Modal paraconsistent tableau systems of logic

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Abstract: The aim of this paper is to show how to simply define paraconsistent tableaus by liberalization of construction of complete tableaus. The presented notions allow us to list all tableau inconsistencies that appear in a complete tableau. Then we can easily choose these inconsistencies that are effects of interactions between premises and a conclusion, simultaneously excluding other inconsistencies. A general technique we describe is presented here for the case of modal logic, as one of the most interesting, and with many applications in communication theory and data mining. In case we have an inconsistent set of information and apply a logic that includes classically accepted patterns of reasoning, we may correctly conclude any arbitrary proposition. To avoid this we propose paraconsistent tableau proofs that list all relevant premises which guarantee truth of conclusion under consideration, but at the same time are consistent.

Key–Words: blind rule, modal logic, paraconsistent tableaus, t-inconsistency, tableau rules, tableaus Typing manuscripts, LaTeX

1 Introduction

It is a well-known property that any system of logic that is superclassical contains a law of explosion (*ex falso quodlibet*, "from a falsehood, anything follows"). It means that when we have an inconsistent set of premises X (for example $X = \{A, \neg A\}$, where $\neg$ is a classical negation) we can correctly draw any conclusion: $X \models B$, for any B in a language of considered logic. Of course, we usually accept most of classical arguments, but in case we have an inconsistent set of propositions, we would not like to accept an arbitrary conclusion. It is a specially striking fact in data mining and analysis of communication, where very often there appear inconsistent parts of information among other pieces that seems quite reliable.

There exist many attempts of modifications of classical logic to prevent from a law of explosion. They are usually quite complicate and sophisticated. Unlike to them we propose an clear and simple method of defining a subsystems of modal logics that do not have the property of law of explosion. Our starting point are suitable tableau counterparts od modal systems, but the goals are paraconsistent tableau subsystems.

Usually tableau methods are at the same time effective and non-formal. Although we prepared a formal theory of tableaus [3] that prevents from a jejune way of applications of tableau rules, so — for example – rules cannot be applied to inconsistent branches, here we define rules as *blind*. It means that tableau inconsistencies that occur in tableaus do not stop developing of a given tableau. When we decompose all expressions, we stop a proof. It is because in the case of paraconsistent arguments we look for a special kind of inconsistency, that follows from incompatibility of premises and a negated conclusion. In order to identify suitable inconsistencies in a tableau, we need to decompose all formulas to the level of literals in such a way that it gives an answer whether there is a collision between premises and a negated conclusion.

In the paper we describe a mechanism of building such tableaus and choosing suitable inconsistencies. In further part we analyze some metatheoretical properties of this proposal. This type of approach can be used for other logics, being generalized as long as tableau rules are defined in the proposed style.

2 Basic notions

First we remind some semantical notions for modal logic and tableau notions we require to formulate and prove facts about paraconsistent tableaus.

2.1 Semantics

Let For be the set of all formulas build finitely over the following alphabet: $Var = \{p, q, r, p_1, q_1, r_1, \dots\}$

and $Con = \{\neg, \Box, \Diamond, \wedge, \vee, \rightarrow, \leftrightarrow\}$, where $\neg$, $\Box$ and $\Diamond$ are unary and $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ are binary connectives. It is a language of modal logic, expressions with Boolean connectives we read normally, expressions of the form $\Box A$ and $\Diamond B$ we read, respectively, *it is necessary that* A and *it is possible that* A (or differently, if we have epistemic, temporal or other intention).

A *model* $\mathfrak{M}$ (or *modal model*) for the set For is a quadruple $\langle W, R, V, w\rangle$, where W is any set (called usually *a set of possible worlds*), R is a binary relation defined on $W \times W$ (so $R \subseteq W \times W$, R is usually called *accessibility relation*), V is a function from $W \times$ For into the set of logical values $\{1, 0\}$ (called *a valuation in possible worlds*), such that any $A, B \in For$ and any $u \in W$ satisfies conditions:

$V(u, \neg A) = 1$ iff $V(u, A) = 0$
$V(u, A \wedge B) = 1$ iff $V(u, A) = 1$ and $V(u, B) = 1$
$V(u, A \vee B) = 1$ iff $V(u, A) = 1$ or $V(u, B) = 1$
$V(u, A \rightarrow B) = 1$ iff $V(u, A) = 0$ or $V(u, B) = 1$
$V(u, A \leftrightarrow B) = 1$ iff $V(u, A) = V(u, B)$.
$V(u, \Box A) = 1$ iff $\forall_{z \in W}(uRz \Longrightarrow V(z, A) = 1)$
$V(u, A) = 1$ iff $\exists_{z \in W}(uRz \,\&\, V(z, A) = 1)$,

and $w \in W$ (so W is non-empty).

