Post-Newtonian Equations of Motion for Inertial Guided Space APT Systems

JOSE M. GAMBI
University Carlos III de Madrid
Gregorio Millan Institute
Avda. de la Universidad, 30, 28911 Leganes
SPAIN
gambi@math.uc3m.es

MARIA L. GARCIA DEL PINO
I.E.S. Alpajes
Department of Mathematics
C. de las Moreras, 28, 28300 Aranjuez
SPAIN
lgarcia@educa.madrid.org

Abstract: The equations of motion derived in this paper are aimed to determine according to the post-Newtonian framework the relative motion of Earth satellites with respect to inertial guided space Acquisition, Pointing and Tracking systems. The equations are suitable for satellites that are even far from the systems. The tool used to derive them is Synge’s world-function for the Earth surrounding space. Hence the equations are written in terms of the quasi-Cartesian coordinates tied to the systems.

Key–Words: Earth satellites, Intersatellite laser links, Acquisition, Pointing and Tracking systems, Earth post-Newtonian framework

1 Introduction

The post-Newtonian framework of the Earth surrounding space is the framework that actually meets the present needs in accurate Positioning and Navigation [1]-[4]. In fact, this is the framework used to synchronize the atomic clocks on board the GPS satellites, so as to determine the round-trip times taken by the laser beams in SLR and LLR [5]-[8].

The emerging importance of space-based systems in locating radio transmitters, both on the Earth surface or in space, is also leading to build up the Geolocation models within this framework. The reason is that increasing accuracy in locating emitters is becoming a must, and, after all, the Geolocation problem can be posed, and solved, equally well than the Navigation problem. In fact, the Geolocation problem is mathematically the inverse of the Navigation problem [9]. Hence some post-Newtonian formulae related to Geolocation are considered to be standard, such as Soffel’s frequency shift formula is considered by Montenbruck and Gill [10]-[11].

Likewise, the implementation of accurate Acquisition, Pointing, and Tracking (APT) systems is becoming a relevant task. In particular, systems with Laser technology merit more and more attention due to the fact that this technology has matured substantially in recent years [12]-[13].

However, the post-Newtonian framework is not used yet by the Sat-to-Sat laser communication systems, despite one important issue into the major tasks of engineering inertial guided laser terminals is to provide accurate tracking procedures for systems endowed with very narrow laser beams [14].

Now, since this fact is not due to the lack of accuracy of the APT hardware, we may reasonably conclude that it could be due to the difficulty for the Newtonian procedures to account in real time for the different curvatures of the Earth surrounding space at the positions of the targeted satellite and the APT system. That is, in Newtonian terms, it can be due to the difficulty to account in real time for the different tidal effects of the Earth on the respective orbital positions, particularly when the target is far from the system.

The post-Newtonian equations introduced below account for these small but important differences. Therefore, they can help increase the standard accuracy in the determination of the relative motion of the target, let us say $S_2$ from now on, with respect to the APT system, say $S_1$.

To derive the equations keeping consistency with previous works, the structure of the space-time about the Earth assumed in this paper is the same assumed by some authors and/or recommended in Geodesy and Geolocation (so as in other fields, such as in Electronic Warfare) [15]-[27]. The structure is the weak approximation to the Schwarzschild field generated by the Earth.

The equations are derived from Synge’s equations of geodesics, which are written in terms of Fermi coordinates. Hence they involve Synge’s world-function. Now, despite this function is a genuine relativistic tool, and so powerful that it is considered
nowadays to be universal [28], it is not certainly a familiar tool. For this reason, we start showing the procedure followed to derive the equations by introducing this function, together with its most relevant properties (used in the paper) in Section 2. Then the equations are derived in Section 3. Finally, numerical simulations showing some typical relative trajectories, so as the validity of the equations, are shown in Section 4.

2 The World-function

The world-function is an old function. In fact, it was introduced into tensor calculus by Ruse [29], but it was only after Synge that it appeared as an outstanding tool to work with within space-time frameworks [30]. Since then it is known as Synge’s world-function, and many relevant results have been obtained with it, among them those in [1].

