On a two-variable Functional Equation arising from Databases

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Abstract: Functional equations offer a tool for narrowing the models used to describe many phenomena. Recently, a certain class of functional equations stems when obtaining the generating functions of queueing systems distributions. This paper has been motivated by an issue considered by L. Flatto and S. Hahn in [SIAM Journal on Applied Mathematics 44(5), (1984), 1041–1053]. The functional equation obtained there has been converted into a set of conditions on the two-unknowns, which in turn lead to the determination of the main unknown. Another motivation of solving such functional equation comes from the fact that the underlying queueing system has applications in the inventory control of database systems. Unfortunately that solution seems to be a bit too general with many technical assumptions. In this paper we introduce a solution using boundary value problem approach. Our solution is validated, obtained by assuming full symmetry on the system under study, and by the reduction to a Riemann-Hilbert boundary value problem.

Key–Words: Queueing systems, Functional equations, Boundary value problems, Inventory control of database systems.

1 Introduction

Functional equations (FEs) are defined as the equations where the unknowns are functions rather than simple variables [1, 2, 3, 4, 5, 6, 7]. They are more than 200 years old subject of mathematics, but their theory has flourished principally through the work of the prolific mathematician J. Aczél [7, 9, 10, 11, 12] who identified many of their classes, illustrating efficient methods for their solutions as well as criteria for the existence and uniqueness of those solutions [8].

FEs arise abundantly in models of various fields, such as population ethics [13], astronomy [14], neural networks [15], economics [32], digital filtering [33], and the experimental sciences [16].

Specifically, each of these models can be formulated so as to eventually lead to an FE that can yield precise quantitative relationships. FEs can be in one variable or two variables, depending on whether the underlying model is one-dimensional or two dimensional. There is no universal solution technique for these FEs, but rather almost each equation is solved differently than the others. The problem of finding some performance measures to some communication, and networks systems comes with a certain interesting class of functional equations.

In general using the literature there are many approaches available to solve such problem e.g., experimental approach, numerical approach, simulation approach, and the analytical approach see [17]. It is the right place to state that Malyshev [28] pioneered the approach of solving functional equations using the theory of boundary value problems in early 1970s.

The idea of reducing functional equations for the generating function to a standard Riemann-Hilbert boundary value problem stems from the work of Fayolle and Iasnogorodski [19] on two parallel \(M/M/1\) queues with coupled processors. Extensive treatments of the boundary value technique for functional equations can be found in Cohen and Boxma [20] and Fayolle et al. [21]. In particular, the following general class of two-variable functional equations

\[
C_1(x, y)P(x, y) = C_2(x, y)P(x, 0) + C_3(x, y)P(0, y) + C_4(x, y)P(0, 0),
\]

(1)

where \(C_i(x, y), i = 1, 2, 3, 4\) are given polynomials in two complex variables \(x, y\), arises from different communication and networks systems. The unknown functions \(P(x, y), P(x, 0), P(0, y)\) are defined as fol-
The functional equation

This equation arises [18] from a double queue model, illustrated in Figure (1), where the arriving customers simultaneously place two demands handled independently by two servers. The arrivals are assumed to be a Poisson process with unit mean, and the two servers have exponential service times with rates $\alpha, \beta$, with the stability condition $1 < \alpha \leq \beta$. The probability generating function (PGF) $P(x, y)$ of the two-dimensional distribution characterizing the system yields the two-variable FE

$$Q(x, y)P(x, y) = N(x, y),$$  \hspace{1cm} (2)

where

$$Q(x, y) = (1 + \alpha + \beta)xy - \alpha y - \beta x - x^2 y^2,$$

and

$$N(x, y) = \beta x(y - 1)P(x, 0) + \alpha y(x - 1)P(0, y).$$

Equation (2) has been solved using the analytic continuation. First it can be rewritten as follows

$$\left((1 + \alpha + \beta)xy - \alpha y - \beta x - x^2 y^2\right)P(x, y) = \beta x(y - 1)P(x, 0) + \alpha y(x - 1)P(0, y).$$ \hspace{1cm} (3)

It is obvious that the above equation is a special case of the general class of equations given by (1) where

$$C_4(x, y) = 0,$$

$$C_1(x, y) = ((1 + \alpha + \beta)xy - \alpha y - \beta x - x^2 y^2),$$

$$C_2(x, y) = \beta x(y - 1),$$

and

$$C_3(x, y) = \alpha y(x - 1).$$

The authors in [18] solved (3) by parameterizing the curve defined by $Q(x, y) = 0$ by a pair of elliptic functions $x = x(t), y = y(t)$. The functional equation for $P(x, y)$ is converted into a set of conditions

$$C_4(x(t), y(t)) = 0,$$

$$C_1(x(t), y(t)) = ((1 + \alpha + \beta)x(t)y(t) - \alpha y - \beta x - x^2 y^2),$$

$$C_2(x(t), y(t)) = \beta x(t)(y(t) - 1),$$

and

$$C_3(x(t), y(t)) = \alpha y(t)(x(t) - 1).$$
on \( P(x(t),0), P(0,y(t)) \), which in turn leads to the determination of \( P(x,y) \) in the form

\[
P(x,0) = \frac{\beta - 1}{\beta} \frac{A(x)}{A(1)} \quad |x| \leq 1,
\]

where the function \( A(x) \) is given by

\[
A(x) = \frac{\sqrt{a_3 - x} + \sqrt{a_3 - 1}}{[\sqrt{a_3 - x} + \sqrt{a_3 - \alpha/\beta}][\sqrt{a_3 - x} - \sqrt{a_3 - \alpha}],}
\]

where \( a_3 \) is some real number greater than one, and the other unknown is given by

\[
P(0,y) = \frac{\alpha - 1}{\alpha} \frac{B(y)}{B(1)} \quad |y| \leq 1,
\]

where the function \( B(y) \) is given by

\[
B(y) = \frac{\sqrt{\alpha' - y} + \sqrt{\alpha' - 1}}{[\sqrt{\alpha' - y} + \sqrt{\alpha' - \alpha/\beta}][\sqrt{\alpha' - y} + \sqrt{\alpha' - \beta}],}
\]

where \( \alpha' \) is some real number greater than one, for \( \alpha = \beta \) the solution is given by

\[
P(x,0) = P(0,x) = \frac{(\alpha - 1)^{3/2}}{\alpha(\alpha - x)^{1/2}}
\]

### 3 The Boundary Value Problem Model of Eq.(2) and its Solution

In order to solve equation (2), the main idea stems from the fact that the main unknown function \( P(x,y) \) is an analytic function in the unit disk, this means that if the \( Q(x,y) \) is zero then also the right hand side containing the other unknowns must be zero. Let

\[
Q(x,y) := (1 + \alpha + \beta)xy - \alpha y - \beta x - x^2y^2 = 0 \quad (4)
\]

then also

\[
\beta(x-1)P(x,0) + \alpha y(x-1)P(0,y) = 0 \quad (5)
\]

The solution of the original functional equation (2) is now reduced to the solution of the functional equation (5). Assume full symmetry in the system under study, i.e., let \( \alpha = \beta \) so that equation (5) can be rewritten as follows

\[
\alpha x(y-1)P(x,0) + \alpha y(x-1)P(0,y) = 0 \quad (6)
\]

dividing the above equation by \((x-1)(y-1)\) \(\neq 0\) to get

\[
\frac{\alpha x}{x-1} P(x,0) + \frac{\alpha y}{y-1} P(0,y) = 0 \quad (7)
\]

Introduce the function

\[
g(x) := \frac{\alpha x}{x-1} P(x,0),
\]

then equation (7) can be rewritten as

\[
g(x) + g(y) = 0, \quad (8)
\]

where the function \( g(.) \) is an analytic function except for a simple pole at 1.

Now we reduced the solution of the main functional equation to the solution of (8) in \( \{ (x,y) : Q(x,y) = 0 \} \). But \( Q(x,y) = 0 \) offers a very large number of ordered pairs, in our symmetric case it is natural to consider the set:

\[
M^* := \{ (x,\bar{x}) : Q(x,\bar{x}) = 0 \},
\]

where \( \bar{x} \) is the complex conjugate of \( x \). Using this special set we can rewrite equation (8) in the form

\[
g(x) + g(\bar{x}) = 0 \implies \Re g(x) = 0, \quad (9)
\]

where \( \Re \) stands or the real part of the complex variable.

Now we have a boundary value problem: The problem of determining a function \( g(.) \) which satisfies:

- \( g(.) \) Analytic everywhere except for a simple pole at 1
- \( \Re g(.) = 0 \) on \( M \setminus \{1\} \)
- \( \lim_{x \to 1} (x-1)g(x) = \alpha - 1 \)

In order to solve the boundary value problem constructed, let \( \phi \), with inverse \( \psi \), be the conformal mapping of the unit disk onto the region bounded by \( M^* \) with normalization conditions \( \phi(0) = K, \phi(1) = 1 \) for some real number \( K \). This mapping exists by the Riemann mapping theorem see [30], the curve \( M^* \) is simply connected only for large values of \( \alpha \). Define \( h(w) := g(\phi(w)) \). We then obtain a relatively simple Riemann Hilbert boundary value problem with a pole, for \( h(.) \) on the unit circle \( D \) (actually, it is a Dirichlet problem with a pole see [20]):

- \( \Re h(w) = 0, \) on \( w \in D \setminus \{1\} \)
- \( \lim_{w \to 1} (w-1)h(w) = \frac{\alpha - 1}{\phi(1)} \) where \( \phi(.) = \frac{d\phi}{dw} \)

with \( h(.) \) analytic on \( D \), continuous on \( \overline{D} \setminus \{1\} \). The solution of this boundary value problem is

\[
h(w) = \frac{1}{2} \frac{\alpha - 1}{\phi(1)} \frac{w + 1}{w - 1}, \quad w \in D
\]

which determines

\[
g(x) = h(\psi(x)) = \frac{1}{2} \frac{\alpha - 1}{\phi(1)} \frac{\psi(x) + 1}{\psi(x) - 1},
\]
inside the curve $M$; Substitution in the original equation finally yields
\[
P(x, y) = (\alpha - 1)\psi'(1) \frac{(x-1)(y-1)}{(\psi(x)-1)(\psi(y)-1)} \times \frac{\psi(x)\psi(y)-1}{(1+2\alpha)xy-\alpha x-\alpha y-x^2y^2}.
\]
(10)

The above equation represents a possible solution to the original equation, noting that we validate that it is a possible solution because it satisfies the normalization condition $P(1, 1) = 1$ through applying the L’Hôpital’s rule. In the subsequent subsections we investigate the conformal mapping $\psi(.)$, and the contour $M^*$ respectively.

3.1 The function $\psi(.)$

In order to find an explicit form to the function $\psi$, we compare our solution with the solution obtained in the original paper [18] to get:

The original one $P(x, 0)$ for $\alpha = \beta$ is given by:
\[
P(x, 0) = \frac{(\alpha - 1)^{3/2}}{\alpha(x - 1)^{1/2}}.
\]
(11)

Using our solution (10), the function $P(x, 0)$ for $\alpha = \beta$ is given by:
\[
P(x, 0) = (\alpha - 1)\psi'(1) \frac{(x-1)(y-1)}{(\psi(x)-1)(\psi(y)-1)} \times \frac{\psi(x)\psi(y)-1}{x}\psi(x)\psi(y)-1.
\]
(12)

If we equate the two solutions we get that
\[
\sqrt{\frac{\alpha - 1}{\alpha - x}} = \psi'(1) \frac{(x-1)(y-1)}{(\psi(x)-1)(\psi(y)-1)} \times \frac{\psi(x)\psi(y)-1}{x}.
\]

solving for $\psi(x)$ to get in the final form
\[
\psi(x) = \frac{x\sqrt{\frac{\alpha - 1}{\alpha - x}}(\psi(x) - 1) - \psi(1)(x - 1)}{\sqrt{\frac{\alpha - 1}{\alpha - x}}(\psi(x) - 1) - \psi(0)\psi(1)(x - 1)}.
\]

3.2 The curve $M^*$

The curve defined by $M^* := \{(x, \tilde{x}) : Q(x, \tilde{x}) = 0\}$, can be rewritten as follows
\[
Q(x, \tilde{x}) = (1 + 2\alpha)x\tilde{x} - \alpha x - \alpha x - \tilde{x}^2(\tilde{x})^2 = 0,
\]
which can be rewritten as
\[
Q(x, \tilde{x}) = (1 + 2\alpha)|x|^2 - 2\alpha(\Re x) - (|x|^2)^2 = 0.
\]

If we assume that $x = a + ib$ the curve can be written as
\[a^4 + b^4 + 2a^2b^2 - (1 + 2\alpha)a^2 - (1 + 2\alpha)b^2 + 2\alpha a = 0,
\]
whenever $\alpha$ is large enough the curve $M^*$ describes a simply connected domain. So it is possible to map the interior of the curve $M^*$ conformally to the unit disk in the case that $\alpha$ is large enough.

4 Expectations

In this section we find the expected number of files in the database in one of the queues using the corresponding generating functions. It is easy to see using (3) that the generating function of the number of files in the first queue is given by
\[
P(x, 1) = \sum_{m=0}^{\infty} P(N_1 = m)x^m
\]
\[
= \frac{\alpha(x-1)P(0,1)}{(1+\alpha+\beta)x-\alpha-\beta x-x^2},
\]
(13)

where $N_1$ is the number of files in the first queue. Using the normalization condition $P(1, 1) = 1$ in the above equation to find $P(0, 1)$ after some nontrivial manipulations including applying l’Hôpital’s rule to get
\[
P(0, 1) = \frac{\alpha - 1}{\alpha}.
\]

From (13) we will compute the expected number of files in queue 1. It is well known see e.g. [31] that the expected number of packets in queue 1 is given by
\[
E[N_1] = \frac{\partial}{\partial x} P(x, 1)|_{x=1}
\]
\[
= P(0, 1) \frac{\alpha(x-1)}{(1+\alpha+\beta)x-\alpha-\beta x-x^2}
\]
\[
= \frac{0}{\beta},
\]
therefore by applying the L’Hôpital’s rule we get
\[
E[N_1] = \frac{1}{\alpha - 1}.
\]
(14)

The above equation represents the expected number of database items in one of the queues. It is clear
from (14) that the higher the service rate the lower the expected number of items will be in the first queue which clearly makes perfect sense. By symmetry we can compute the expected number of database items in the second queue using the same procedure. In figure (2) we plot the expected number of database items $E[N_i]$ versus the service rate $\alpha$. It is clear from the figure that when the service rate is large enough then the expected number of waiting items in the queue will be sufficiently small which practically makes sense.

5 Conclusion

In this article, we managed to introduce a solution of a two-variable functional equation arising from a queueing model. This is done by considering the fully symmetric case on the underlying system and by reduction to a boundary value problem. Another contribution is using generating functions to find some expectations of interest. Possible extension of this work could be to use boundary value problems to find a solution to the asymmetric case. Another extension is to find a general solution methodology to such interesting class of equations.

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