Let $\mathfrak{M} = \langle W, R, V, w\rangle$ be a model. For any formula A and any set of formulas X we define:

A is *true* in $\mathfrak{M}$ (in short: $\mathfrak{M} \models A$) iff $V(w, A) = 1$
A is *false* in $\mathfrak{M}$ (in short: $\mathfrak{M} \not\models A$) iff $V(w, A) = 0$
X is *true* in $\mathfrak{M}$ (in short: $\mathfrak{M} \models X$) iff $\mathfrak{M} \models B$, for all $B \in X$
X is *false* in $\mathfrak{M}$ (in short: $\mathfrak{M} \not\models X$) iff $\mathfrak{M} \not\models B$, for some $B \in X$.

Now we define a central notion of any logic, a consequence relation. Let $\mathbf{M}$ be a set of models. Let A be a formula and X be a set of formulas. A is a *consequence* of X *modulo* $\mathbf{M}$ (in short: $X \models_{\mathbf{M}} A$) iff $\forall_{\mathfrak{M} \in \mathbf{M}}$ (if $\mathfrak{M} \models X$ then $\mathfrak{M} \models A$).[1]

As we know we can determine various classes of models by imposing constraints on accessibility relation R, for example, taking models — among others — with the conditions:

Reflexivity	$\forall_{u \in W} uRu$
Symmetry	$\forall_{u,z \in W}(uRz \Rightarrow zRu)$
Transitivity	$\forall_{u,z,x \in W}(uRz \,\&\, zRx \Rightarrow zRx)$.

Each of the conditions and more similar ones (and of course possible combinations of them) define some classes of models: $\mathbf{M}_{Ref}$, $\mathbf{M}_{Sym}$, $\mathbf{M}_{Trans}$, etc., and so some modal logics that can be identified with suitable consequence relations $\models_{\mathbf{M}_{Ref}}$, $\models_{\mathbf{M}_{Sym}}$, $\models_{\mathbf{M}_{Trans}}$ etc.

[1]By those conditions we have defined here so called *normal modal* logics [2].

Now we define a notion of inconsistent set of formulas. A set X of formulas is *inconsistent* iff there is no model $\mathfrak{M}$ such that $\mathfrak{M} \models X$.[2] A special kind of inconsistent set of formulas is a set that contains formulas A, $\neg A$, for some formula A.

As a corollary by definition of $\models_{\mathbf{M}}$ we have:

Corollary 1 *Let* $\boldsymbol{M}$ *be a set of models. Let* X *be an inconsistent set of formulas. Then* $X \models_{\boldsymbol{M}} A$*, for any formula* A*.*

The corollary 1 expresses a fact that all normal modal logics explode in case we have inconsistent premises. Of course, a paraconsistent logic should not have the above property.

2.2 Tableau systems for modal logics

In works [4] and especially in [3] we presented a formal theory of tableau systems for a class of logics defined by some syntactical and semantical conditions[3]. Hence, we have precise tableau notions that incorporate standard, intuitive notions. The precise notions (of a tableau rule and various kinds of branches, tableaus) with a notion of tableau system are necessary, when we generalize results, looking for some abstract properties of tableau methods.

However, here we dealt with tableaus for modal language, so we just use well-known intuitive tableau notions presented for example in [1] or [2]. We remind them in turns, giving a handful of definitions.

First of all tableaus are defined on some extended language, we call it a *set of expressions* and denote by Ex. Let $\mathbb{N}$ be the set of natural numbers. Ex is a union of the sets:

$For \times \mathbb{N}$,
$\{irj : i, j \in \mathbb{N}\}$,
$\{\sim irj : i, j \in \mathbb{N}\}$,
$\{i = j : i, j \in \mathbb{N}\}$,
$\{\sim i = j : i, j \in \mathbb{N}\}$,

that respectively denote formulas in possible worlds ($\langle A, i\rangle$, we will write them as pairs A, i), accessability/inaccessability between possible worlds (irj, $\sim irj$), identity/non-identity of possible worlds ($i = j$, $\sim i = j$).

Having a set of formulas X and a number i, by $\langle X, i\rangle$ we denote a set of expressions $\{\langle A, i\rangle : A \in X\}$, or just shortly X, i.

[2]We use a word *inconsistent* instead — for example — *contradictory*, since it enables us a direct transition between semantical and tableau notions.

[3]We mean such logics that are logics of terms or propositions, and are two–valued.

We assume that a set of expressions X is *t-inconsistent* iff for some $A \in For$ and some $i \in \mathbb{N}$, X contains $\langle A, i\rangle$ and $\langle \neg A, i\rangle$, or for some $i, j \in \mathbb{N}$, irj and $\sim irj \in X$, or $i = j$ and $\sim i = j \in X$. A set of expresions X is *t-consistent* iff X is not *t-inconsistent*.

