On some properties of certain subclasses of analytic functions defined by using the subordination principle

RABHA EL-ASHWAH
Department of Mathematics,
Faculty of Science,
Damietta University,
Damietta 34517
EGYPT
r_elashwah@yahoo.com

ALAA HASSAN
Department of Mathematics,
Faculty of Science,
Zagazig University,
Zagazig 44519
EGYPT
alaahassan1986@yahoo.com

Abstract: - In this paper, we introduce some new subclasses of analytic functions related to starlike, convex, close-to-convex and quasi-convex functions defined by using a generalized operator and the differential subordination principle. Inclusion relationships for these subclasses are established. Moreover, we introduce some integral-preserving properties.

Key-Words: - Starlike function; Convex function; Close-to-convex function; Quasi-convex function; Subordination principle.

1 Introduction

Let $\mathbb{A}$ denotes the class of functions $f(z)$ which are analytic in $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, for $0 \leq \alpha, \beta < 1$, let $S^*(\alpha), C(\alpha), K(\beta, \alpha)$ and $K^*(\beta, \alpha)$ denote, respectively, the well-known subclasses of $\mathbb{A}$ consisting of univalent functions which are starlike of order $\alpha$, convex of order $\alpha$, close-to-convex of order $\beta$ and type $\alpha$ and quasi-convex of order $\beta$ and type $\alpha$ (see [23], [28], [32], [34], [38], [40], [43], and [44] etc.).

Let $M$ be the class of all functions $\varphi$ which are analytic and univalent in $U$ and for which $\varphi(U)$ is convex with $\varphi(0) = 1$ and $\text{Re}(\varphi(z)) > 0$; $z \in U$.

We begin with recalling the principle of subordination between analytic functions.

Definition 1. For two functions $f(z)$ and $g(z)$, analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$ in $U$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $U$, satisfying the following conditions: $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if $g(z)$ is univalent in $U$, then $f \prec g$, if and only if (see [31] and [6]) $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 2. Making use of Definition 1, several authors have investigated the subclasses $S^*(\alpha; \varphi), C(\alpha; \varphi), K(\beta, \alpha; \varphi, \psi)$ and $K^*(\beta, \alpha; \psi, \varphi)$ of the class $\mathbb{A}$ for $0 \leq \alpha, \beta < 1$ and $\varphi, \psi \in M$, which are defined as follows (see [9], [10], [11], [20], and [27]):

$$S^*(\alpha; \varphi) = \left\{ f : f(z) \in \mathbb{A} \text{ and } \frac{zf'(z)}{f(z)} - \alpha < \varphi(z) \right\},$$

$$C(\alpha; \varphi) = \left\{ f : f(z) \in \mathbb{A} \text{ and } 1 + \frac{zf'(z)}{f(z)} - \alpha < \varphi(z) \right\},$$

$$K(\beta, \alpha; \psi, \varphi) = \left\{ f : f(z) \in \mathbb{A} \text{ and } \exists g(z) \in S^*(\alpha; \varphi); \right\}$$

$$\frac{1}{1-\beta} \left( \frac{zf'(z)}{g(z)} - \beta < \psi(z) \right) \left( 0 \leq \alpha, \beta < 1, \beta, \psi \in M, z \in U \right),$$

and

$$K^*(\beta, \alpha; \psi, \varphi) = \left\{ f : f(z) \in \mathbb{A} \text{ and } \exists g(z) \in C(\alpha; \varphi); \right\}$$

$$\frac{1}{1-\beta} \left( \frac{zf'(z)}{g(z)} - \beta < \psi(z) \right) \left( 0 \leq \alpha, \beta < 1, \psi \in M, z \in U \right),$$

...
In particular, for \( \varphi(z) = \psi(z) = (1+z)/(1-z) \), we obtain the familiar classes \( S^*(\alpha), C(\alpha), K(\beta, \alpha) \) and \( K^*(\beta, \alpha) \), respectively. Furthermore, if we set \( \alpha = 0 \) and \( \varphi(z) = \psi(z) = (1+Az)/(1-Bz) \) \((-1 \leq B < A \leq 1)\), we obtain the following function classes:

\[
S^\prime \left(0, \frac{1+Az}{1-Bz}\right) = S^*(A, B) \quad \text{and} \quad C \left(0, \frac{1+Az}{1-Bz}\right) = C(A, B),
\]

which were introduced by Janowski [18] (see also [17]).

Following the recent work of El-Ashwah and Aouf [14] and [13, with \( p = 1 \)], for \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \lambda > 0, l > -1 \) and for function \( f(z) \in \mathcal{A} \) given by (1.1), the integral operator \( L^{m}_{\alpha, \lambda} : \mathcal{A} \to \mathcal{A} \) is defined as follows:

\[
L^{m}_{\alpha, \lambda} f(z) = \int_{\alpha}^{z} \frac{f(t) dt}{(l+1) - \lambda (n-1)+1}, \quad m = 1, 2, \ldots.
\]

It is clear from (1.2) that:

\[
L^{m}_{\alpha, \lambda} f(z) = z + \sum_{n=\alpha}^{\infty} \frac{\lambda^{n}}{(\lambda+1)n!} f^{(n)}(0) \lambda^{n} \quad (\alpha = c) \quad (1.3)
\]