To show this function, so as its properties (used in this paper), let us first consider, as an example, the 3-D space about the Earth is Euclidean. Then the world-function, and many relevant results have been obtained with it, among them those in [1].

Let us then assume that \( x^{\alpha 1} \) and \( x^{\alpha 2} \) (\( \alpha = 1, 2, 3 \)) are the Cartesian coordinates in \( E_3 \) of two spots \( P_1 \) and \( P_2 \), supposed to be occupied, not necessarily at the same instant, by two satellites that are moving in the Earth surrounding space. (To get as closer as possible to our problem, let us evoke it by denoting the satellites by \( S_1, S_2 \).) Let us now assume that the geometry of the space about the Earth is Euclidean. Then the world-function \( \Omega(P_1, P_2) \) is given by

\[
\Omega(P_1, P_2) = \frac{1}{2} [(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2],
\]

or, in compact form, by

\[
\Omega(P_1, P_2) = \frac{1}{2} \Delta x^\alpha \Delta x^\alpha = \frac{1}{2} \delta_{\alpha \beta} \Delta x^\alpha \Delta x^\beta,
\]

where \( \Delta x^\alpha = x^{\alpha 2} - x^{\alpha 1} \). Thus, \( \delta_{\alpha \beta} \) appears as what it actually is: the metric tensor of \( E_3 \) in Cartesian coordinates.

The most important characteristic of \( \Omega(P_1, P_2) \) emerges from the most significant feature of Euclidean spaces: since there is only one straight line in \( E_3 \) joining \( P_1 \) and \( P_2 \), we have that if \( S_1, S_2 \) move along smooth paths, and \( P_1, P_2 \) are spots successively occupied by \( S_1, S_2 \) (not necessarily at the same instants), \( \Omega(P_1, P_2) \) results in a smooth function, non-negative in this case, of the three coordinates of \( P_1 \) and of the three of \( P_2 \). Or shortly said, \( \Omega(P_1, P_2) \) result to be a two-point smooth scalar function of \( P_1 \) and \( P_2 \), which are the end points of the segment \( P_1 P_2 \), as suggested by the notation (Fig.1).

Let us now assume that the space-time about the Earth is flat, that is, that the structure of the space-time about the Earth is Minkowskian. This leads to assume that the 3-D space about the Earth is still Euclidean; also, that the model for the gravitational action of the Earth on the satellites is Newtonian, and finally, that the speed of any electromagnetic signal in vacuo is \( c \), so that the metric tensor for this space-time in Earth Centered Inertial (ECI) coordinates, \( (x^\alpha, ct) \), is \( \eta_{ij} = \text{diag}(1, 1, 1, -1) \). (Latin indices range from 1 to 4). Then, unlike in (2), \( P_1 \) and \( P_2 \) are now events, and the world function relates events \( P_1 \) to events \( P_2 \) as for any other space-time, although this time according to the following rule: if \( (x^{\alpha 1}, ct^1) \), \( (x^{\alpha 2}, ct^2) \) are the ECI coordinates of \( P_1, P_2 \), then

\[
\Omega(P_1, P_2) = \frac{1}{2} [\Delta x^\alpha \Delta x^\alpha - c^2 \Delta t^2],
\]

(3)

or in compact form, to show the role of \( \eta_{ij} \),

\[
\Omega(P_1, P_2) = \frac{1}{2} \eta_{ij} \Delta x^i \Delta x^j,
\]

(4)

where \( \Delta x^i = x^{i 2} - x^{i 1} \), and now \( \Delta t = t^2 - t^1 \). Or in compact form, to show the role of \( \eta_{ij} \),

\[
\Omega(P_1, P_2) = \frac{1}{2} \eta_{ij} \Delta x^i \Delta x^j,
\]

(4)

This function is illustrated in Fig.2, where \( L_1 \) represents the time history of events, or world line, of \( S_1 \), and \( L_2 \), the world line of \( S_2 \). Therefore, the projections of \( L_1 \) and \( L_2 \) onto the 3-D Euclidean space (spanned by the three space axes at the bottom of the picture) represent the trajectories of \( S_1 \) and \( S_2 \) in space.