By **R** we denote a set of rules for a given tableau system. It always contains three kinds of tableau rules: a) all standard tableau rules for Boolean connectives — we have nine rules in **R**, four positive rules (for $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$) and five negative (for $\neg\neg$, $\neg\wedge$, $\neg\vee$, $\neg\rightarrow$, $\neg\leftrightarrow$); b) moreover, we have rules for $\Box$ and $\Diamond$ as well as for $\neg\Box$ and $\neg\Diamond$; c) the last rules we mention are rules for properties of relation of accessibility that depends on a given class of models (this subset may be empty, if we do not impose any conditions). These of those rules rules we remind in the next section where we modify them to obtain paraconsistent tools.

The further tableau notions are shortly as follows. A *root* is a set of expressions that contains premises and a negation of conclusion with the same number. A *branch* is a sequence of expressions that starts from a root. The rest of the branch contains results of applying of rules to former expressions. A branch is *complete* iff all applicable tableau rules were used [4]. A branch is *incomplete* iff it is not *complete*. Branches can be also closed or open. A branch is *closed* iff it contains t-incosistent set of expressions; it is *open* iff is not closed.

Tableau can be treated as sets of branches with the same roots. Tableaus that include all suitable and only complete branches are called *complete*. *Complete tableaus* can be closed or open. A tableau is *closed* iff it is complete and all branches that includes are closed; it is *open* iff is not closed.

Let **R** be a set of tableau rules for some modal, normal logic. Now, for any set of formulas X and a formula A we define a tableau consequence relation $\rhd_{\mathbf{R}}$, by putting:

Definition 2 *$X \rhd_{\mathbf{R}} A$ iff there exist a finite subset Y of X and a closed tableau with a root $\langle Y \cup \{\neg A\}, i\rangle$, for some $i \in \mathbb{N}$.*

Of course, the relation $\rhd_{\mathbf{R}}$ is fully determined by tableau rules of **R**. So, by $\langle For, \rhd_{\mathbf{R}}\rangle$ we understand a tableau system determined by tableau rules **R**.

[4]Generally, we divide complete branches into open and closed ones, since in our formal theory of tableau methods in [3] our aim is always to complete a branch, so a branch itself is just a technical concept. At the same time an occurrence of a t-inconsistency completes a branch. In the paper we change our point of view a bit: applying of rules is allowed as far as it is possible, ignoring any t-inconsistency — later we will come back to the idea, when explaining exactly what we mean by 'blind rules'.

If a class of models **M** and a set of tableau rules **R** fit each others then a relation $\rhd_{\mathbf{R}}$ is equal to $\models_{\mathbf{M}}$, which means that a suitable adequateness theorem holds:[5]

Theorem 3 *For all $X \subseteq For$, $A \in For$:*
$X \rhd_{\mathbf{R}} A$ iff $X \models_{\mathbf{M}} A$.

From the corollary 1 and the theorem 3 we have a conclusion:

Corollary 4 *For all $X \subseteq For$, if $\rhd_{\mathbf{R}} = \models_{\mathbf{M}}$ and X is an inconsistent set of formulas then for all $A \in For$ $X \rhd_{\mathbf{R}} A$.*

The corollary says that within an adequate tableau system when starting from an inconsistent set of premisses we can built a closed tableau for any conclusion A. So this is a syntactical, tableau counterpart of law of explosion.

As we said our aim is to define a paraconsistent tableau inference that defines some paraconsistent subrelation of classical propositional consequence relation. However, firstly we introduce some auxiliary notions.

3 Paraconistent tableaus

Let $P(\mathbb{N})$ be the powerset of the set of natural numbers. Its members we call *indexes*. The subsetes will serve as superscripts of expressions.

By $P(\mathbb{N})'$ we mean a set $\{\{x\} : \{x\} \in P(\mathbb{N}), x \neq 0\}$ — so a set of all singletons without $\{0\}$. We distinguish an index — the singleton $\{0\}$ for conclusions of tableau proofs.

Next, we define a set of expressions indexed by superscripts: $Ex' = \{A^{\Phi} : A \in Ex, \Phi \in P(\mathbb{N})\}$ — the expressions from Ex' will represent formulas from Ex in tableau proofs — and we define its subset $Ex'' = \{A^{\Phi} : A \in Ex, \Phi \in P(\mathbb{N})'\}$ — they will represent expressions in a root. Notice that any formula in Ex'' neither has a superscript $\emptyset$, nor a superscript containing $\{0\}$.

By the function $\bullet : Ex' \longrightarrow P(\mathbb{N})$, defined with the condition $\bullet(A^{\Phi}) = \Phi$, we can choose superscripts that occur in expressions from Ex'.