Also, for \( \mu > 0 \) and \( a, c \in \mathbb{C} \), are such that \( \Re\{c-a\} \geq 0, \Re\{a\} > -\mu \) Raina and Sharma [39] defined the integral operator \( J^{\mu}_{\alpha} : \mathcal{A} \to \mathcal{A} \), as follows:

\[
J^{\mu}_{\alpha} f(z) = \int_{\gamma}^{1} \frac{f(t^\alpha \lambda) dt}{(1-t^\alpha \lambda)} \quad \text{where} \quad \Re\{c-a\} > 0.
\]

For \( f(z) \) defined by (1.1), it is easily from (1.4) that:

\[
J^{\mu}_{\alpha} f(z) = z + \sum_{n=\alpha}^{\infty} \frac{\lambda^{n}(\lambda + \mu)}{(\lambda+1)n!} a_{n} z^{n}.
\]

(\( \mu > 0, \alpha, c \in \mathbb{C}; \Re\{c-a\} \geq 0; \Re\{a\} > -\mu \))

By combining the two linear operators \( L^{m}_{\alpha, \lambda} \) and \( J^{\mu}_{\alpha} \), we define the generalized operator

\[
I^{\mu, \alpha}_{\beta, \gamma} : \mathcal{A} \to \mathcal{A},
\]

is defined for the purpose of this paper as following:

\[
I^{\mu, \alpha}_{\beta, \gamma}(a, c, \mu, \beta, \alpha) f(z) = L^{\mu}_{\alpha, \gamma}(J^{\mu}_{\alpha} f(z)) = J^{\mu}_{\alpha} \left( L^{\mu}_{\alpha, \gamma}(f(z)) \right),
\]

which can be easily expressed as follows:

\[
I^{\mu, \alpha}_{\beta, \gamma}(a, c, \mu, \beta, \alpha) f(z) = z + \sum_{n=\alpha}^{\infty} \frac{\lambda^{n}(\lambda + \mu)}{(\lambda+1)n!} a_{n} z^{n},
\]

(\( \mu > 0, a, c \in \mathbb{C}; \Re\{c-a\} \geq 0; \Re\{a\} > -\mu, \lambda > 0, l > -1, m \in \mathbb{N}_0 \)).

In view of (1.3), (1.5) and (1.6), it is clear that:

\[
I^{\mu}_{\alpha, \gamma}(a, c, \mu, \beta, \alpha) f(z) = I^{\mu}_{\alpha, \gamma}(f(z)) \quad \text{and} \quad I^{\mu}_{\alpha, \gamma}(a, a, \mu, \beta, \alpha) f(z) = L^{\mu}_{\alpha, \gamma}(f(z)).
\]

The importance of the operator \( I^{\mu, \alpha}_{\beta, \gamma} \) comes from its generalization of a lot of previous operators, as follows:

(i) \( I^{\mu}_{\alpha, \gamma}(v-1,0,1) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\lambda > 0; l > -1; v > 0; m \in \mathbb{N}_0) \) (see Aouf and El-Ashwah [2]);
(ii) \( I^{\mu}_{\alpha, \gamma}(v-1,0,1) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (l > -1; v > 0; s \in \mathbb{R}) \) (see Cho and Kim [9]);
(iii) \( I^{\mu}_{\alpha, \gamma}(v-1,0,1) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\lambda > 0; v > 0; m \in \mathbb{Z}) \) (see Aouf et al. [4]);
(iv) \( I^{\mu}_{\alpha, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (l > -1; n \in \mathbb{N}_0) \) (see Catas [8]);
(v) \( I^{\mu}_{\alpha, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\lambda > 0; l > -1; n \in \mathbb{Z}) \) (see Patell [37]);
(vi) \( I^{\mu}_{\alpha, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\sigma > 0) \) (see Jung et al. [19], see also Liu [24]);
(ix) \( I^{\mu}_{\alpha, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\mu > 0; \alpha > 0) \) (see Komatu [21], see also Aouf [1]);
(viii) \( I^{\mu}_{\alpha, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\lambda > 0; n \in \mathbb{Z}) \) (see Patell [37]);
(x) \( I^{\mu, \alpha}_{\beta, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\sigma > 0) \) (see Aouf et al. [4]);
(xi) \( I^{\mu}_{\alpha, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (l > -1) \) (see Gao et al. [16]);
(xii) \( I^{\mu, \alpha}_{\beta, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (l > -1) \) (see Owa and Srivastava [36] and Srivastava and Owa [45]);
(xiii) \( I^{\mu, \alpha}_{\beta, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\lambda > 0; \alpha > 0) \) (see Aouf et al. [3]);
(xiv) \( I^{\mu, \alpha}_{\beta, \gamma}(a,a,\mu) f(z) = I^{\mu, \alpha}_{\beta, \gamma}(f(z)) \quad (\beta > 0) \) (see Liu and Owa [25], see also Jung et al. [19] and Li [22]);
Using (1.7), we can obtain the following recurrence relations, which are needed for our proofs in following two sections:

\[ z \left( I_{a,m}^{m+1}(a,c,\mu)f(z) \right) = \frac{1+l}{\lambda} I_{a,m}^{m}(a,c,\mu)f(z) - \frac{1+l-\lambda}{\lambda} I_{a,m}^{m+1}(a,c,\mu)f(z), \quad (8) \]

\[ z \left( I_{a,m}^{m}(a,c,\mu)f(z) \right) = \frac{a+\mu}{\mu} I_{a,m}^{m}(a+1,c,\mu)f(z) - \frac{a}{\mu} I_{a,m}^{m+1}(a,c,\mu)f(z). \quad (9) \]