Let us now note that the expression in (4) contains all the information on the intrinsic geometry of Minkowskian space. In fact, it contains the information in \( \text{finite} \) form. Thus we have for the classification of events in Minkowskian space that \( \Omega(P_1, P_2) \) can be positive, negative, or null. In fact, if the geometry of the space-time about the Earth is assumed to be Minkowskian, then for a given event \( P_1 \in S_1 \) there are infinite events \( P_2 \in S_2 \) for which \( \Omega(P_1, P_2) < 0 \),

Figure 1: The world-function for the 3-D Euclidean space.

Figure 2: The world-function for the 3-D Minkowskian space.
where $g_{ij} = \omega_{ij} + \gamma_{ij}$, and $\gamma_{ij}$ is the geodesic curvature,
	herefore$ the world-function is, to within the factor $\epsilon$, nothing else but half the square of the measure of the geodesic (assumed unique) that joins any two events in a given space-time. Hence, as a consequence, we have: (I) $\Omega(P_1, P_2)$ is single-valued, and does not depend on $\Gamma_{P_1, P_2}$; it only depends on the eight coordinates $x^{1i}, x^{2i}$, of the events $P_1, P_2$, which are the end points of $\Gamma_{P_1, P_2}$, as suggested by the notation; (II) $\Omega(P_1, P_2)$ is a two-point scalar function of $x^{1i}$ and $x^{2i}$, i.e. it is invariant under coordinate transformations both at $P_1$ and $P_2$; (III) successive covariant derivatives of $\Omega(P_1, P_2)$ can be taken unambiguously with respect to the coordinates of $P_1$ and/or with respect to the coordinates of $P_2$ (following Synge, these derivatives will be indicated with simple subscripts, that is to say, without the usual stroke); (IV) the partial derivatives of $\Omega(P_1, P_2)$ with respect to $P_1$, i.e. $\Omega_{i_1}(P_1, P_2)$, are equal to $-U_{i_1}$, and analogously, $\Omega_{i_2}(P_1, P_2) = U_{i_2}$ (the minus sign in the first expression is consistent with the fact that $\Omega_{i_1}$ and $\Omega_{i_2}$ are the derivatives of $\Omega(P_1, P_2)$ at the end points $P_1$, $P_2$). In particular, if $\Gamma_{P_1, P_2}$ is not null ($\epsilon = \pm 1$), then $\Omega_{i_1}(P_1, P_2) = -L\lambda_{i_1}$ and $\Omega_{i_2}(P_1, P_2) = L\lambda_{i_2}$, where $L = \int_{P_1}^{P_2} ds$, and $\lambda_{i_1}$, $\lambda_{i_2}$ are the unit tangent vectors to $\Gamma_{P_1, P_2}$ at $P_1$ and $P_2$ respectively; (V) the norms of $\Omega_{i_1}$ and $\Omega_{i_2}$ are $\Omega_i$, that is, $g^{1i}, g^{2i} = g^{ij}, g^{kj}$ are the contravariant metric tensors at $P_1$ and $P_2$ respectively; (VI) the second covariant derivatives $\Omega_{ij}(P_1, P_2)$ and $\Omega_{ij}(P_1, P_2)$ are equal to $\delta\Omega_{ij}/\delta x^{j} - \Gamma^{a}_{ij}, \Omega_{a}$ and to $\delta\Omega_{ij}/\delta x^{j} - \Gamma^{a}_{ij}, \Omega_{a}$ respectively, where the Christoffel symbols are taken at $P_1$ and $P_2$ respectively, as suggested by the notation; (VII) $\Omega_{ij}(P_1, P_2) = \partial\Omega_{ij}/\partial x^{j}$, $\Omega_{ij}(P_1, P_2) = \partial\Omega_{ij}/\partial x^{j}$, and $\Omega_{ij}(P_1, P_2) = \delta\Omega_{ij}/\partial x^{j} - \Gamma^{a}_{ij}, \Omega_{a}$; (VIII) $\Omega_i(P_1, P_2) = g^{ij}, \Omega_{ij}$ and $\Omega^{ij}(P_1, P_2) = g^{ij}, \Omega_{ij}$ (IX) in flat space-time with Cartesian coordinates $x^i$,