Let X be a subset of Ex. By $X(x)$ we mean such a non-empty subset of powerset of Ex'' that for all $Y \in X(x)$ and all $A, B \in Ex$ the conditions are fulfilled:

1. $A \in X$ iff for some $\Phi \in P(\mathbb{N})'$, $A^{\Phi} \in Y$

[5]A general mechanism for proving adequateness theorem for tableau systems of normal modal logics can be found in [4].

2. for any $\Phi, \Psi \in P(\mathbb{N})'$, if $A^{\Phi}, B^{\Psi} \in Y$ then one of the below conditions holds:

   (a) $A \neq B$ and $\Phi \cap \Psi = \emptyset$

   (b) $A = B$ and $\Phi = \Psi$.

Of course, for a set of expressions X there are usually many sets satisfying $X(x)$–conditions, so writing $Y \in X(x)$ we mean some arbitrary, but fixed set from $X(x)$ that we take into consideration.

Now we give a notion of a particular kind of t-inconsistency. We mean an inconsistency that is a result of expressions with some fixed indexes. Surely, this notion is based on a usual notion of inconsistency (defined here 2.2), so it is still about a set of expressions that is normally inconsistent, but additionally both inconsistent expressions should have indexes of some kind.

Formally, let $i, j \in \mathbb{N}$ and $X \subseteq Ex'$. X is $t^{i,j}$-*inconsistent* iff for some A, B:

1. $\{A, B\} \subseteq X$
2. $\{A, B\}$ is a t-inconsistent set of expressions
3. $i \in \bullet(A)$ and $j \in \bullet(B)$.

It is a particular kind of t-inconsistency, because it refers to some superscripts omitting t-inconsistencies with other superscripts. Hence, a set Y of expressions with superscripts can be t-inconsistent, but not $t^{i,j}$-inconsistent, for some $i, j \in \mathbb{N}$, since no pair of t-inconsistent expressions in Y contains in superscripts i, j. On the other hand, the opposite relationship holds: if a set is $t^{i,j}$-inconsistent, for some $i, j \in \mathbb{N}$, it is also just t-inconsistent. As we see the expressions collect numbers in superscripts. Those collections trace an origin of expressions. This will be more clear, when we introduce modified tableau rules.

Let **R** be a set of tableau rules for some modal logic. So now, we reformulate the tableau rules of **R**. A new set of rules **R**$'$ is defined on Ex'. For all $i, j \in \mathbb{N}$ and all $\Phi, \Psi \in P(\mathbb{N})$ the schemas of new rules are as below:

(a) tableau rules for classical Boolean connectives

$$R_{\wedge}: \frac{(A \wedge B)^{\Phi}, i}{A^{\Phi} i; B^{\Phi}, i} \qquad R_{\vee}: \frac{(A \vee B)^{\Phi}, i}{A^{\Phi}, i || B^{\Phi}, i}$$

$$R_{\rightarrow}: \frac{(A \rightarrow B)^{\Phi}, i}{\neg A^{\Phi}, i || B^{\Phi}, i} \qquad R_{\leftrightarrow}: \frac{(A \leftrightarrow B)^{\Phi}, i}{A^{\Phi}, i; B^{\Phi}, i || \neg A^{\Phi}, i; \neg B^{\Phi}, i}$$

$$R_{\neg\neg}: \frac{\neg\neg A^{\Phi}, i}{A^{\Phi}, i} \qquad R_{\neg\wedge}: \frac{\neg(A \wedge B)^{\Phi}, i}{\neg A^{\Phi}, i || \neg B^{\Phi}, i}$$

$$R_{\neg\vee}: \frac{\neg(A \vee B)^{\Phi}, i}{\neg A^{\Phi}, i; \neg B^{\Phi}, i} \qquad R_{\neg\rightarrow}: \frac{\neg(A \rightarrow B)^{\Phi}, i}{A^{\Phi}, i; \neg B^{\Phi}, i}$$

$$R_{\neg\leftrightarrow}: \frac{\neg(A \leftrightarrow B)^{\Phi}, i}{\neg A^{\Phi}, i; B^{\Phi}, i || A^{\Phi}, i; \neg B^{\Phi}, i}$$

(b) tableau rules for modal connectives

$R_{\Box}: \frac{\Box A^{\Phi}, i; irj^{\Psi}}{A^{\Phi \cup \Psi}, j}$

$R_{\Diamond}: \frac{\Diamond A^{\Phi}, i}{irk^{\Phi}; A^{\Phi}, k}$, where k is a new number on a branch

$R_{\neg\Box}: \frac{\neg\Box A^{\Phi}, i}{\Diamond\neg A^{\Phi}, i}$

$R_{\neg\Diamond}: \frac{\neg\Diamond A^{\Phi}, i}{\Box\neg A^{\Phi}, i}$

(c) tableau rules for properties of accessability relation (here we have only examples, since a reformulation of this kind of rules essentially depends on which rules are in **R**).

$R_{Ref}: \frac{}{iri^{\emptyset}}$, where i occurs on a branch

$R_{Sym}: \frac{irj^{\Phi}}{jri^{\Phi}}$

$R_{Trans}: \frac{irj^{\Phi}, jrk^{\Psi}}{irk^{\Phi \cup \Psi}}$

The tableau rules in **R**$'$ have such a property that they preserve superscripts. For example, when we decompose an expression $\neg\neg p^{\Phi}, i$ by rule for $\neg\neg$, we obtain p^{Φ}, i; if we decompose an expression $(p \rightarrow q)^{\Phi}, i$ by rule for $\rightarrow$, we obtain on the left branch $\neg p^{\Phi}, i$ and on the right branch q^{Φ}, i etc., for any subset of natural numbers Φ and any number i. The technique allows us to trace a process of decomposition of expressions and find out the origin of t-inconsistencies. But the rules need some comments.

In all given tableau rules a conclusion of a rule inherits a set of indexes that is a superscript of premises, and then we still know where it comes from. In a case we have a more-than-one-premise rule a conclusion inherits a union of all superscripts, since it comes from more than one expression (like in rule $R_{\Box}$ or R_{Trans}).

We do not write all possible rules for accessability relation, but only some examples. As we know here we construct paraconsistent tableau systems for these modal tableau systems that are adequate to some semantically determined modal logic (theorem 3), so tableau rules for a given modal logic must be separately reformulated according to described mechanism and given examples of tableau rules.

An interesting case of such a rule is R_{Ref}, The rule can introduce an expression $iri^{\emptyset}$ to a proof for any i that has already occurred on a branch. Clearly, because the expression is from “nothing” — the logic has just models with a reflexive relation of accessability — so it does not provide any track of origin — that is why we have empty set as a superscript.

As we already said we resign here from internal mechanism nested in rules that blocks applying rules to branches including t-inconsistencies (it was one of distinguishing features of our last works [3], [4]). Here, we want to develop branches as long as it is possible in order to get all t-inconsistencies that a branch can generate. Now it is clear why we call these rules 'blind' — they just do not see that a branch is closed, which normally is a sufficient fact to stop applying rules.

Moreover, we assume all definitions for tableaus for modal logics — obviously, now the notions depend on the new set of tableau rules $\mathbf{R}'$, for any normal modal logic that is defined by some set of tableau rules $\mathbf{R}$. However, we add one more definition for testing a property of paraconsistency.

3.1 Paraconsistent tableau consequence relation

Therefore we define a notion of paraconsistently closed tableau to capture some inferences we like, exclude some inferences we do not like (various forms of law of explosion), and at the same time get intuitive semantics for new paraconsistent consequence relations we study.

Definition 5 *Let $X \subseteq For \times \{i\}$ and $B, i \in For \times \{i\}$, for some $i \in \mathbb{N}$. Let $Y \in X(x)$. A tableau T with a root $Y \cup \{\neg B^{\{0\}}, i\}$ is paraconsistently closed iff:*

1. *T is complete*
2. *there exists such a subset $Z \subseteq Y$ that:*
   (a) *for any branch b in T there are such indexes $i, j \in \bigcup \bullet(Z \cup \{\neg B^{\{0\}}\})$ that $t^{i,j}$-inconsistent set of expressions belongs to b.*
   (b) *there is a branch b in T that for any pair of indexes $i, j \in \bigcup \bullet(Z)$ no $t^{i,j}$-inconsistent set of expressions belongs to b.*

Now, we explain the conditions in definition 5 one by one. Firstly, we have some set of formulas X and a formula B that is supposed to follow from X — all of them are with index i, for some $i \in \mathbb{N}$.

We do not assume that X is a finite set, since defining a suitable tableau consequence relation we will impose a constraint that there must exist a finite set as a root for some closed tableau (like in the case of normal modal tableau consequence relation 2), so below we give examples only for finite cases.

We take a set $Y \in X(x)$, so Y contains all and only expressions from X, each one with a different superscript that is a singleton $\{x\}$, for some $x \in \mathbb{N}$. Next we build a complete tableau with the root $Y \cup \{\neg B^{\{0\}}, i\}$.

There should exist a subset $Z \subseteq Y$ such that any branch in the tableau contains $t^{i,j}$-inconsistent subset of expressions for some $i, j \in \bigcup \bullet(Z \cup \{\neg B^{\{0\}}\})$, although on some branch in the tableau for no $i, j \in \bigcup \bullet(Z)$ the branch contains $t^{i,j}$-inconsistent subset of expressions. Hence some subset of premisses must be inconsistent with a negation of conclusion, but simultaneously consistent itself.

Now we present few simple examples of paraconsistently closed tableaus (according to our last definition 5) for some key cases.