**Definition 3.** For \( \mu > 0, a, c \in \mathbb{C}; \text{Re} \{ c-a \} \geq 0, \text{Re} \{ a \} > -\mu, \lambda > 0, l > -1, 0 \leq \alpha, \beta < 1, m \in \mathbb{N}_0 \) and the operator \( I_{a,m}^{m}(a,c,\mu)f(z) \) defined by (1.12), we introduce the following subclasses of the normalized analytic functions class \( \mathcal{A} \), as follows:

\[ S_{a,l,m}^{\mu}(\alpha; a, c, \mu; \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in S^{\ast}(\alpha; \varphi) \right\}, \]

\[ C_{a,l,m}^{\mu}(\alpha; a, c, \mu; \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in C(\alpha; \varphi) \right\}, \]

\[ K_{a,l,m}^{\mu}(\beta; a, c, \mu; \psi, \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in K(\beta; \alpha; \psi, \varphi) \right\}, \]

\[ K_{a,l,m}^{\mu}(\beta; a, c, \mu; \psi, \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in K^{\ast}(\beta; \alpha; \psi, \varphi) \right\}. \]

For the subclasses defined above, we note that:

\[ f(z) \in C_{a,l,m}^{\mu}(\alpha; a, c, \mu; \varphi) \Leftrightarrow z^\mu f(z) \in S_{a,l,m}^{\mu}(\alpha; a, c, \mu; \varphi), \quad (10) \]

\[ f(z) \in K_{a,l,m}^{\mu}(\beta; a, c, \mu; \psi, \varphi) \Leftrightarrow z^\mu f(z) \in K_{a,l,m}^{\mu}(\beta; a, c, \mu; \psi, \varphi). \quad (11) \]

**Remark 1.** If we set \( a = c \) in Definition 1, we obtain the following subclasses of \( \mathcal{A} \):

\[ S_{a,l}^{\mu}(\alpha; a, c, \mu; \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in S^{\ast}(\alpha; \varphi) \right\}, \]

\[ C_{a,l}^{\mu}(\alpha; a, c, \mu; \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in C(\alpha; \varphi) \right\}, \]

\[ K_{a,l}^{\mu}(\beta; a, c, \mu; \psi, \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in K(\beta; \alpha; \psi, \varphi) \right\}, \]

\[ K_{a,l}^{\mu}(\beta; a, c, \mu; \psi, \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l,m}^{m}(a,c,\mu)f(z) \in K^{\ast}(\beta; \alpha; \psi, \varphi) \right\}. \]

Where \( L_{a,l}^{m}f(z) \) is defined by (1.7).

**Remark 2.** If we set \( m = 0 \) in Definition 1, we obtain the following subclasses of \( \mathcal{A} \):

\[ S^{\ast}(\alpha; a, c, \mu; \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l}^{m}(a,c,\mu)f(z) \in S^{\ast}(\alpha; \varphi) \right\}, \]

\[ C(\alpha; a, c, \mu; \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l}^{m}(a,c,\mu)f(z) \in C(\alpha; \varphi) \right\}, \]

\[ K(\beta; a, c, \mu; \psi, \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l}^{m}(a,c,\mu)f(z) \in K(\beta; \alpha; \psi, \varphi) \right\}, \]

\[ K^{\ast}(\beta; a, c, \mu; \psi, \varphi) = \left\{ f : f(z) \in \mathcal{A} \text{ and } I_{a,l}^{m}(a,c,\mu)f(z) \in K^{\ast}(\beta; \alpha; \psi, \varphi) \right\}. \]

Where \( J_{a,l}^{m}f(z) \) is defined by (1.10).

In order to introduce our main results, we shall need the following lemmas.

**Lemma 1** (see [12]). Let \( h \) be a convex univalent function in \( U \) with \( h(0) = 1 \) and \( \text{Re}\{\mu h(z) + \nu\} > 0 \) \((\mu, \nu \in \mathbb{C})\). If \( p \) is an analytic function in \( U \) with \( p(0) = 1 \), then

\[ p(z) + \frac{zp'(z)}{\mu p(z) + \nu} < h(z); \quad z \in U, \]

implies that

\[ p(z) < h(z); \quad z \in U. \]

**Lemma 2** (see [29] and [30]). Let \( h \) be a convex function in \( U \) with \( h(0) = 1 \). Suppose also that \( w \) be an analytic function in \( U \) with \( \text{Re}\{w(z) + \nu\} \geq 0 \) \((z \in U)\). If \( p \) is an analytic function in \( U \) with \( p(0) = 1 \), then

\[ p(z) + w(z)p'(z) < h(z); \quad z \in U, \]

implies that

\[ p(z) < h(z); \quad z \in U. \]

2 Inclusion Relationships

Unless otherwise mentioned, we shall assume throughout the paper that \( \mu > 0, \quad a, c \in \mathbb{C}, \)
In this section, we give several inclusion relationships for analytic function classes, which are associated with the generalized operator $I_{\lambda,l}^m(a,c,\mu)$ defined by (1.12).