$$\Omega(P_1, P_2) = \frac{1}{2} \int_0^{\epsilon} g_{ij} U^i U^j d\omega,$$

(5)

taken along $\Gamma_{P_1, P_2}$, where $\Gamma_{P_1, P_2}$ is given by $x^i = x^i(\omega)$, $\omega$ being an affine parameter satisfying $0 \leq \omega \leq 1$, so that $P_1 \equiv x^i(0), P_2 \equiv x^i(1)$ and $U^j = dx^j/d\omega$. (As before, Latin indices range from 1 to 4, and Greek from 1 to 3).

It is then straightforward to deduce from the definition that if $ds$ is the arc length element of $\Gamma_{P_1, P_2}$, we have

$$\Omega(P_1, P_2) = \frac{1}{2} \epsilon \left( \int_{P_1}^{P_2} ds \right)^2,$$

(6)

where $\epsilon = +1$, if $\Gamma_{P_1, P_2}$ is space-like; $\epsilon = -1$, if $\Gamma_{P_1, P_2}$ is time-like; and $\epsilon = 0$, if $\Gamma_{P_1, P_2}$ is null;

Figure 2: The world-function for Minkowskian space.
(0 ≤ ω ≤ 1), and ε is a small dimensionless parameter such that ε^2 is of the order of v^2 and U, where v is in our case the characteristic Classical 3-speed of the satellites in orbit about the Earth with respect to the Earth, and U the Newtonian potential of the Earth in the neighborhood of the Earth. Thus, the first part in (9) is the world-function in (7), and the remainder is O(ε^2) (note that we are now taking c = G = 1); and (XI) the world-functions in (7) and (9), so as their derivatives, can be expanded about P1 and/or P2 with the usual methods of approximation without abandoning the facilities of tensor calculus.

Since −Ωi(t(P1, P2)) is the Minkowskian 4-position vector of P2 with respect to P1 for the particular case in (7), and analogously, −Ωi2(P1, P2) is the 4-position vector of P1 with respect to P2, we are entitled to consider −Ωi1 and −Ωi2 as 4-position vectors for the more general case in (9). In fact, when P1 and P2 occur in the vicinity of the Earth, the heads of −Ωi1 and −Ωi2 can be thought as “not too far apart” from P2 and P1 respectively, provided that “far apart” is meant in the 4-Euclidean sense, that is to say, with the 4-Euclidean topology associated to Minkowskian space. Likewise, the second-order covariant derivatives yield relative velocities, and the third-order derivatives, relative accelerations [31].

3 The Equations of Motion

Let E be a space-time with metric gij(x^k), i.e. gij(x^a, t), and world-function Ω(x^k1, x^k2). Let (\lambda^i_1(s_1), \lambda^i_2(s_1)) be an orthogonal tetrad of unit vectors Fermi-transported along a time-like (base) world line, let us say M1(x^k1(s_1)), with \lambda^i_1(s_1) = A^i(s_1) = ds^k_1/ds_1, where s_1 is the proper time of C1, i.e. of the object whose world-line is M1, so that A^i_1(s_1) = \lambda^i_1(s_1) = ds_1/ds_1; let P2(x^k_2) be an arbitrary event in a time-like geodesic, say M2(x^k_2(s_2)), where s_2 is the proper time of C2, the object whose world-line is M2; and let (X^i_1, s_1) = (X_1, s_1) be the Fermi coordinates of P2(x^k_2) with respect to C1 (see e.g. [30]). If b_1(s_1), the first curvature of M1, is null for all s_1; if Ωi1j1 and Ωi1j2 are the third-order covariant derivatives of Ω(x^k_1, x^k_2) taken as indicated by the indices, i.e. with respect to x^k_1, x^k_2 and x^k_1, in the first case, and with respect to x^k_2 for the third derivative in the second case; if, furthermore, H^k_2 = A^k_2(ds_2/ds_1) with A^k_2 = dx^k_2/ds_2, and finally, if