Example 6 *Consider a set of premises $X = \{p \wedge \neg p\}$ and a possible conclusion q. We take a root $\{(p \wedge \neg p)^{\{1\}}, 1;\ \neg q^{\{0\}}, 1\}$ and draw a complete tableau, using only rules for Boolean connectives, exactly the rule for $\wedge$.*

$$\{(p \wedge \neg p)^{\{1\}}, 1;\ \neg q^{\{0\}}, 1\}$$
$$|$$
$$p^{\{1\}}, 1$$
$$|$$
$$\neg p^{\{1\}}, 1$$

It is a classically closed tableau, but — according to definition 5 — as we see it is not a paraconsistently closed tableau. In any branch (there is of course only one branch) there is $t^{i,i}$-inconsistency, for an index $i \in \bigcup \bullet(\{(p \wedge \neg p)^{\{1\}}, 1\})$. Admittedly, we have a t-inconsistent set $\{p^{\{1\}}, 1;\ \neg p^{\{1\}}, 1\}$ in any branch. On the other hand the branch does not contain a $t^{0,0}$-inconsistency. The example shows that a consequence relation completely determined by the notion of paraconsistently closed tableau 5 is robust to unlimited ex falso quodlibet.

Another positive point of the presented approach is that we can of course infer any tautology. Formula $\Box p \rightarrow p$ is a tautology of modal logic with reflexive relation of accessability.

Example 7 *Consider a set of premises $X = \emptyset$ and a possible conclusion $\Box p \rightarrow p$. We take a root $\{\neg(\Box p \rightarrow p)^{\{0\}}, 1\}$ and draw a complete tableau, using rules for negation of $\rightarrow$, R_{Ref} and $R_{\Box}$.*

$$\{\neg(\Box p \rightarrow p)^{\{0\}}\}, 1$$
$$|$$
$$\Box p^{\{0\}}, 1$$
$$|$$
$$\neg p^{\{0\}}, 1$$
$$|$$
$$1r1^{\emptyset}$$
$$|$$
$$p^{\{0\}}, 1$$

As we see it is a paraconsistently closed tableau by definition 5. In any branch (there is of course only one branch) there is $t^{i,0}$-inconsistency, for some index i. Admittedly, we have a t-inconsistent set $\{p^{\{0\}}, 1; \neg p^{\{0\}}, 1\}$, but for a set of premises $\emptyset$ we have a branch without any such $t^{i,j}$-inconsistency that $i, j \in \bullet(\emptyset)$, so subset $Z = \emptyset$.

Example 8 *Consider a set of premises $X = \{\Diamond(p \wedge \neg p) \vee q)\}$ and a possible conclusion q. We take a root $\{((p \wedge \neg p) \vee q))^{\{1\}}, 1; \neg q^{\{0\}}, 1\}$ and draw a complete tableau, using rules for $\vee$, $R_\Diamond$, and for $\wedge$.*

$$\{(\Diamond(p \wedge \neg p) \vee q))^{\{1\}}, 1; \neg q^{\{0\}}, 1\}$$

$$\begin{array}{cc} \Diamond(p \wedge \neg p)^{\{1\}}, 1 & q^{\{1\}}, 1 \\ | & \\ 1r2^{\{1\}} & \\ | & \\ (p \wedge \neg p)^{\{1\}}, 2 & \\ | & \\ p^{\{1\}}, 2 & \\ | & \\ \neg p^{\{1\}}, 2 & \end{array}$$

The tableau is complete, since all possible rules of decomposition were used. It is a classically closed tableau, and — according to the definition 5 — it is a paraconsistently closed tableau, since on all branches we have some t-inconsistency, and at the same time on the right branch we do not have $t^{1,1}$-inconsistency, and as a consequence the condition 2 of definition 5 is satisfied. A subset Z of definition 5 is of course equal to $\{((p \wedge \neg p) \vee q))^{\{1\}}, 1\}$.

In the end we present an example, where the mentioned liberalization of rules really works. We take a modal logic with transitive relation of accessability.

Example 9 *Consider a set of premises $X = \{\Box q, p \wedge \neg p\}$ and a possible conclusion $\Box\Box q$. We take a root $\{\Box q^{\{1\}}, 1; (p \wedge \neg p)^{\{2\}}, 1; \neg\Box\Box q^{\{0\}}, 1\}$ and draw a tableau, using the rule for $\wedge$.*

$$\begin{array}{c} \{\Box q^{\{1\}}, 1; (p \wedge \neg p)^{\{2\}}, 1; \neg\Box\Box q^{\{0\}}, 1\} \\ | \\ p^{\{2\}}, 1 \\ | \\ \neg p^{\{2\}}, 1 \end{array}$$

Classically, this is a closed and complete tableau, if we assume we cannot apply tableau rules to inconsistent sets of premisses. There is only one branch and we have $t^{2,2}$-inconsistency on it.

Moreover, it is not a paraconsistently closed tableau, since on each branch there is $t^{i,j}$-inconsistency for indexes $i, j \in \bigcup\bullet(\{\Box q^{\{1\}}, 1; (p \wedge \neg p)^{\{2\}}, 1)$. So there would seem there was no paraconsistently closed tableau for conclusion $\Box\Box q$. But it is not true, we can still make the tableau longer and obtain some interesting expressions as below. We use the rules: $R_{\neg\Box}$, $R_\Diamond$, $R_\Box$, and R_{Trans}.

$$\begin{array}{c} \{\Box q^{\{1\}}, 1; (p \wedge \neg p)^{\{2\}}, 1; \neg\Box\Box q^{\{0\}}, 1\} \\ | \\ p^{\{2\}}, 1 \\ | \\ \neg p^{\{2\}}, 1 \\ | \\ \Diamond\neg\Box q^{\{0\}}, 1 \\ | \\ 1r2^{\{0\}} \\ | \\ \neg\Box q^{\{0\}}, 2 \\ | \\ q^{\{0,1\}}, 2 \\ | \\ \Diamond\neg q^{\{0\}}, 2 \\ | \\ 2r3^{\{0\}} \\ | \\ \neg q^{\{0\}}, 3 \\ | \\ 1r3^{\{0\}} \\ | \\ q^{\{0,1\}}, 3 \end{array}$$

As we see now it is a paraconsistently closed tableau according to 5, because for a subset of premisses $\{\Box q^{\{1\}}, 1\}$ we have a branch without $t^{1,1}$-inconsistency, while for $\{\Box q^{\{1\}}, 1; \neg\Box\Box q^{\{0\}}, 1\}$ we have on all branches $t^{1,0}$-inconsistency and the conditions of the definition is satisfied.

The tableau we get, because we can apply tableau rules, even if we have some t-inconsistency. We should not worry about this, since as we have already said we shall define a paraconsistent tableau consequence relation in such a way that a formula A is a consequence of X iff for some finite subset Y of X we have a paraconsistently closed tableau. So although the example is an example of paraconsistently closed tableau, from the premisses it follows the conclusion, one can built a paraconsistently tableau with a root $\{\Box q^{\{1\}}, 1; \neg\Box\Box q^{\{0\}}, 1\}$.

Now, we have an interesting and important corollary about a relationship between classical and paraconsistent tableaus.

Corollary 10 *Let $X \subseteq For \times \{i\}$ and $B, i \in For \times \{i\}$, for some $i \in \mathbb{N}$. Let $Y \in X(x)$. A tableau T_1 with the root $Y \cup \{\neg B^{\{0\}}, i\}$ is paraconsistently closed iff for some $Z \subseteq X$:*

1. *there is a closed tableau T_3 with a root $Z \cup \{\neg B\}, i$.*
2. *there is a complete and open tableau T_2 with a root Z.*

Proof: The proof is by conditions (a) and (b) of the definition 5. □

It means that we could replace definition 5 by the statements 1 and 2 of the corollary 10 as definitional conditions. Theoretically, it would be simpler. However, practically it is difficult to choose a suitable subset of premises that generates complete and open tableau, but with a negated conclusion generates a closed tableau. In the presented approach we consider all possible decompositions, tracking superscripts and kinds of t-inconsistencies that appear, and finally we can choose a suitable and consistent set of premisses (if any exists) which in interactions with a negated conclusion generates some t-inconsistency.

Now, having a tableau consequence relation $\rhd_{\mathbf{R}}$, for some set of tableau rules **R**, we can define a paraconsistent tableau consequence relation $\rhd_{\mathbf{R}'}$.

Definition 11 *Let $X \subseteq For$ and $A \in For$. $X \rhd_{\boldsymbol{R}'} A$ iff there exist a finite subset Y of X, i and a paraconsistently closed tableau with a root $Z \cup \{\neg A^{\{0\}}, i\}$, for some $Z \in Y(y)$ and $i \in \mathbb{N}$.*

A demanded fact is that the paraconsistent, tableau consequence relation $\rhd_{\mathbf{R}'}$ is a proper subrelation of a modal tableau consequence relation $\rhd_{\mathbf{R}}$.

Corollary 12 $\rhd_{\boldsymbol{R}'} \subset \rhd_{\boldsymbol{R}}$.

Proof: Let $X \rhd_{\mathbf{R}'} A$, for some $X \subseteq For$ and $A \in For$. Then by definition 11, there exist a finite subset Y of X, i and a paraconsistently closed tableau with a root $Z \cup \{\neg A^{\{0\}}, i\}$, for some $Z \in Y(y)$ and $i \in \mathbb{N}$.

By corollary 10 there exist a finite subset U of Y and a closed tableau with a root $U \cup \{\neg A, i\}$. So, according to the definition 2, $X \rhd_{\mathbf{R}} A$, and $\rhd_{\mathbf{R}'} \subseteq \rhd_{\mathbf{R}}$.

On the other hand, we have an example of a closed tableau (example 6), that is not paraconsistently closed. Hence, by definitions 2 and 11, we get $\rhd_{\mathbf{R}} \not\subseteq \rhd_{\mathbf{R}'}$. □

Having a relation $\rhd_{\mathbf{R}'}$, we straightforwardly determine a paraconsistent tableau system $\langle For, \rhd_{\mathbf{R}'} \rangle$ of a sublogic of a suitable normal, modal logic determined by tableau rules **R**.

4 Semantics

As quick as there appears a question about semantics for $\langle For, \rhd_{\mathbf{R}'} \rangle$, we get a natural answer. A natural and commonsense approach to the problem of paraconsistency in modal language is to define a paraconsistent semantic relation of consequence by a class of models **M** as follows:

Definition 13 *For all $X \subseteq For$ and $A \in For$, $X \models'_{\boldsymbol{M}} A$ iff there is such $Y \subseteq X$ that Y is a consistent set of formulas and $Y \models_{\boldsymbol{M}} A$.*

Surely, the relation $\models'_{\mathbf{M}}$ is identical to our relation $\rhd_{\mathbf{R}'}$, when $\models_{\mathbf{M}} = \rhd_{\mathbf{R}}$, so we have a theorem.

Theorem 14 *Let **R** be a set of modal tableau rules. Let **M** be a class of modal models. If $\models_{\boldsymbol{M}} = \rhd_{\boldsymbol{R}}$, then $\models'_{\boldsymbol{M}} = \rhd_{\boldsymbol{R}'}$.*

Proof: Let **R** be a set of modal tableau rules and **M** be a class of modal models. We assume $\models_{\mathbf{M}} = \rhd_{\mathbf{R}}$ and take some $X \subseteq For$, $A \in For$.

Firstly, we assume that $X \models'_{\mathbf{M}} A$. Then, by definition 13, there exists such $Y \subseteq X$ that Y is a consistent set of formulas, $Y \models_{\mathbf{M}} A$ and Y is finite — by compactness of $\models_{\mathbf{M}}$, since normal, modal logics are compact.

By fact 3 we have $Y \rhd A$, so there is a closed tableau with a root $Y \cup \{\neg A\}$, i, for some $i \in \mathbb{N}$. But because Y is a consistent set, so there is a complete and open tableau with a root Y, i.

As a consequence, by corollary 10, a tableau with a root $Z \cup \{\neg A^{\{0\}}, i\}$, for some $Z \in U(u)$, where $U = Y, i$, is paraconsistently closed. Hence, by 11, $X \rhd_{\mathbf{R}'} A$.

Secondly, we assume that $X \rhd_{\mathbf{R}'} A$. By 11 there exist a finite subset Y of X and a paraconsistently closed tableau with a root $Z \cup \{\neg A^{\{0\}}, i\}$, for some $i \in \mathbb{N}$ and some $Z \in U(u)$, where $U = Y, i$.

By corollary 10 Y is consistent, since there is a complete and open tableau with a root Y, i, and $Y \rhd_{\mathbf{R}'} A$. Hence, $Y \rhd_{\mathbf{R}} A$, by corollary 12, and $Y \models_{\mathbf{M}} A$, by fact 3. As a consequence, since $Y \subseteq X$, Y is consistent and $Y \models_{\mathbf{M}} A$, $X \models'_{\mathbf{M}} A$. □

By theorem 4 and corollary 12 we have some final conclusion:

Corollary 15 *Let **R** be a set of modal tableau rules. Let **M** be a class of modal models. If $\models_{\boldsymbol{M}} = \rhd_{\boldsymbol{R}}$, then $\rhd_{\boldsymbol{R}'} = \rhd_{\boldsymbol{R}} \cap \{\langle X, A\rangle : \langle X, A\rangle \subseteq 2^{For} \times For, \exists_Y \subseteq X$, Y is consistent and $Y \models_{\boldsymbol{M}} A\}$.*

5 Further applications

The presented mechanism can be used to other tableau systems/logics. Through a formal theory of tableau systems [3] we should aim at a general theorem:

$$\text{if } \models = \rhd, \text{ then } \models' = \rhd'$$

where $\models$ and $\rhd$ are semantical and tableau consequence relations of a given logic, while $\models'$ and $\rhd'$ are their paraconsistent tableau counterparts.

Acknowledgements: This work was completed with the support of Polish National Center of Science NCN UMO-2012/05/E/HS1/03542.

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