**Theorem 1.** Let $\max_{c\in\mathbb{D}} \Re\{f(z)\} < \min\left\{\frac{\Re\{\frac{a+\lambda}{a-1}\}}{a-1}\right\}$.

Then

$$S^{n}_{\lambda,l}(\alpha;\alpha+1,c,\mu;\varphi) \subseteq S^{n}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi) \subseteq S^{n}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi).$$

**Proof.** We begin with proving that

$$S^{n}_{\lambda,l}(\alpha;\alpha+1,c,\mu;\varphi) \subseteq S^{n}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi).$$

Let $f(z) \in S^{n}_{\lambda,l}(\alpha;\alpha+1,c,\mu;\varphi)$ and set

$$\frac{1}{1-\alpha} \left( z I_{\lambda,l}^m(a+1,c,\mu) f(z) \right) - \alpha = f(z).$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in $U$ and $p(z) \neq 0$ for all $z \in U$. Applying (9) and (14), we obtain

$$\frac{a+\mu}{\mu} \frac{I_{\lambda,l}^m(a+1,c,\mu) f(z)}{I_{\lambda,l}^m(a,c,\mu) f(z)} = \left(1-\alpha\right) p(z) + \frac{a}{\mu} + \alpha.$$  

By using the logarithmic differentiation on both sides of (15), we obtain

$$z \left( I_{\lambda,l}^m(a+1,c,\mu) f(z) \right)' = \frac{z \left( I_{\lambda,l}^m(a,c,\mu) f(z) \right)'}{I_{\lambda,l}^m(a,c,\mu) f(z)} + \frac{(1-\alpha) zp'(z)}{(1-\alpha) p(z) + \frac{a}{\mu} + \alpha},$$

by using (14) again, we have

$$\frac{1}{1-\alpha} \left( z I_{\lambda,l}^m(a+1,c,\mu) f(z) \right)' - \alpha = f(z).$$

Thus, by using Lemma 1 and (14), we observe that

$$p(z) + \frac{z p'(z)}{(1-\alpha) p(z) + \frac{a}{\mu} + \alpha} < \varphi(z) \quad (z \in U).$$

which implies that

$$f(z) \in S^{n}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi),$$

which proves the first inclusion relationship (13). Now, we prove the second inclusion relationship, asserted as following

$$S^{n}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi) \subseteq S^{n+1}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi).$$

Let $f(z) \in S^{n}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi)$ and set

$$\frac{1-z I_{\lambda,l}^m(a,c,\mu) f(z)}{1-\alpha} = q(z),$$

where $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ is analytic in $U$ and $q(z) \neq 0$ for all $z \in U$. Then, by using arguments similar to those detailed above with (8), it follows that

$$q(z) < \varphi(z) \quad (z \in U),$$

which implies that

$$f(z) \in S^{n+1}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi),$$

which proves the second inclusion relationship (17). Combining the inclusion relationships (13) and (17), we complete the proof of Theorem 1.

**Theorem 2.** Let $\max_{c\in\mathbb{D}} \Re\{f(z)\} < \min\left\{\frac{\Re\{\frac{a+\lambda}{a-1}\}}{a-1}\right\}$.

Then

$$C^{\alpha}_{\lambda,l}(\alpha;\alpha+1,c,\mu;\varphi) \subseteq C^{\alpha}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi) \subseteq C^{\alpha}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi).$$

**Proof.** Applying (10) and Theorem 1, we observe that

$$f(z) \in C^{\alpha}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi) \quad \Rightarrow I_{\lambda,l}^m(a+1,c,\mu) f(z) \in C(\alpha;\varphi)$$

$$\Rightarrow z(I_{\lambda,l}^m(a+1,c,\mu) f(z))' \in S^*(\alpha;\varphi)$$

$$\Rightarrow I_{\lambda,l}^m(a+1,c,\mu) f(z) \in S^*(\alpha;\varphi)$$

$$\Rightarrow z f(z) \in S^*_\lambda(\alpha;\alpha+1,c,\mu;\varphi)$$

$$\Rightarrow z f(z) \in S^*_\lambda(\alpha;\alpha,c,\mu;\varphi)$$

$$\Rightarrow I_{\lambda,l}^m(a,c,\mu) f(z) \in S^*(\alpha;\varphi)$$

$$\Rightarrow z(I_{\lambda,l}^m(a,c,\mu) f(z))' \in S^*(\alpha;\varphi)$$

$$\Rightarrow f(z) \in C^{\alpha}_{\lambda,l}(\alpha;\alpha,c,\mu;\varphi).$$
and 
\[ f(z) \in C_{\lambda,1}^m(\alpha; a, c, \mu; \varphi) \]
\[ \iff zf'(z) \in S_{\lambda,1}^m(\alpha; a, c, \mu; \varphi) \]
\[ \implies zf'(z) \in S_{\lambda,1}^{m+1}(\alpha; a, c, \mu; \varphi) \]
\[ \implies z(f_{\lambda,1}\alpha(a, c, \mu))f(z) \in S^*(\alpha; \varphi) \]
\[ \implies f(z) \in C_{\lambda,1}^{m+1}(\alpha; a, c, \mu; \varphi). \]

Which evidently proves Theorem 2.

**Theorem 3.** Let \( \max_{z \in U} \Re \{\varphi(z)\} < \min \left\{ \frac{\Re \left[ \frac{1}{\mu} \right] + \frac{1}{\lambda} + \frac{1}{\mu}}{a - 1} \right\} \).