\[
dL_1(dS_1) = \chi L_1 + \Omega_{i_1j_1j_2} A^{i_1} A^{j_1} H^{j_2} + \Omega_{i_1j_2j_2} A^{i_1} H^{j_2} H^{j_2},
\]

with L_1 = Ω_{i_1j_2} A^{i_1} A^{j_2}, where Ω_{i_1j_2} are the second-order covariant derivatives of Ω(x^k_1, x^k_2), first with respect to x^k_1, and then with respect to x^k_2; x = (d^2s_2/d\xi^2)/(ds_2/ds_1), and Ωi1j1j2, Ωi1j2j2 are the third-order covariant derivatives whose interpretation is similar to those of the previous derivatives, then Syngen’s equations for C2 in terms of the Fermi coordinates associated to M1 read [30]

\[
\begin{align*}
\frac{d^2X^i_1}{ds_1^2} & = -\Omega_{i_1j_1j_2} A^{i_1} A^{j_1} H^{j_2} - \Omega_{i_2j_1j_2} A^{i_2} A^{j_1} H^{j_2} - \frac{dL_1}{ds_1} \\
\end{align*}
\]

According to Syngen the calculations to integrate Equations (11) become unmanageable. However, they become much simpler, and probably useful, if the following assumptions are considered to be reasonable in order to determine the relative motion of S2 with respect to S1: i) the structure of the space-time about the Earth is that of the post-Newtonian approximation of the Earth Schwarzschild field. In ECI coordinates the metric of this field is

\[ g_{ij} = \eta_{ij} + \gamma_{ij}, \]

where

\[
\begin{align*}
\gamma_{\alpha\beta} & = \frac{2m}{r} x^{a_\alpha} x^{a_\beta} + O(\varepsilon^2), \\
\gamma_{\alpha4} & = O(\varepsilon^2), \gamma_{44} = \frac{2m}{r} + O(\varepsilon^2),
\end{align*}
\]

m being the mass of the Earth measured in seconds (c = G = 1) and v^2 = x^{a_\alpha} x^{a_\beta}; ii) ds_2/ds_1 = 1 approx.; and iii) L_2 is nearly parallel to L_1 (physically this means that the relative speed of S2 with respect to S1 is small as compared to v); for in that case (i) \lambda^{i_1}_1(s_1) becomes an inertial guided system co-moving with S1, i.e.

\[
\begin{align*}
\lambda^{i_1}_1(s_1) & = \delta^{i_1}_1, \lambda^{i_2}_1(s_1) = v^{a_1}, \lambda^{i_1}_2(s_1) = v^{a_1}, \\
\lambda^{i_2}_2(s_1) & = 1 + \frac{1}{2} \gamma_{44}(x^{a_\alpha}) + \frac{1}{2} (v_1)^2,
\end{align*}
\]

where v^{a_1} = A^{a_1} = dx^{a_1}/ds_1 and (v_1)^2 = v^{a_1} v^{a_1}, and (ii) the space Fermi coordinates of P2 with respect to \lambda^{i_1}_1(s_1), X^{(a_2)}, become the quasi-Cartesian coordinates of S2 with respect to S1 at s_1. Further, if the coordinates of the events P2, P_2 ∈ L_2 with respect to \lambda^{i_1}_1(s_1) and \lambda^{i_1}_2(s_1 + ds_1) are X^{(a_2)} \approx (X^{(a_2)}(s_1)), and X^{(i_2)} + dX^{(i_2)} \equiv (X^{(a_2)} + dX^{(a_1)}, s_1 + ds_1) respectively, and P_1 is the foot at L_1 of the geodesic drawn from P_2 to cut orthogonally L_1 (or, in other words, if P_2 is in the instantaneous local space of
$P_i \in L_1$), then the structure of the space-time as seen by $S_1$ at $s_1$ is given by

$$2\Omega(P_2, P_2') = g_{(ij)}dX^{(i)}dX^{(j)},$$

(15)

with

$$g_{(ij)} = \delta_{ij} + 2h_{(ij)},$$

$$g_{(44)} = 1 + 2h_{(41)} + 2h_{(42)},$$

$$g_{(44)} = O(\varepsilon^2),$$

(16)

where

$$h_{(ij)} = \frac{3}{2}X^{(i)}X^{(j)} \int_0^1 (1 - u)uS_{(ij)}du,$$

$$h_{(41)} = \frac{3}{2}X^{(i)}X^{(j)} \int_0^1 (1 - u)^2S_{(41)}du,$$

$$h_{(42)} = \frac{3}{2}X^{(i)}X^{(j)} \int_0^1 (1 - u)^2S_{(42)}du.$$

(17)

The integrals being taken along the straight line $C$ described above ($x^\nu = x^\nu(1 - u) + x^\nu_0u$); $S_{(abcd)} = S_{(abcd)}(x^\nu(u))$

$$S_{(abcd)} = S_{(abcd)}[x^\nu(u)]$$

$$= S_{ijkl}(x^\nu(u))\lambda_{ij}^k\lambda_{kl}^m\lambda_{mn}^i(x^\nu(u)).$$

(18)

where $\lambda_{ij}^k(x^\nu(u))$ are the tetrads obtained by parallel transport from $\lambda_{ij}^k$ along $C$ according to the metric (12), and

$$S_{ijkl}(x^\nu(u)) = \frac{1}{3}(R_{ijkl} + R_{jimk})(x^\nu(u)),$$

(19)

where $R_{ijkl}(x^\nu(u))$ is the Riemann tensor of (12) at $x^\nu(u).$

In fact, under these hypothesis Equations (11) become

$$\frac{d^2X^{(\alpha)}}{ds_1^2} = -\Omega_{(\alpha 4141)} - 2\Omega_{(\alpha 4142)} - \Omega_{(\alpha 1424)},$$

(20)

where

$$\Omega_{(\alpha 4141)} = -\frac{1}{3}X^{(\gamma)} \int_0^1 (1 - u)^2R_{(4\gamma 4)}du,$$

$$\Omega_{(\alpha 4142)} = \frac{1}{3}X^{(\gamma)} \int_0^1 u^2R_{(4\gamma 4)}du,$$

$$\frac{d^2X^{(\alpha)}}{ds_1^2} = \frac{m}{r_1^2}\left[(3\cos^2 M_1 - 1)X^{(1)} + (3\cos M_1 \sin M_1)X^{(2)}\right],$$

$$\frac{d^2X^{(\alpha)}}{ds_1^2} = \frac{m}{r_1^2}\left[(3\cos M_1 \sin M_1)^2X^{(1)} + (3\sin^2 M_1 - 1)X^{(2)}\right].$$

(21)

In this regard, let us note that $h_{(ij)}$ in (17) need not be used to derive (20), nor $\gamma_{ij}$ in (13) to derive (22). Note, finally, that the equations (20) reduce to the equations of the geodesic deviation for nearby satellites. These equations are

$$\frac{d^2X^{(\alpha)}}{ds_1^2} = -R_{(\alpha 4\gamma 4)}X^{(\gamma)},$$

(23)

with $R_{(\alpha 4\gamma 4)}$ evaluated at $x^\nu_1(s_1).$

## 4 Numerical Simulations

To show the qualitative behavior of the solutions of (20), it is enough to assume that $S_1$ and $S_2$ are in coplanar circular orbits, taking care of not exceeding two time limits, which are unavoidable and characteristics of each simulation: the first is an upper bound that fix the mathematical validity of the equations in each case; the second is the upper bound from which the relative position of $S_2$ with respect to $S_1$ cannot be materialized by $S_1.$ Otherwise, the fact is that the differences between the Newtonian relative motions and the solutions of (20) for arbitrary orbital motions, so as between the solutions of (23) and those of the linear approximation to (20), which are only valid for small differences of orbital radii and short time intervals, are significant enough, and straightforwardly analyzable. The reason is that, whatever the orbital elements are, those differences essentially depend on the semi-major axis and eccentricities of the orbits of $S_1$ and $S_2.$

Under the restrictions mentioned above, (23) becomes

$$\frac{d^2X^{(1)}}{ds_1^2} = \frac{m}{r_1^2}\left[(3\cos^2 M_1 - 1)X^{(1)} + (3\cos M_1 \sin M_1)X^{(2)}\right],$$

$$\frac{d^2X^{(2)}}{ds_1^2} = \frac{m}{r_1^2}\left[(3\cos M_1 \sin M_1)^2X^{(1)} + (3\sin^2 M_1 - 1)X^{(2)}\right].$$

(24)

1The quasi-Cartesian angle directions do not differ from the Euclidean angles (see the components of $\lambda_{ij}^k$ in (14)). The quasi-Cartesian coordinates differ from the Cartesian coordinates only in that, instead of computing the range from $S_1$ to $S_2$ by means of the principal terms in (15), thus getting Cartesian coordinates, the range required to derive quasi-Cartesian coordinates is computed by means of (15)-(19).
the linear approximation to (20) for short time intervals and small radial distances is
\[
\frac{d^2 X_{(1)}}{ds_1^2} = \frac{m}{r_1^3} \left(1 - \frac{7}{4} \eta \right) \left(3 \cos^2 M_1 - 1 \right) X^{(1)}
\]
\[+ (3 \cos M_1 \sin M_1) X^{(2)},
\]
\[
\frac{d^2 X_{(2)}}{ds_1^2} = \frac{m}{r_1^3} \left(1 - \frac{7}{4} \eta \right) \left(3 \cos M_1 \sin M_1 \right) X^{(1)}
\]
\[+ \left(3 \sin^2 M_1 - 1 \right) X^{(2)},
\]
and the linear equations corresponding to (20) are
\[
\frac{d^2 X_{(1)}}{ds_1^2} = m \left[ \int_0^1 \left( (3r_1^2(1-u)^2 \cos^2 M_1 
\]
\[+ 6r_1r_2(1-u)u \cos M_1 \cos M_2 
\]
\[+ 3r_2^2u^2 \cos^2 M_2 \right) / \left( r_1^2(1-u)^2 + 2r_1r_2(1-u)u \right)
\]
\[\cos(M_1 - M_2) + r_2^2u^2 \right)^{5/2} \right] (1 - 2u + 3u^2) du \] X^{(1)}
\]
\[+ m \left[ \int_0^1 \left( (3r_1^2(1-u)^2 \sin M_1 \cos M_1 
\]
\[+ 3r_1r_2(1-u)u \sin(M_1 + M_2) 
\]
\[+ 3r_2^2u^2 \sin M_2 \cos M_2 \right) / \left( r_1^2(1-u)^2 + 2r_1r_2(1-u)u \right)
\]
\[\cos(M_1 - M_2) + r_2^2u^2 \right)^{5/2} \right] (1 - 2u + 3u^2) du \] X^{(2)},
\]
\[
\frac{d^2 X_{(2)}}{ds_1^2} = m \left[ \int_0^1 \left( (3r_1^2(1-u)^2 \sin M_1 \cos M_1 
\]
\[+ 3r_1r_2(1-u)u \sin(M_1 + M_2) 
\]
\[+ 3r_2^2u^2 \sin M_2 \cos M_2 \right) / \left( r_1^2(1-u)^2 + 2r_1r_2(1-u)u \right)
\]
\[\cos(M_1 - M_2) + r_2^2u^2 \right)^{5/2} \right] (1 - 2u + 3u^2) du \] X^{(1)}
\]
\[+ m \left[ \int_0^1 \left( (3r_1^2(1-u)^2 \sin^2 M_1 
\]
\[+ 6r_1r_2(1-u)u \sin M_1 \sin M_2 
\]
\[+ 3r_2^2u^2 \sin^2 M_2 \right) / \left( r_1^2(1-u)^2 + 2r_1r_2(1-u)u \right)
\]
\[\cos(M_1 - M_2) + r_2^2u^2 \right)^{5/2} \right] (1 - 2u + 3u^2) du \] X^{(2)},
\]
where \(X^{(1)}, X^{(2)}\) are the plane orbital coordinates; \(M_1 = M_1(s_1)\) is the mean anomaly of \(S_1\) at \(s_1\); \(r_1^2 = x^6, x^6; r_2^2 = x^6, x^6\) as before, and \(\eta = ((r_2 - r_1)/(r_1)) \ll 1\).

![Figure 3: Relative orbits from \(S_1\).](image1)

![Figure 4: ECI centered orbits.](image2)

We note with respect to (25) that \(\eta\) does not depend on \(s_1\). It can also be verified, as a matter of checking, that if \(\eta = 0\), and the initial condition are \(X_{(1)}(0) = X_{(2)}(0) = 0\), \((dX_{(1)}/ds_1)_0 = (dX_{(2)}/ds_1)_0 = 0\), then \(X_{(1)}(s_1) = X_{(2)}(s_1) = 0\), as expected.

Finally, we note that when \(S_1\) and \(S_2\) are in opposition with respect to the Earth, then the line integrals in (26) become singular, since then \(\cos(M_1 - M_2) = -1\), and there is a value of \(u (u = r_1/(r_1 + r_2))\) for which the denominators in the integrands are zero.
Therefore, from the mathematical point of view the equations in (26) are valid until the instant at which that configuration is reached. But this implies that these equations are always applicable, since fortunately that limit is far beyond the physical limits mentioned above, which are due to the Earth size (as is known, these limits correspond to the time intervals within which \( S_2 \) is in the line of sight of \( S_1 \), so that the largest limit is reached when \( S_1 \) is geostationary).

Since \( c = G = 1 \) in all the expressions and equations derived in the paper, the data involved in the simulations corresponding to Figs. 3-8 have been introduced in seconds. In particular, \( m \) has been assumed to amount \( 1.479 \cdot 10^{-11} \) sec. Figs. 3-5 have been generated by means of (26) for \( r_1 = 14.002 \cdot 10^{-2} \) sec, \( X_0^1 = r_2 - r_1 = -3 \cdot 10^{-3} \) sec, and \( X_0^2 = 0 \) sec, for the time interval \([0, 400000]\) sec. Figs. 6-8 have been generated for \( r_1 = 4.0 \cdot 10^{-2} \) sec, \( X_0^1 = r_2 - r_1 = -8 \cdot 10^{-4} \) sec, and \( X_0^2 = 0 \) sec, for the time interval \([0, 50000]\) sec. Figs. 3 and 6 show the Newtonian and post-Newtonian relative orbits of \( S_2 \) with respect to \( S_1 \). The respective ECI orbits are shown in Figs. 4, 7, and to generate them, the transformations from the inertial local system given in (14) have been used to the respective order of approximation. In comparing Figs. 3, 6 with Figs. 4, 7 it can be seen, particularly when these last are sequentially plotted, that the small loops in Figs. 3, 6 correspond to delays and advances of the post-Newtonian motion of \( S_2 \) with respect to the Newtonian prediction. Finally, it can be deduced from Figs. 4, 7 and Figs. 5, 8 the following fact: the integrals in (21) manifest themselves in the oscillatory motion, i.e. in the tidal motion, of the post-Newtonian ECI orbit of \( S_2 \) about its Newtonian orbit.
5 Conclusion

The calculations to integrate (20) are certainly manageable, and so, feasible to increment the accuracy of the APT laser systems, provided that, to keep consistency, the initial data are obtained by means of post-Newtonian Geolocation formulae or by close tracking. In fact, the main characteristic of (20) is that when they include the Earth tidal effects and reduce to the Newtonian Geolocation formulae or by close tracking the APT laser systems, provided that, to keep consistency, the initial data are obtained by means of post-Newtonian Geolocation formulae or by close tracking. In fact, the main characteristic of (20) is that

Finally, nearby
equations in (26), (25), (24), and (23) successively when $S_2$ is respectively closer and closer, up to be, finally, nearby $S_1$. This is the reason we can expected that the differences between the Newtonian and post-Newtonian predictions for $S_1$ may be very large according to (20), even up to tens of meters, as Figs. 5, 8 suggest; therefore, numerical integration of (20) and error analysis are required to provide accurate quantitative predictions.

Acknowledgements: The authors thank Prof. M.M. Tung (Univ. Politecnica de Valencia) and J. Gschwindl (Technische Univ. Wien) for their contributions in getting these and other numerical simulations.

References:


