$$

$$\Delta_2(n) = [(4-n)(-1)^n - 3]G_0 + [2(n-3)(-1)^n - 1]G_1,$$
$$\Delta_3(n) = [(n-1)G_{n-1} - nG_0]^2.$$

Thus the proof is completed. □

Corollary 14 *Let* $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$ *and* $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$, *where* $\{\mathbf{F}_j\}_{0\leq j\leq n-1}$ *and* $\{\mathbf{M}_j\}_{0\leq j\leq n-1}$ *denote Fermat sequence and Mersenne sequence respectively, then the spectral norm of Hadamard product of* A *and* B *is valid the inequality*

$$\|A \circ B\|_2 \leq 4(\mathbf{F}_n + n - 2)(\mathbf{M}_n - n).$$

Proof: Since $\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2$, the proof is trivial by Theorem 6 and Theorem 9. □

Corollary 15 *Supposed that* $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$ *and* $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$, *where* $\{\mathbf{F}_j\}_{0\leq j\leq n-1}$ *and* $\{\mathbf{M}_j\}_{0\leq j\leq n-1}$ *denote Fermat sequence and Mersenne sequence respectively, then the Frobenius norm of Kronecker product of* A *and* B *is*

$$\|A \otimes B\|_F = \frac{1}{18}\sqrt{(\alpha_1 + \alpha_2 + \alpha_3)(\beta_1 + \beta_2 + \beta_3)},$$

where

$$\begin{aligned}\alpha_1 &= 24(2n-1)(\mathbf{F}_{n-1}^2 + \mathbf{F}_{n-1}) - 32(\mathbf{F}_{n-2}^2 + \mathbf{F}_{n-2}),\\ \alpha_2 &= 2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n + 6\mathbf{F}_{n-1}),\\ \alpha_3 &= 6(20-12n)\mathbf{F}_{n-1} - 3(9n^2 - 33n + 136),\\ \beta_1 &= 24(2n-1)(\mathbf{M}_{n-1}^2 - \mathbf{M}_{n-1})\\ &+2(8-3n)(3\mathbf{M}_{n-1}^2 - 2\mathbf{M}_{n-1}\mathbf{M}_{n-2}),\\ \beta_2 &= 36(n-4)\mathbf{M}_{n-1} - 32(\mathbf{M}_{n-2}^2 - \mathbf{M}_{n-2}),\\ \beta_3 &= 3(3n^2 + 21n + 16).\end{aligned}$$

Proof: Since $\|A \otimes B\|_F = \|A\|_F \|B\|_F$, the proof is trivial by Theorem 5 and Theorem 8. □

Corollary 16 *Let* $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$ *and* $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, *where* $\{\mathbf{M}_j\}_{0\leq j\leq n-1}$ *and* $\{G_j\}_{0\leq j\leq n-1}$ *denote Mersenne sequence and Gaussian Fibonacci number respectively, then the spectral norm of Hadamard product of* B *and* C *is valid the inequality*

$$\|B \circ C\|_2 \leq 4(\mathbf{M}_n - n)(G_{n+1} - G_n).$$

Proof: Since $\|B \circ C\|_2 \leq \|B\|_2 \|C\|_2$, the proof is trivial by Theorem 9 and Theorem 12. □

Corollary 17 *Supposed that* $B = \mathrm{RFMLRcircfr}(\mathbf{M}_0, \mathbf{M}_1, \cdots, \mathbf{M}_{n-1})$ *and* $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, *where* $\{\mathbf{M}_j\}_{0\leq j\leq n-1}$ *and* $\{G_j\}_{0\leq j\leq n-1}$ *denote Mersenne* *sequence and Gaussian Fibonacci number respectively, then the Frobenius norm of Kronecker product of B and C is*

$$\|B \otimes C\|_F = \sqrt{\frac{1}{18}(\beta_1 + \beta_2 + \beta_3)} \times \sqrt{2(2\gamma_1 + \gamma_3 - \gamma_4) + (2n-7)\gamma_2 + \gamma_5},$$

where

$$\begin{gathered}\beta_1 = 24(2n-1)(\mathbf{M}_{n-1}^2 - \mathbf{M}_{n-1}) \\ +2(8-3n)(3\mathbf{M}_{n-1}^2 - 2\mathbf{M}_{n-1}\mathbf{M}_{n-2}), \\ \beta_2 = 36(n-4)\mathbf{M}_{n-1} - 32(\mathbf{M}_{n-2}^2 - \mathbf{M}_{n-2}), \\ \beta_3 = 3(3n^2 + 21n + 16), \\ \gamma_1 = G_n G_{n-1}, \\ \gamma_2 = G_n G_{n-2}, \\ \gamma_3 = G_{n-2} G_{n-3}, \\ \gamma_4 = (n-1) G_0 G_{n-1},\end{gathered}$$

and

$$\begin{gathered}\gamma_5 = [(4-n)(-1)^n - 3]G_0 \\ +[2(n-3)(-1)^n - 1]G_1 - nG_2.\end{gathered}$$

Proof: Since $\|B \otimes C\|_F = \|B\|_F \|C\|_F$, the proof is trivial by Theorem 8 and Theorem 11. □

Corollary 18 *Let $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$ and $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, where $\{\mathbf{F}_j\}_{0 \le j \le n-1}$ and $\{G_j\}_{0 \le j \le n-1}$ denote Fermat sequence and Gaussian Fibonacci number respectively, then the spectral norm of Hadamard product of A and C is valid the inequality*

$$\|A \circ C\|_2 \le 4(\mathbf{F}_n + n - 2)(G_{n+1} - G_n).$$

Proof: Since $\|A \circ C\|_2 \le \|A\|_2 \|C\|_2$, the proof is trivial by Theorem 6 and Theorem 12. □

Corollary 19 *Supposed that $A = \mathrm{RFMLRcircfr}(\mathbf{F}_0, \mathbf{F}_1, \cdots, \mathbf{F}_{n-1})$ and $C = \mathrm{RFMLRcircfr}(G_0, G_1, \cdots, G_{n-1})$, where $\{\mathbf{F}_j\}_{0 \le j \le n-1}$ and $\{G_j\}_{0 \le j \le n-1}$ denote Fermat sequence and Gaussian Fibonacci number respectively, then the Frobenius norm of Kronecker product of A and C is*

$$\|A \otimes C\|_F = \sqrt{\frac{1}{18}(\alpha_1 + \alpha_2 + \alpha_3)} \sqrt{2(2\gamma_1 + \gamma_3 - \gamma_4) + (2n-7)\gamma_2 + \gamma_5},$$

where

$$\begin{gathered}\alpha_1 = 24(2n-1)(\mathbf{F}_{n-1}^2 + \mathbf{F}_{n-1}) \\ -32(\mathbf{F}_{n-2}^2 + \mathbf{F}_{n-2}), \\ \alpha_2 = 2(14-6n)(\mathbf{F}_{n-1}\mathbf{F}_n + 6\mathbf{F}_{n-1}), \\ \alpha_3 = 6(20-12n)\mathbf{F}_{n-1} - 3(9n^2 - 33n + 136), \\ \gamma_1 = G_n G_{n-1}, \\ \gamma_2 = G_n G_{n-2}, \\ \gamma_3 = G_{n-2} G_{n-3}, \\ \gamma_4 = (n-1) G_0 G_{n-1},\end{gathered}$$

and

$$\begin{gathered}\gamma_5 = [(4-n)(-1)^n - 3]G_0 \\ +[2(n-3)(-1)^n - 1]G_1 - nG_2.\end{gathered}$$

Proof: Since $\|A \otimes C\|_F = \|A\|_F \|C\|_F$, the proof is trivial by Theorem 5 and Theorem 11. □

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