Then
\[
K_{\lambda,1}^m(\beta, \alpha; a+1, c, \mu; \psi, \varphi) \\
\subseteq K_{\lambda,1}^m(\beta, \alpha; a, c, \mu; \psi, \varphi) \\
\subseteq K_{\lambda,1}^{m+1}(\beta, \alpha; a, c, \mu; \psi, \varphi). \tag{20}
\]

**Proof.** We begin with proving that
\[
K_{\lambda,1}^m(\beta, \alpha; a+1, c, \mu; \psi, \varphi) \subseteq K_{\lambda,1}^m(\beta, \alpha; a, c, \mu; \psi, \varphi). \tag{21}
\]

Let \( f(z) \in K_{\lambda,1}^m(\beta, \alpha; a+1, c, \mu; \psi, \varphi) \). Then, there exists a function \( r(z) \in S^*(\alpha; \varphi) \) such that
\[
1 - \beta \left( z \left( f_{\lambda,1}(a+1, c, \mu) f(z) \right) \right) \cdot r(z) < \varphi(z) \quad (z \in U). \tag{22}
\]

Choose the function \( g(z) \) such that
\[
I_{\lambda,1}^m(a+1, c, \mu) g(z) = r(z), \]
so that we have \( g(z) \in S_{\lambda,1}^m(\alpha; a+1, c, \mu; \varphi) \) and
\[
1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) < \varphi(z) \quad (z \in U). \tag{23}
\]

Next, we set
\[
1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta = p(z), \tag{24}
\]

where \( p(z) = 1 + p_1 z + p_2 z^2 + ... \) is analytic in \( U \) and \( p(z) \neq 0 \) for all \( z \in U \). Thus, by using the identity (9), we obtain
\[
1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta = 1 - \beta \left( I_{\lambda,1}^m(a+1, c, \mu) \right) \cdot \left( z f'(z) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta.
\]

Moreover, since
\[
g(z) \in S_{\lambda,1}^m(\alpha; a+1, c, \mu; \varphi) \subset S_{\lambda,1}^m(\alpha; a, c, \mu; \varphi),
\]
by using Theorem 1, we can put
\[
1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta = 1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta
\]

Taking the derivatives of (23) and (24), we have
\[
I_{\lambda,1}^m(a+1, c, \mu) g(z) - I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta = p(z).
\]

Upon differentiating both sides of (26), we have
\[
1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta = (1-\beta) \varphi(z) + [(1-\beta) p(z) + \beta][(1-\alpha) G(z) + \alpha]
\]

Making use of (22), (27), and (28), we get
\[
1 - \beta \left( z \left( I_{\lambda,1}^m(a+1, c, \mu) f(z) \right) \right) \cdot I_{\lambda,1}^m(a+1, c, \mu) g(z) - \beta = p(z) + \frac{z \varphi(z)}{(1-\alpha) G(z) + \alpha + \frac{a}{\lambda}} < \psi(z) \quad (z \in U). \tag{29}
\]

Using \( \max_{z \in U} \Re \{\varphi(z)\} < \frac{\Re \left[ \frac{1}{\mu} \right] + \frac{1}{\lambda} + \frac{a}{\lambda}}{a - 1} \) and \( G(z) \varphi(z) \)
\((\varphi \in M, z \in U) \), then we have
\[
\text{Re}\left\{(1-\alpha)G(z)+\alpha+a/\mu\right\}>0 \quad (z \in U).
\]

Hence, upon taking
\[
w(z)=\frac{1}{(1-\alpha)G(z)+\alpha+a/\mu}
\]
in (29), and applying Lemma 2, we obtain that
\[p(z)<\psi(z) \quad (z \in U),\]
then, in view of (23) we deduce that
\[f(z) \in K^m_{\ell,j}(\beta,\alpha; a, c, \mu; \psi, \phi)\]
which proves (21). For the second part, by using arguments similar to those detailed above with (8), thus we choose to omit the details. The proof of Theorem 3 is completed.

**Theorem 4.** Let
\[\max_{z \in U} \text{Re}\{\psi(z)\} < \min\left\{\frac{\text{Re}\{\varphi(z)\}+a}{\alpha-1}, \frac{\text{Re}\{\varphi(z)\}+a}{\alpha-1}\right\}.
\]

Then
\[K^m_{\ell,j}(\beta,\alpha; a+1, c, \mu; \psi, \phi)
\subseteq K^m_{\ell,j}(\beta,\alpha; a, c, \mu; \psi, \phi)
\subseteq K^m_{\ell,j+1}(\beta,\alpha; a, c, \mu; \psi, \phi).
\]

**Proof.** Just, as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (10). Similarly, we can prove Theorem 4 as a consequence of Theorem 3 in conjunction with the equivalence (11). Therefore, again, we choose to omit the details involved.

**Remark 3.** (i) Taking \(a=\nu-1(\nu>0)\), \(c=0\) and \(\mu=1\) in Theorems 1-3, we obtain the results obtained by Aouf and El-Ashwah [2, Theorems 1-3];
(ii) Taking \(m=s(s \in \mathbb{R}), \lambda=1, a=\nu-1(\nu>0), c=0\) and \(\mu=1\) in Theorems 1-3, we obtain the results obtained by Cho and Kim [9, Theorems 2.1-2.3];
(iii) Taking \(l=0, a=\nu-1(\nu>0), c=0\), and \(\mu=1\) in Theorems 1-3, we obtain the results obtained by Aouf et al. [4, Theorems 1-3].

Taking \(a=c\) in Theorems 1-4, we obtain the following corollary.

**Corollary 1.** For the subclasses \(S^m_{\ell,j}(\alpha;\phi), C^m_{\ell,j}(\alpha;\phi), K^m_{\ell,j}(\beta,\alpha;\psi,\phi)\) and \(K^m_{\ell,j}(\beta,\alpha;\psi,\phi)\)
defined in Remark 1, we have the following inclusion relations.
\[S^m_{\ell,j}(\alpha;\phi) \subseteq S^m_{\ell,j+1}(\alpha;\phi),
C^m_{\ell,j}(\alpha;\phi) \subseteq C^m_{\ell,j+1}(\alpha;\phi),
K^m_{\ell,j}(\beta,\alpha;\psi,\phi) \subseteq K^m_{\ell,j+1}(\beta,\alpha;\psi,\phi),
K^m_{\ell,j}(\beta,\alpha;\psi,\phi) \subseteq K^m_{\ell,j+1}(\beta,\alpha;\psi,\phi).
\]

**Remark 4.** (i) Taking \(\lambda=1, m=\mu(\mu>0), l=a-1(\alpha>0)\) and \(\varphi(z)=\psi(z)=\frac{1}{1+z}\) in Corollary 1, we obtain the results obtained by Aouf [1, Theorems 1-4];
(ii) Taking \(\lambda=1, m=\sigma(\sigma>0)\) and \(\varphi(z)=\psi(z)=\frac{1}{1+z}\) in Corollary 1, we obtain the results obtained by Liu [24, Theorems 1-4].

Taking \(m=0\) in Theorems 1-4, we obtain the following corollary.

**Corollary 2.** For the subclasses \(S^*(\alpha;a,c,\mu;\phi), C(\alpha;a,c,\mu;\phi), K(\beta,\alpha;a,c,\mu;\psi,\phi)\) and \(K^*(\beta,\alpha;a,c,\mu;\psi,\phi)\)
defined in Remark 2, we have the following inclusion relations.
\[S^*(\alpha;a+1, c, \mu; \phi) \subseteq S^*(\alpha;a, c, \mu; \phi),
C(\alpha;a+1, c, \mu; \phi) \subseteq C(\alpha;a, c, \mu; \phi),
K(\beta,\alpha;a+1, c, \mu; \psi, \phi) \subseteq K(\beta,\alpha;a, c, \mu; \psi, \phi),
K^*(\beta,\alpha;a+1, c, \mu; \psi, \phi) \subseteq K^*(\beta,\alpha;a, c, \mu; \psi, \phi).
\]

**Remark 5.** Taking \(\alpha=\beta=0, a=\nu-1(\nu>0), c=\lambda(\lambda>-1)\) and \(\mu=1\) in Corollary 2, we obtain the results obtained by Choi et al. [11, Theorems 1-3].

### 3 Integral-Preserving Properties

Now, we recall the definition of the generalized Bernardi-Libera-Livingston integral operator \(L_\sigma : \mathbb{A} \rightarrow \mathbb{A}\), as following (see [36]):
\[L_\sigma f(z) = \frac{\sigma+1}{\zeta-1} \int_0^\zeta f(t)dt \quad (\sigma>-1, f(z) \in \mathbb{A}). \quad (31)
\]
The operator \(L_\sigma f(z) (\sigma \in \mathbb{N})\) was introduced by Bernardi [5]. In particular, the operator \(L_1 f(z)\) was studied earlier by Libera [23] and
Livingston [26]. Using (7) and (31), it is clear that $L_a f(z)$ satisfies the following relationship:

$$z \left( I_{n_1}(a, c, \mu) L_a f(z) \right) = (\sigma + 1) I_{n_1}(a, c, \mu) f(z) - \sigma I_{n_1}(a, c, \mu) L_a f(z).$$

Now, we begin the Integral-preserving property involving the integral operator $L_\sigma$ by the following theorem.

**Theorem 5.** Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \), then \( L_\sigma f(z) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \).

**Proof.** Let \( f(z) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \) and set

$$1_{-1} \left( \frac{z \left( I_{n_1}(a, c, \mu) L_a f(z) \right)}{I_{n_1}(a, c, \mu) L_a f(z)} - \alpha \right) = p(z),$$

where \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) is analytic in \( U \) and \( p(z) \neq 0 \) for all \( z \in U \). By applying (32) and (33), we have

$$(\sigma + 1) \frac{I_{n_1}(a, c, \mu) f(z)}{I_{n_1}(a, c, \mu) L_a f(z)} = (1 - \alpha) p(z) + \alpha + \sigma.$$ \( (34) \)

By using the logarithmic differentiation on both side of (34), we have

$$1_{-1} \left( \frac{z \left( I_{n_1}(a, c, \mu) f(z) \right)}{I_{n_1}(a, c, \mu) f(z)} - \alpha \right) = p(z) + \frac{zp(z)}{(1 - \alpha) p(z) + \alpha + \sigma}. \quad (35)$$

Since \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \) and \( f(z) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \), from (35), we have

$$\Re \{ (1 - \alpha) \varphi(z) + \alpha + \sigma \} > 0$$

and

$$p(z) + \frac{zp(z)}{(1 - \alpha) p(z) + \alpha + \sigma} < \varphi(z) \quad (z \in U).$$

Hence, by Using Lemma 1, we obtain

$$p(z) < \varphi(z) \quad (z \in U),$$

then, in view of (33) we deduce that \( L_\sigma f(z) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \), which completes the proof of Theorem 5.

Taking \( a = c \) in Theorem 5, we obtain the following corollary.

**Corollary 3.** Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in S_{a, I}^{m}(\alpha; \varphi) \), then \( L_{a} f(z) \in S_{a, I}^{m}(\alpha; \varphi) \).

**Remark 6.** (i) Taking \( \lambda = 1, \ m = v(v > 0) \), \( l = a - 1(a > 0) \) and \( \varphi(z) = \frac{1}{1-z} \) in Corollary 3, we obtain the results obtained by Aouf [1, Theorem 5];

(ii) Taking \( \lambda = l = 1, \ m = \sigma(\sigma > 0) \) and \( \varphi(z) = \frac{1}{1-z} \) in Corollary 3, we obtain the results obtained by Liu [24, Theorem 5].

Taking \( m = 0 \) in Theorem 5, we obtain the following corollary.

**Corollary 4.** Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in S^*(\alpha; a, c, \mu; \varphi) \), then \( L_{a} f(z) \in S^*(\alpha; a, c, \mu; \varphi) \).

**Remark 7.** Taking \( \alpha = 0, \ a = v(\nu > 0) \), \( c = \lambda(\lambda > -1) \) and \( \mu = 1 \) in Corollary 4, we obtain the results obtained by Choi et al. [11, Theorem 4].

The next Integral-preserving property involving the integral operator $L_\sigma$ is given by the following theorem

**Theorem 6.** Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in C_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \), then \( L_{a} f(z) \in C_{a, I}^{m}(\alpha; a, c, \mu; \varphi) \).

**Proof.** Applying (10) and Theorem 5, we observe that

$$f(z) \in C_{a, I}^{m}(\alpha; a, c, \mu; \varphi)$$

$$\iff zf'(z) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi)$$

$$\iff L_{\sigma} \left( zf'(z) \right) \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi)$$

$$\iff z \left( L_{\sigma} f(z) \right)' \in S_{a, I}^{m}(\alpha; a, c, \mu; \varphi)$$

$$\iff L_{\sigma} f(z) \in C_{a, I}^{m}(\alpha; a, c, \mu; \varphi).$$

The proof of Theorem 6 is evidently completed.

Taking \( a = c \) in Theorem 6, we obtain the following corollary.
Corollary 5. Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in C_{a,l}^m(\alpha; \varphi) \), then \( L_{\sigma} f(z) \in C_{a,l}^m(\alpha; \varphi) \).

Remark 8. (i) Taking \( \lambda = 1, m = \mu(\mu > 0), l = a - 1(a > 0) \) and \( \varphi(z) = \frac{1}{z^2} \) in Corollary 5, we obtain the results obtained by Aouf [1, Theorem 6];
(ii) Taking \( \lambda = 1, m = \sigma(\sigma > 0) \) and \( \varphi(z) = \frac{1}{z^2} \) in Corollary 5, we obtain the results obtained by Liu [24, Theorem 6].

Taking \( m = 0 \) in Theorem 6, we obtain the following corollary.

Corollary 6. Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in C(\alpha; a, c, \mu; \varphi) \), then \( L_{\sigma} f(z) \in C(\alpha; a, c, \mu; \varphi) \).

Remark 9. Taking \( \alpha = 0, a = \mu - 1(\mu > 0), c = \lambda(\lambda > -1) \) and \( \mu = 1 \) in Corollary 6, we obtain the results obtained by Choi et al. [11, Theorem 5].

Also, an Integral-preserving property involving the integral operator \( L_{\sigma} \) is given by the following theorem.

Theorem 7. Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in K_{a,l}^m(\beta, \alpha; a, c, \mu; \psi; \varphi) \), then
\[ L_{\sigma} f(z) \in K_{a,l}^m(\beta, \alpha; a, c, \mu; \psi; \varphi). \]

Proof. Let \( f(z) \in K_{a,l}^m(\beta, \alpha; a, c, \mu; \psi; \varphi) \). Then, in view of (1.4), there exists a function \( g(z) \in S_{a,l}^m(\alpha; a, c, \mu; \psi; \varphi) \) and
\[ 1 - \beta \left( \frac{z(I_{a,l}^m(a,c,\mu)f(z))}{I_{a,l}^m(a,c,\mu)g(z)} \right) < \psi(z) \quad (z \in U). \] (36)

Set
\[ 1 - \beta \left( \frac{z(I_{a,l}^m(a,c,\mu)L_{\sigma} f(z))}{I_{a,l}^m(a,c,\mu)L_{\sigma} g(z)} \right) = p(z) \] (37)
where \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) is analytic in \( U \) and \( p(z) \neq 0 \) for all \( z \in U \). Applying (33), we obtain
\[ 1 - \beta \left( \frac{z(I_{a,l}^m(a,c,\mu)f(z))}{I_{a,l}^m(a,c,\mu)g(z)} \right) = \frac{1}{1 - \beta} \left( \frac{z(I_{a,l}^m(a,c,\mu)L_{\sigma} f(z))}{I_{a,l}^m(a,c,\mu)L_{\sigma} g(z)} \right) \]

(38)

Since \( g(z) \in S_{a,l}^m(\alpha; a, c, \mu; \varphi) \), by using Theorem 5, we have \( L_{\sigma} g(z) \in S_{a,l}^m(\alpha; a, c, \mu; \varphi) \), then we obtain
\[ 1 - \beta \left( \frac{z(I_{a,l}^m(a,c,\mu)L_{\sigma} g(z))}{I_{a,l}^m(a,c,\mu)L_{\sigma} g(z)} + \sigma(I_{a,l}^m(a,c,\mu)L_{\sigma} g(z)) \right) = H(z) < \psi(z) \quad (z \in U). \] (39)

Then, by using the same techniques as in the proof of Theorem 3, we conclude from (36), (37), (38) and (39) that
\[ 1 - \beta \left( \frac{z(I_{a,l}^m(a,c,\mu)f(z))}{I_{a,l}^m(a,c,\mu)g(z)} \right) = p(z) + \frac{z p(z)}{(1 - \alpha)H(z) + \alpha + \sigma} < \psi(z) \quad (z \in U). \] (40)

Hence, upon setting
\[ \psi(z) = \frac{1}{(1 - \alpha)H(z) + \alpha + \sigma}, \]
in (40), in view of Lemma 2, we obtain
\[ p(z) < \psi(z) \quad (z \in U), \]
which leads to
\[ L_{\sigma} f(z) \in K_{a,l}^m(\beta, \alpha; a, c, \mu; \psi; \varphi), \]
which completes the proof of Theorem 7.

Taking \( a = c \) in Theorem 7, we obtain the following corollary.

Corollary 7. Let \( \max_{z \in U} \Re \{ \varphi(z) \} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in K_{a,l}^m(\beta, \alpha; \psi; \varphi) \), then \( L_{\sigma} f(z) \in K_{a,l}^m(\beta, \alpha; \psi; \varphi) \).

Remark 10. (i) Taking \( \lambda = 1, m = \mu(\mu > 0), l = a - 1(a > 0) \) and \( \varphi(z) = \psi(z) = \frac{1}{z^2} \) in Corollary 7, we obtain the results obtained by Aouf [1, Theorem 7];
ii) Taking \( \lambda = l = 1, m = \sigma (\sigma > 0) \) and \( \phi(z) = \psi(z) = \frac{i \pi}{l} \) in Corollary 7, we obtain the results obtained by Liu [24, Theorem 7].

Taking \( m = 0 \) in Theorem 7, we obtain the following corollary.

**Corollary 8.** Let \( \max_{z \in \mathbb{D}} \text{Re}\{\phi(z)\} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in K(\beta, \alpha; a, c, \mu; \psi, \varphi) \), then
\[
L_\sigma f(z) \in K_\lambda^{\infty}(\beta, \alpha; a, c, \mu; \psi, \varphi).
\]

**Remark 11.** Taking \( \alpha = \beta = 0, a = v - 1(v > 0), c = \lambda(\lambda > -1) \) and \( \mu = 1 \) in Corollary 6, we obtain the results obtained by Choi et al. [11, Theorem 6].

**Theorem 8.** Let \( \max_{z \in \mathbb{D}} \text{Re}\{\phi(z)\} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in K^{\infty}_\lambda(\beta, \alpha; a, c, \mu; \psi, \varphi) \), then
\[
L_\sigma f(z) \in K^{\infty}_\lambda(\beta, \alpha; a, c, \mu; \psi, \varphi).
\]

Proof. Just as we derived Theorem 6 from Theorem 5 by using (10). Easily, we can deduce Theorem 8 from Theorem 7 by using (11). So we choose to omit the proof.

**Remark 12.** (i) Taking \( m = s(s \in \mathbb{R}), \lambda = 1, a = v - 1(v > 0), c = 0 \) and \( \mu = 1 \) in Theorems 5-7, we obtain the results obtained by Cho and Kim [9, Theorems 3.1-3.3];

(ii) Taking \( a = v - 1(v > 0), c = 0 \) and \( \mu = 1 \) in Theorems 5-7, we obtain the results obtained by Aouf and El-Ashwah [2, Theorems 4-6];

(iii) Taking \( l = 0, a = v - 1(v > 0), c = 0 \) and \( \mu = 1 \) in Theorems 5-7, we obtain the results obtained by Aouf et al. [4, Theorems 4-6].

Taking \( a = c \) in Theorem 8, we obtain the following corollary.

**Corollary 9.** Let \( \max_{z \in \mathbb{D}} \text{Re}\{\phi(z)\} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in K^{\infty}_\lambda(\beta, \alpha; \psi, \varphi) \), then
\[
L_\sigma f(z) \in K^{\infty}_\lambda(\beta, \alpha; \psi, \varphi).
\]

**Remark 13.** (i) Taking \( \lambda = 1, m = \mu (\mu > 0), l = a - 1(a > 0) \) and \( \phi(z) = \psi(z) = \frac{i \pi}{l} \) in Corollary 9, we obtain the results obtained by Aouf [1, Theorem 8];

(ii) Taking \( \lambda = l = 1, m = \sigma (\sigma > 0) \) and \( \phi(z) = \psi(z) = \frac{i \pi}{l} \) in Corollary 9, we obtain the results obtained by Liu [24, Theorem 8].

Taking \( m = 0 \) in Theorem 8, we obtain the following corollary.

**Corollary 10.** Let \( \max_{z \in \mathbb{D}} \text{Re}\{\phi(z)\} < \frac{\alpha + \sigma}{\alpha - 1} \). If \( f(z) \in K^{\ast}(\beta, \alpha; a, c, \mu; \psi, \varphi) \), then
\[
L_\sigma f(z) \in K^{\ast}(\beta, \alpha; a, c, \mu; \psi, \varphi).
\]

References:


