Optimal time-consistent investment in the dual risk model with diffusion

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Abstract: The objective of this paper is to investigate optimal investment strategy for a pharmaceutical or petroleum company under mean-variance criterion. The surplus of the company is modeled as a dual risk model. We assume that the company can invest into a risk-free asset and n risky assets. Short-selling and borrowing money are allowed. Since this problem is time-inconsistent, we study it within a game theoretical framework. The subgame perfect Nash equilibrium strategies (namely time-consistent strategies) are derived by solving the Extended Hamilton-Jacobi-Bellman(HJB) equations in the classic diffusion dual model and in the diffusion dual approximation model, separately. Surprisingly when a coefficient parameter $\rho = 0$, optimal time-consistent investment strategies and the value functions have the same expressions in both cases. Finally, We present economics implications and provide sensitivity analysis for our results by numerical examples.

Key–Words: Optimal time-consistent investment; Extended Hamilton-Jacobi-Bellman equation; Mean-variance criterion

1 Introduction

Recently, quite a few interesting papers have been published on the dual risk model. The dual risk model can be natural for companies that have occasional profits. For companies such as pharmaceutical or petroleum companies, the jump can be interpreted as the net present value of future income from an invention or discovery. Many scholars investigated the dual risk model under different criterions. For example, Zhu and Yang [22] studied the ruin probability under the dual model, while Avanzi et al. [1] [2] considered the dividend payment problems in the dual model. Dai et al.[8] and Yao et al.[18] investigated the dual model under the objectives of maximizing the expected present value of the dividends minus capital injections, while the latter considered the transaction costs.

The objective of this paper is to investigate the optimal investment problem in the dual model under mean-variance criterion. Most of the literatures exploited an embedding technique to deal with dynamic mean-variance problem. The embedding technique was proposed by Li and Ng(2000) and Zhou and Li(2000). Since the iterated expectation property does not hold for the variance operator, the optimal strategy(precommitted strategy) derived by the embedding technique does not satisfy the Bellman Optimality Principle and is time-inconsistent. This precommitted strategy implies that the strategy computed at t will not necessarily coincide with the strategy derived at $t + \Delta t$. As a result, at $t + \Delta t$ the rational investor will implement the strategy computed at t. A rational investor will find a time-consistent strategy which ensures him to keep a consistent satisfaction. Another possibility is to take the time-inconsistency more seriously and study the problem within a game theoretical framework. The basic idea is that when we decide on a strategy at t we should explicitly take into account that at future times we will have a different objective functional, which means our preferences change in a temporally inconsistent way as time goes by, and we can thus view this problem as a non-cooperative game. We then look for a subgame perfect Nash equilibrium point for this game. The game theoretical approach to address general time inconsistency via Nash equilibrium points has a long history starting with [17] where a deterministic Ramsay problem was studied. Further work along this line in continuous and discrete time were provided in [11], [14] and [15].

Recently there has been renewed interest in these problems. In the interesting papers [9] and [10], the authors considered optimal consumption and investment under hyperbolic discounting in deterministic

*School of Management,Tianjin University and stochastic models from the above game theoretical point of view. They provided a precise definition of the game theoretical equilibrium concept in continuous time. Björk and Murgoci [3][4] studied time inconsistency (in discrete and continuous time) and the authors undertake a deep study of the problem within a Wiener driven framework. Zeng and Li [19] studied the diffusion risk model under mean-variance criterion and obtained optimal time-consistent investment and reinsurance policies. Zeng et al.[20] investigated the optimal time-consistent investment problem in the classic risk model. In the following paper, Björk et al.[5] investigated mean-variance portfolio optimization with state dependent risk aversion. Besides, there are some other investment problems under different models, see Li et al.[12], Chang and Lu [6], Chang et al.[7] and references therein.

However, all of the above references do not consider optimal investment problem in the dual risk model under mean-variance criterion. Zhang et al.[21] studied optimal investment strategy in a dual risk model under mean-variance criterion and they assumed that the financial market consisted of a risk-free asset and a risky asset which price was modeled by a diffusion process without jumps. In our paper, we assume the prices of the risky assets are described by a diffusion process with jumps. Short-selling and borrowing money are allowed. Due to lack of Bellman Optimality Principle, we exploit the game theoretical approach to deal with this problem and the time-consistent investment strategies are investigated in the classic diffusion dual model and the diffusion dual approximation model. Similarly as Björk and Murgoci [3], we give a series of elementary definitions and closed expressions for optimal time-consistent investment and the corresponding value functions are derived by solving the extended Hamilton-Jacobi-Bellman equations.

This paper is organized as follows. In Section 2, we describe the classic diffusion dual risk model and the diffusion dual approximation model by considering the investment strategy, define the equilibrium strategy and give the verification theorem for the dual risk model. In Section 3, optimal time-consistent investment strategy and the value function are derived by solving the extended HJB equation in the classic diffusion dual model. In Section 4 we give optimal time-consistent investment strategy and the value function in the diffusion dual approximation model. The final Section provides economics implications and sensitivity analysis for our results by numerical methods.

2 Problem formulation

In this section, we start with the classic diffusion dual model and the surplus process of the company is given by

$$R(t) = x + \sum_{j=1}^{N_1(t)} Z_j - ct + \sigma_0 W_0(t),\ R(0) = x, \quad (1)$$

where x is the initial capital, c is the rate of expenses, $\{N_1(t)\}_{t\geq 0}$ is a Poisson process with intensity λ_1 and Z_j is the size of the jth positive incomes or profits. The incomes are i.i.d.with the first and second moment μ_z and σ_z^2, respectively. $W_0(t)$ is a standard Brownian motion which denotes the uncertainty of profit. The expected increase of the surplus per unit time satisfies the positive loading condition: $\lambda_1\mu_z - c > 0$.

The company is allowed to invest its surplus in a financial market consisting of a risk-free asset and n risky assets. The total amount of money invested in the ith risky asset at time t is described as $l_i(t)$. Denote the price of the risk-free asset S_0 by

$$dS_0(t) = r_0(t)S_0(t)dt,\ S_0(0) = s_0, \quad (2)$$

where s_0 is the initial price of the risk-free asset, $r_0(t)$ represents the risk-free rate and it is a positive continuous bounded function. The price process $S_i(t)$ of the ith risky asset $(i = 1, 2, ..., n)$ satisfies the following stochastic differential equation

$$\begin{aligned} dS_i(t) = S_i(t)\Bigg(& r^i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t) \\ & + d\sum_{j=1}^{N_2(t)} Y_j\Bigg),\ S_i(0) = s_i, \end{aligned} \quad (3)$$

where $r^i(t)>r_0(t)$ is the appreciation rate, $r^i(t)$ and $\sigma_{ij}(t)$ are positive continuous bounded functions. $W(t) = (W_1(t), W_2(t), ..., W_d(t))^T$ is a d-dimensional standard Brownian motion. Y_j is the jth jump amplitude of the risky asset price and $Y_j, j = 1, 2, ...$ are i.i.d. random variables with the finite first-order moment μ_y and second-order moment σ_y^2. Furthermore, assume $r^i(t) + \lambda_2\mu_y>r_0(t)$ which means it is better to invest money into risky asset in the long term and $W_0(t)$, $W(t)$, $\sum_{j=1}^{N_1(t)} Z_j$ and $\sum_{j=1}^{N_2(t)} Y_j$ are independent and $P(Y_i \geq -1) = 1$ which can make the risky asset's price remain positive. Here the superscript "T" denotes the transpose of a matrix or a vector and $d \geq n$. Let $X^l(t)$ denote the resulting surplus process after incorporating strategy l into (1).

The dynamics of $X^l(t)$ can be preserved as follows

$$\begin{aligned} dX^l(t) &= \big(r_0(t)X^l(t) + r^T(t)l(t) - c\big)\, dt \\ &+\sigma_0 dW_0(t) + l^T(t)\sigma(t)dW(t) \\ &+d \sum_{j=1}^{N_1(t)} Z_j + l^T(t)e\, d \sum_{j=1}^{N_2(t)} Y_j, \end{aligned} \tag{4}$$

where $r(t) = (r^1(t) - r_0(t), r^2(t) - r_0(t),...,r^n(t) - r_0(t))^T$, $\sigma(t) = (\sigma_{ij})_{n\times d}$ and $e = (1, 1, ..., 1)^T$. Furthermore, denote $\Sigma(t) = \sigma(t)\sigma^T(t)$ and let I represents the identity matrix. We also assume that $\Sigma(t) + \lambda_2\sigma_y^2 I$ is reversible for all $t \in [0, T]$.

Similarly as Schmidli [16], the diffusion approximation of (1) can be described as

$$\begin{aligned} dR(t) &= (\lambda_1\mu_z - c)dt + \sqrt{\lambda_1\sigma_z^2}dW^*(t) \\ &+\sigma_0 dW_0(t), \end{aligned} \tag{5}$$

where $W^*(t)$ is a standard Brownian motion. Assume that $W_0(t)$, $W^*(t)$, $W(t)$ and $\sum_{j=1}^{N_2(t)} Y_j$ are independent except that $W^*(t)$ is correlated with $W_0(t)$ with correlation coefficient ρ, i.e.,

$$E(W_0(t)W^*(t)) = \rho t. \tag{6}$$

When the company invest its money into the financial market which is described above, the resulting surplus process $Y^l(t)$ can be given by

$$\begin{aligned} dY^l(t) &= \big(r_0(t)Y^l(t) + r^T(t)l(t) + \lambda_1\mu_z \\ &-c\big)\, dt + \sigma_0 dW_0(t) + l^T(t)\sigma(t)dW(t) \\ &+\sqrt{\lambda_1\sigma_z^2}dW^*(t) + l^T(t)e\, d\textstyle\sum_{j=1}^{N_2(t)} Y_j, \end{aligned} \tag{7}$$

Denote $C^{1,2}(Q) = \big\{\phi(t, x)|\phi(t, .)$ is once continuously differentiable on [0, T], and $\phi(., x)$ is twice continuously differentiable on $\mathbb{R}\big\}$ where $Q := [0, T] \times \mathbb{R}$, then for $\phi(t, x) \in C^{1,2}(Q)$, the infinitesimal operator of the surplus process $X^l(t)$ is

$$\begin{aligned} \mathcal{A}^l\phi(t, x) &= \phi_t(t, x) + \phi_x(t, x)\,[r_0(t)x - c \\ &+r^T(t)l(t)] + \tfrac{1}{2}\phi_{xx}(t, x)\left(\sigma_0^2 + l^T(t)\Sigma(t)l(t)\right) \\ &+\lambda_1\,[E\phi(t, x + Z) - \phi(t, x)] \\ &+\lambda_2\,[E\phi(t, x + \textstyle\sum_{i=1}^n l_i(t)Y) - \phi(t, x)], \end{aligned} \tag{8}$$

and the infinitesimal operator of the surplus process $Y^l(t)$ is

$$\begin{aligned} \mathcal{B}^l\phi(t, x) &= \phi_t(t, x) + \phi_x(t, x)\,[r_0(t)x + \lambda_1\mu_z \\ &-c + r^T(t)l(t)] + \tfrac{1}{2}\phi_{xx}(t, x)\big(\sigma_0^2 + \rho\sigma_o\sqrt{\lambda_1\sigma_z^2} \\ &+l^T(t)\Sigma(t)l(t) + \lambda_1\sigma_z^2\big) \\ &+\lambda_2\,[E\phi(t, x + \textstyle\sum_{i=1}^n l_i(t)Y) - \phi(t, x)]\,. \end{aligned} \tag{9}$$

Next, we give a series of definition on the classic diffusion dual process $X^l(t)$.

Definition 1. *A strategy* $l = \{l(t) = (l_1(t), l_2(t), ..., l_n(t))\}_{t\geq 0}$ *is said to be admissible if*
(1) $l(t)$ *is a* $\mathcal{F}_t$*-adapted process;*
(2) l *satisfies the integrability condition:*
$E\int_0^t l^T(t)\Sigma(t)l(t)ds < \infty$ *almost surely, for all* $t \geq 0$*;*
(3) SDE(4) has a unique solution corresponding to l*.*

Denote the set of all the admissible strategies by U. We mainly consider mean-variance criterion and the objective of the company is find the maximization of the following function

$$J(0, x; l) = E_{0,x}\left[X^l(T)\right] - \frac{\gamma}{2}Var_{0,x}[X^l(T)], \tag{10}$$

where γ is a pre-specified risk aversion coefficient, $E_{0,x}[.] = E[.|X^l(0) = x]$ and $Var_{0,x}[.] = Var[.|X^l(0) = x]$. This problem is a static mean-variance problem. Due to this criterion lacking the iterated-expectation property, it leads to a time-inconsistent problem. It means that the Bellman Optimality Principle does not hold. We formulate an investment problem in a game theoretical framework.

Furthermore, we take this problem as a non-cooperate game, with one player for each time t, where player t can be regarded as the future incarnation at time t. For any fixed (t, x), the objective is to find

$$\begin{aligned} \sup_{l\in U} J(t, x; l) = \sup\nolimits_{l\in U} \big\{E_{t,x}\left[X^l(T)\right] - \\ \tfrac{\gamma}{2}Var_{t,x}\left[X^l(T)\right]\big\}\,. \end{aligned} \tag{11}$$

This problem can be viewed as a dynamic mean-variance problem.

Similarly as Bjӧrk and Murgoci [3], we provide the definition of the equilibrium strategy and the verification theorem for problem(11).

Definition 2. (**Equilibrium Strategy**) *For any fixed chosen initial state* $(t, x) \in Q$*, Consider an admissible strategy* $l^*(t, x)$*. Choose two fixed real number* $\tilde{l} > 0$ *and* $\varepsilon > 0$ *and define the following strategy:*

$$l^\varepsilon(s, x) = \begin{cases} \tilde{l},\ for\ (s, x) \in [t, t + \varepsilon] \times \mathbb{R} \\ l^*(s, x),\ for(s, x) \in [t + \varepsilon,\ T] \times \mathbb{R}. \end{cases}$$

If for all $\tilde{l} \in \mathbb{R}^+$ *and* $(t,\ x) \in Q$*, we have*

$$\liminf_{\epsilon\to 0} \frac{J(t, x, l^*) - J(t, x, l^\varepsilon)}{\varepsilon} \geq 0, \tag{12}$$

then $l^*(t, x)$ *is called an equilibrium strategy, and the corresponding equilibrium value function is defined by*

$$\begin{aligned} V(t, x) = J(t, x, l^*) = E_{t,x}\left[X^{l^*}(T)\right] \\ -\tfrac{\gamma}{2}Var_{t,x}\left[X^{l^*}(T)\right]. \end{aligned} \tag{13}$$

By definition 2, we know that the equilibrium strategy is time-consistent. So the equilibrium strategy l^* is called optimal time-consistent strategy for problem (11).

To solve problem (11), we use stochastic analysis techniques described in [3] or [19] to derive the extended Hamilton-Jacobi-Bellman(HJB) equation and the verification theorem.

Theorem 3. ***(Verification Theorem)*** *If there exist two real functions $W(t,x)$, $h(t,x) \in C^{1,2}(Q)$, which satisfy the following extended HJB equation*

$$\sup_{l\in U}\left\{\mathcal{A}^l W(t,x) - \mathcal{A}^l\left(\frac{\gamma}{2}h^2(t,x)\right)\right. \left. + \gamma h(t,x)\mathcal{A}^l h(t,x)\right\} = 0, \tag{14}$$

$$W(T,x) = x, \tag{15}$$

$$\mathcal{A}^{l^*}h(t,x) = 0, \tag{16}$$

$$h(T,x) = x, \tag{17}$$

where

$$\begin{aligned} l^* = \arg\sup\left\{\mathcal{A}^l W(t,x) - \mathcal{A}^l\left(\tfrac{\gamma}{2}h^2(t,x)\right)\right. \\ \left. +\gamma h(t,x)\mathcal{A}^l h(t,x)\right\}. \end{aligned} \tag{18}$$

Then $V(t,x) = W(t,x)$, $E_{t,x}(X^{l^*}(T)) = h(t,x)$ and l^* is optimal time-consistent strategy.

Theorem 3 can be proved by the same procedure stated in [3] or [19] ,while the only difference in the proof is that diffusion process and jump diffusion process have the different infinitesimal generator.

Remark 4. *A series of definitions on $Y^l(t)$ can be omitted, for we can do it in the same way as in classic diffusion dual process X_t.*

In the following two sections, optimal time-consistent strategies and the corresponding value functions can be explicitly derived in the classic diffusion dual model and the diffusion dual approximation model, respectively.

3 Optimal time-consistent strategy and its equilibrium value function in the classic dual model with diffusion

This section studies optimal time-consistent investment strategy and the optimal equilibrium value function in the classic dual model with diffusion. Next, we will construct the solution to problem (11). Assume that there exist two real functions $W(t,x)$ and $h(t,x)$ satisfying the conditions stated in Theorem 3. By virtue of the infinitesimal operator (8), (14) can be rewritten as

$$\begin{aligned} &\sup_{l\in U}\left\{W_t(t,x) + W_x(t,x)\left(r_0(t)x - c\right.\right.\\ &\left.+r^T(t)l(t)\right) + \tfrac{1}{2}(W_{xx}(t,x) - \gamma h_x^2(t,x))\\ &\times\left(\sigma_0^2 + l^T(t)\Sigma(t)l(t)\right) + \lambda_1 E\left[W(t,x+Z)\right.\\ &\left.-\tfrac{\gamma}{2}h(t,x+Z)\times(h(t,x+Z) - 2h(t,x))\right]\\ &+\lambda_2 E\left[W(t,x+l(t)Y) - \tfrac{\gamma}{2}h(t,x+Y\textstyle\sum_{i=1}^n\right.\\ &\left. l_i(t))\times(h(t,x+\textstyle\sum_{i=1}^n l_i(t)Y) - 2h(t,x))\right]\\ &\left.-(\lambda_1+\lambda_2)\left[W(t,x) + \tfrac{\gamma}{2}g^2(t,x)\right]\right\} = 0 \end{aligned} \tag{19}$$

(16) becomes

$$\begin{aligned} &h_t(t,x) + h_x(t,x)\left(r_0(t)x - c + r^T(t)l^*(t)\right)\\ &+\tfrac{1}{2}h_{xx}(t,x)\times\left(\sigma_0^2 + l^T(t)\Sigma(t)l^*(t)\right)\\ &+\lambda_1 E\left[h(t,x+Z) - h(t,x)\right]\\ &+\lambda_2 E\left[h(t,x+\textstyle\sum_{i=1}^n l_i^*(t)Y) - h(t,x)\right] = 0 \end{aligned} \tag{20}$$

where l^* is determined below. Since the linear structure of (19) and (20), and the boundary conditions of $W(t,x)$ and $h(t,x)$ given by (15) and (17) are linear in x, it is natural to conjecture that

$$\begin{aligned} W(t,x) &= M(t)x + N(t),\\ M(T) &= 1, N(T) = 0,\\ h(t,x) &= m(t)x + n(t),\\ m(T) &= 1, n(T) = 0. \end{aligned} \tag{21}$$

The corresponding partial derivatives are

$$\begin{aligned} W_t(t,x) = \dot{M}(t)x + \dot{N}(t),\ W_x(t,x) = M(t),\\ h_t(t,x) = \dot{m}(t)x + \dot{n}(t),\ h_x(t,x) = m(t),\\ W_{xx}(t,x) = 0,\ h_{xx}(t,x) = 0. \end{aligned} \tag{22}$$

Inserting (21)-(22) into (19), it yields

$$\begin{aligned} &\sup_{l\in U}\left\{\dot{M}(t)x + \dot{N}(t) + M\left(r_0(t)x - c\right.\right.\\ &\left.+r^T(t)l(t) + \lambda_1\mu_z + \lambda_2\mu_y l^T(t)e\right)\\ &-\tfrac{\gamma}{2}m^2(t)\left[\left(\sigma_0^2 + \lambda_1\sigma_z^2\right) + l^T(t)\Sigma(t)l(t)\right.\\ &\left.\left.+\lambda_2\sigma_y^2 l^T(t)l(t)\right]\right\} = 0. \end{aligned} \tag{23}$$

Next, we construct a function

$$\begin{aligned} L(l) = &\dot{M}(t)x + \dot{N}(t) + M\left(r_0(t)x - c\right.\\ &+r^T(t)l(t) + \lambda_1\mu_z + \lambda_2\mu_y l^T(t)e\big)\\ &-\tfrac{\gamma}{2}m^2(t)\left[\left(\sigma_0^2+\lambda_1\sigma_z^2\right) + l^T(t)\Sigma(t)l(t)\right.\\ &\qquad\left.+\lambda_2\sigma_y^2 l^T(t)l(t)\right]. \end{aligned} \tag{24}$$

Differentiating $L(l)$ with respect to l and setting the derivative to zero, we get

$$\begin{aligned} &M(t)\left(r(t)+\lambda_2\mu_y\, e\right) - \gamma m^2(t)\left(\Sigma(t)\right.\\ &\qquad\left.+\lambda_2\sigma_y^2 I\right) l(t) = 0. \end{aligned} \tag{25}$$

It follows from (25) that

$$l^*(t) = \frac{\left(\Sigma(t)+\lambda_2\sigma_y^2 I\right)^{-1}\left(r(t)+\lambda_2\mu_y\, e\right)}{\gamma\left(M(t)\right)^{-1}m^2(t)}. \tag{26}$$

Inserting (26) into (19) and (20), we have

$$\begin{aligned} &\left(\dot{M}(t)+r_0(t)M(t)\right)x + \dot{N}(t) - cM(t)\\ &+\lambda_1\mu_z M(t) - \tfrac{\gamma}{2}m^2(t)(\sigma_0^2+\lambda_1\sigma_z^2)\\ &+\tfrac{M^2(t)\xi(t)}{2\gamma m^2(t)} = 0,\\ &\left(\dot{m}(t)+r_0(t)m(t)\right)x + \dot{n}(t) - cm(t)\\ &+\lambda_1\mu_z m(t) + \tfrac{M(t)\xi(t)}{\gamma m(t)} = 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \xi(t) = &\left(r(t)+\lambda_2\mu_y e\right)^T\left(\Sigma(t)+\lambda_2\sigma_y^2 I\right)^{-1}\\ &\qquad\left(r(t)+\lambda_2\mu_y e\right). \end{aligned} \tag{28}$$

To ensure the above equations hold, we require

$$\begin{aligned} &\dot{M}(t)+r_0(t)M(t)=0,\ M(T)=1,\\ &\dot{N}(t)+\left(\lambda_1\mu_z-c\right)M - \tfrac{\gamma}{2}m^2(t)(\sigma_0^2+\lambda_1\sigma_z^2)\\ &+\tfrac{M^2(t)\xi(t)}{2\gamma m^2(t)} = 0, N(T)=0,\\ &\dot{m}(t)+r_0(t)m(t)=0,\ m(T)=1,\\ &\dot{n}(t)+\left(\lambda_1\mu_z-c\right)m(t)+\tfrac{M(t)\xi(t)}{\gamma m(t)} = 0,\ n(T)=0. \end{aligned}$$

Solving the system of ordinary equations, we have

$$\begin{aligned} &M(t) = e^{\int_t^T r_0(s)ds},\\ &N(t) = \left(\lambda_1\mu_z-c\right)\textstyle\int_t^T e^{\int_s^T r_0(u)du}ds\\ &-\tfrac{\gamma}{2}(\sigma_0^2+\lambda_1\sigma_z^2)\textstyle\int_t^T e^{2\int_s^T r_0(u)du}ds\\ &+\tfrac{1}{2\gamma}\textstyle\int_t^T \xi(s)ds,\\ &m(t) = e^{\int_t^T r_0(s)ds},\\ &n(t) = \left(\lambda_1\mu_z-c\right)\textstyle\int_t^T e^{\int_s^T r_0(u)du}ds\\ &+\tfrac{1}{\gamma}\textstyle\int_t^T \xi(s)ds. \end{aligned} \tag{29}$$

Substituting (29) into (26), we have

$$l^*(t) = \frac{\left(\Sigma(t)+\lambda_2\sigma_y^2 I\right)^{-1}\left(r(t)+\lambda_2\mu_y\, e\right)}{\gamma e^{\int_t^T r_0(s)ds}}. \tag{30}$$

According to the argument listed above, we can derive the explicit expressions for $W(t,x)$ and $h(t,x)$ and the results can be summarized as the following theorem.

Theorem 5. *In the classic dual model, optimal time-consistent strategy l^* is given by (30), and the equilibrium value function is given by*

$$\begin{aligned} &V(t,x) = e^{\int_t^T r_0(s)ds}x + (\lambda_1\mu_z - c)\\ &\times\textstyle\int_t^T e^{\int_s^T r_0(u)du}ds - \tfrac{\gamma}{2}(\sigma_0^2+\lambda_1\sigma_z^2)\\ &\times\textstyle\int_t^T e^{2\int_s^T r_0(u)du}ds + \tfrac{1}{2\gamma}\int_t^T \xi(s)ds \end{aligned} \tag{31}$$

and

$$\begin{aligned} &E_{t,x}(X^{l^*}(T)) = e^{\int_t^T r_0(s)ds}x + (\lambda_1\mu_z - c)\\ &\qquad\times\textstyle\int_t^T e^{\int_s^T r_0(u)du}ds + \tfrac{1}{\gamma}\int_t^T \xi(s)ds. \end{aligned} \tag{32}$$

By Theorem 5 and the definition of the corresponding value function given by (13), we have

$$\begin{aligned} &Var_{t,x}(X^{l^*}(T)) = \tfrac{2}{\gamma}\left[h(t,x)-V(t,x)\right]\\ &= (\sigma_0^2+\lambda_1\sigma_z^2)\textstyle\int_t^T e^{2\int_s^T r_0(u)du}ds\\ &+\tfrac{1}{\gamma^2}\textstyle\int_t^T \xi(s)ds. \end{aligned} \tag{33}$$

Remark 6. *It is easy to see that the optimal strategy does not depend on the wealth process $X^l(t)$ and the parameters of the surplus process have no impact on the optimal strategy; The risk aversion coefficient and the coefficients of financial market decide the optimal strategy together.*

Remark 7. *The efficient frontier for problem (11) at initial state (t,x) can be derived from (32) and (33).*

$$\begin{aligned} &E_{t,x}(X^{l^*}(T)) = e^{\int_t^T r_0(s)ds}x + (\lambda_1\mu_z - c)\\ &\times\textstyle\int_t^T e^{\int_s^T r_0(u)du}ds + \Big\{\left[Var_{t,x}(X^{l^*}(T))\right.\\ &\left.-(\sigma_0^2+\lambda_1\sigma_z^2)\times\textstyle\int_t^T e^{2\int_s^T r_0(u)du}ds\right]\\ &\textstyle\int_t^T \xi(s)ds\Big\}^{\frac{1}{2}}. \end{aligned} \tag{34}$$

This efficient frontier is not a straight line but a hyperbola in the mean-standard deviation plane.

4 Optimal time-consistent strategy and its equilibrium value function in the dual diffusion approximation model

In this section, optimal time-consistent investment strategy and the optimal equilibrium value function can be derived in the dual diffusion approximation model by the same way as that in Section 3. Denote the set of all admissible strategies for $Y(t)$ by U^*.

Next, we will construct the solution to this problem. Assume that there exist two real functions $Q(t,x)$ and $g(t,x)$ satisfy the Extended HJB equation which have the same expression as in Theorem 3 except the different infinitesimal operator.

$$\begin{aligned}&\sup_{l\in U^*}\Big\{Q_t(t,x)+Q_x(t,x)\,(r_0(t)x+\lambda_1\mu_z\\&-c+r^T(t)l(t)\big)+\tfrac{1}{2}(Q_{xx}(t,x)-\gamma g_x^2(t,x))\times\\&\Big(\sigma_0^2+\rho\sigma_o\sqrt{\lambda_1\sigma_z^2}+\lambda_1\sigma_z^2+l^T(t)\Sigma(t)l(t)\Big)\\&+\lambda_2 E\big[Q(t,x+l(t)Y)-\tfrac{\gamma}{2}g(t,x+\textstyle\sum_{i=1}^n Y\\&l_i(t))\times\big(g(t,x+\textstyle\sum_{i=1}^n l_i(t)Y)-2g(t,x)\big)\big]\\&-\lambda_2\big[Q(t,x)+\tfrac{\gamma}{2}g^2(t,x)\big]\Big\}=0,\end{aligned}\tag{35}$$

$$Q(T,x)=x. \tag{36}$$

$$\begin{aligned}&g_t(t,x)+g_x(t,x)\,(r_0(t)x+\lambda_1\mu_z\\&-c+r^T(t)l(t)\big)+\tfrac{1}{2}g_{xx}(t,x)\times\\&\Big(\sigma_0^2+\rho\sigma_o\sqrt{\lambda_1\sigma_z^2}+\lambda_1\sigma_z^2+l^T(t)\Sigma(t)l(t)\Big)\\&+\lambda_2 E\left[g(t,x+\textstyle\sum_{i=1}^n l_i(t)Y)-g(t,x)\right]=0,\end{aligned}\tag{37}$$

$$g(T,x)=x. \tag{38}$$

where l is determined by the incoming equation (42). Since (35)-(38) are linear in x, we can conjecture that $Q(t,x)$ and $g(t,x)$ have the following structures

$$\begin{aligned}&Q(t,x)=D(t)x+F(t),\\&D(T)=1,\ \ F(T)=0,\\&g(t,x)=d(t)x+f(t),\\&d(T)=1,\ \ f(T)=0.\end{aligned}\tag{39}$$

By a simple calculation, the corresponding partial derivatives are given by

$$\begin{aligned}&Q_t(t,x)=\dot{D}(t)x+\dot{F}(t),Q_x(t,x)=D(t),\\&g_t(t,x)=\dot{d}(t)x+\dot{f}(t),g_x(t,x)=d(t),\\&Q_{xx}(t,x)=0,g_{xx}(t,x)=0.\end{aligned}\tag{40}$$

By inserting(39)-(40) into (35), (35) can be reduced to the following equation

$$\begin{aligned}&\sup_{l\in U^*}\Big\{\dot{D}(t)x+\dot{F}(t)+D\,(r_0(t)x-c\\&+r^T(t)l(t)+\lambda_1\mu_z+\lambda_2\mu_y l^T(t)e)\\&-\tfrac{\gamma}{2}d^2(t)\Big[\Big(\sigma_0^2+\rho\sigma_o\sqrt{\lambda_1\sigma_z^2}+\lambda_1\sigma_z^2\Big)\\&+l^T(t)\Sigma(t)l(t)+\lambda_2\sigma_y^2 l^T(t)l(t)\Big]\Big\}=0.\end{aligned}\tag{41}$$

Differentiating the function in the left bracket of (41) with respect to l and setting the derivative to zero, we get

$$l(t)=\frac{D(t)\left(\Sigma(t)+\lambda_2\sigma_y^2 I\right)^{-1}(r(t)+\lambda_2\mu_y\, e)}{\gamma d^2(t)}. \tag{42}$$

Inserting (42) into (35) and (37), we have

$$\begin{aligned}&\big(\dot{D}(t)+r_0(t)D(t)\big)x+\dot{F}(t)\\&-cD(t)+\lambda_1\mu_z D(t)+\tfrac{D^2(t)\xi(t)}{2\gamma d^2(t)}\\&-\tfrac{\gamma}{2}d^2(t)(\sigma_0^2+\rho\sigma_o\sqrt{\lambda_1\sigma_z^2}+\lambda_1\sigma_z^2)=0,\end{aligned}\tag{43}$$

$$\begin{aligned}&\Big(\dot{d}(t)+r_0(t)d(t)\Big)\,x+\dot{f}(t)-cd(t)\\&+\lambda_1\mu_z d(t)+\tfrac{D(t)\xi(t)}{\gamma d(t)}=0,\end{aligned}\tag{44}$$

where

$$\begin{aligned}\xi(t)=&\,(r(t)+\lambda_2\mu_y e)^T\left(\Sigma(t)+\lambda_2\sigma_y^2 I\right)^{-1}\\&\times(r(t)+\lambda_2\mu_y e)\,.\end{aligned}\tag{45}$$

In order to ensure the above equations hold, we require the functions $D(t),\ F(t),\ d(t),\ f(t)$ satisfy the following equations

$$\begin{aligned}&\dot{D}(t)+r_0(t)D(t)=0,\ \ D(T)=1,\\&\dot{F}(t)+(\lambda_1\mu_z-c)\,D+\tfrac{D^2(t)\xi(t)}{2\gamma d^2(t)}-\tfrac{\gamma}{2}d^2(t)\\&\times(\sigma_0^2+\rho\sigma_o\sqrt{\lambda_1\sigma_z^2}+\lambda_1\sigma_z^2)=0,F(T)=0,\\&\dot{d}(t)+r_0(t)d(t)=0,\ d(T)=1,\\&\dot{f}(t)+(\lambda_1\mu_z-c)\,f(t)+\tfrac{D(t)\xi(t)}{\gamma d(t)}=0,\\&f(T)=0.\end{aligned}\tag{46}$$

Solving the system of equations, we have

$$\begin{aligned}
&D(t) = e^{\int_t^T r_0(s)ds},\\
&F(t) = (\lambda_1\mu_z - c)\int_t^T e^{\int_s^T r_0(u)du}ds\\
&-\tfrac{\gamma}{2}(\sigma_0^2 + \rho\sigma_o\sqrt{\lambda_1\sigma_z^2} + \lambda_1\sigma_z^2)\\
&\times \int_t^T e^{2\int_s^T r_0(u)du}ds + \tfrac{1}{2\gamma}\int_t^T \xi(s)ds,\\
&d(t) = e^{\int_t^T r_0(s)ds},\\
&f(t) = (\lambda_1\mu_z - c)\int_t^T e^{\int_s^T r_0(u)du}ds\\
&+\tfrac{1}{\gamma}\int_t^T \xi(s)ds.
\end{aligned} \tag{47}$$

Substituting (47) into (42), we have

$$l(t) = \frac{\left(\Sigma(t) + \lambda_2\sigma_y^2 I\right)^{-1}(r(t) + \lambda_2\mu_y\, e)}{\gamma e^{\int_t^T r_0(s)ds}}. \tag{48}$$

Summarizing the results discussed above, the explicit expressions for $Q(t,x)$ and $g(t,x)$ can be given by the following theorem.

Theorem 8. *For the diffusion dual approximation model, optimal time-consistent strategy l is given by*

$$l(t) = \frac{\left(\Sigma(t) + \lambda_2\sigma_y^2 I\right)^{-1}(r(t) + \lambda_2\mu_y\, e)}{\gamma e^{\int_t^T r_0(s)ds}}. \tag{49}$$

and the equilibrium value function is given by

$$\begin{aligned}
&V(t,x) = Q(t,x) = e^{\int_t^T r_0(s)ds}x\\
&+(\lambda_1\mu_z - c)\int_t^T e^{\int_s^T r_0(u)du}ds\\
&-\tfrac{\gamma}{2}(\sigma_0^2 + \rho\sigma_o\sqrt{\lambda_1\sigma_z^2} + \lambda_1\sigma_z^2)\\
&\times\int_t^T e^{2\int_s^T r_0(u)du}ds + \tfrac{1}{2\gamma}\int_t^T \xi(s)ds
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
&E_{t,x}(X^{l^*}(T)) = g(t,x) = e^{\int_t^T r_0(s)ds}x\\
&+(\lambda_1\mu_z - c)\int_t^T e^{\int_s^T r_0(u)du}ds + \tfrac{1}{\gamma}\int_t^T \xi(s)ds.
\end{aligned} \tag{51}$$

Remark 9. *From (50) and (51), the relationship between the expectation and the variance of the terminal wealth can be obtained as below:*

$$\begin{aligned}
&E_{t,x}(X^{l^*}(T)) = e^{\int_t^T r_0(s)ds}x + (\lambda_1\mu_z - c)\\
&\times\int_t^T e^{\int_s^T r_0(u)du}ds + \Big\{\Big[Var_{t,x}(X^{l^*}(T)) -\\
&(\sigma_0^2 + \rho\sigma_o\sqrt{\lambda_1\sigma_z^2} + \lambda_1\sigma_z^2)\int_t^T e^{2\int_s^T r_0(u)du}ds\Big]\\
&\int_t^T \xi(s)ds\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{52}$$

(52) shows that the efficient frontier is also a hyperbola in the mean-standard deviation plane.

Remark 10. *It follows from (30) and (49) that optimal time-consistent investment strategies have the same expressions in the classic diffusion dual model and the diffusion dual approximation model. When $W^*(t)$ is independent with $W_0(t)$, in other words $\rho = 0$, the value functions and the same efficient frontiers are the same in both cases. Thus, we can only present economics implications and provide sensitivity analysis for our results in the dual diffusion approximation model.*

Remark 11. *A special case is considered in order to analyze the effect of the parameters on the time-consistent strategy and the equilibrium value function. Assume that the wealth can only invest into a risk-free asset and a risky asset where the dimension d of $W(t)$ equals to 1 and all the other parameters are all constants. The optimal investment strategy, the corresponding value function and the efficient frontier are given by the following equations*

$$l(t) = \frac{r + \lambda_2\mu_y}{\gamma e^{r_0(T-t)}\left(\sigma^2 + \lambda_2\sigma_y^2\right)}. \tag{53}$$

$$\begin{aligned}
&V(t,x) = e^{r_0(T-t)}x + \tfrac{(\lambda_1\mu_z - c)}{r_0}\left(e^{r_0(T-t)} - 1\right)\\
&-\tfrac{\gamma(\sigma_0^2 + \rho\sigma_o\sqrt{\lambda_1\sigma_z^2} + \lambda_1\sigma_z^2)}{4r_0}\left(e^{2r_0(T-t)} - 1\right)\\
&+\tfrac{\xi}{2\gamma}(T-t)
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
&E_{t,x}(X^{l^*}(T)) = e^{r_0(T-t)}x + \tfrac{(\lambda_1\mu_z - c)}{r_0}\\
&\times\left(e^{r_0(T-t)} - 1\right) + \Big\{\Big[Var_{t,x}(X^{l^*}(T))\\
&-\tfrac{(\sigma_0^2 + \rho\sigma_o\sqrt{\lambda_1\sigma_z^2} + \lambda_1\sigma_z^2)}{2r_0}\left(e^{2r_0(T-t)} - 1\right)\Big]\\
&\times\xi(T-t)\Big\}^{\frac{1}{2}},
\end{aligned} \tag{55}$$

where

$$r = r_1 - r_0,\ \xi = \frac{(r + \lambda_2\mu_y)^2}{\sigma^2 + \lambda_2\sigma_y^2}. \tag{56}$$

5 Numerical analysis

In the next subsections, we study the effect of parameters on optimal time-consistent strategy and the corresponding value functions in the dual diffusion approximation model and provide some numerical examples to illustrate the effects. For convenience but without loss of generality, assume that $d = 1$,

$n = 1$ and all the other parameters involved are constants. For the following numerical illustrates, unless otherwise stated, the basic parameters are given by $r_0 = 0.06, r_1 = 0.15, c = 0.6, x = 10, \gamma = 0.6, \rho = 0.3, \lambda_1 = 1, \mu_z = 1, \sigma_z = 1.1, \lambda_2 = 0.6, \mu_y = 0.5, \sigma_y^2 = 1, \sigma = 0.3, \sigma_0 = 0.2, T = 10$.

Due to all the parameters are all constants, optimal investment strategy, the corresponding value function and the efficient frontier are given by (53)-(55).

5.1 Analysis of the time-consistent strategy

In this subsection, we will work on numerical analysis of time-consistent strategy in the dual diffusion approximation model. From (53), we can conclude that

(1) $\frac{\partial l}{\partial \gamma} = -\frac{l}{\gamma} < 0$ which illustrates that optimal time-consistent investment strategy is decreasing with respect to γ, namely, the more the company dislikes risk, the less amount the company invests in the risky asset, see Figure 1(a)

(2) $\frac{\partial l}{\partial r_0} = -\frac{1+(r+\lambda_2\mu_y)(T-t)}{\gamma e^{r_0(T-t)}\left(\sigma^2+\lambda_2\sigma_y^2\right)} < 0$ which reveals that optimal time-consistent investment strategy is decreasing with respect to r_0, namely, the smaller the risk-free rate is, the more amount the company invests in the risky asset, see Figure 1(b).

(3) $\frac{\partial l}{\partial r_1} = \frac{1}{\gamma e^{r_0(T-t)}\left(\sigma^2+\lambda_2\sigma_y^2\right)} > 0$ which reveals optimal time-consistent investment strategy is increasing with respect to r_1, namely, when the appreciation rate r_1 increases, the company should invests more money in the risky asset, see Figure 1(c).

(4) $\frac{\partial l}{\partial \sigma^2} = -\frac{l}{\sigma^2+\lambda_2\sigma_y^2} < 0$ which tells that optimal time-consistent investment strategy is decreasing with respect to σ^2, namely, when the volatility of the risky asset increases, the company should invests more money in the risk-free asset, see Figure 1(d).

(5) $\frac{\partial l}{\partial \mu_y} = \frac{\lambda_2}{\gamma e^{r_0(T-t)}\left(\sigma^2+\lambda_2\sigma_y^2\right)} > 0$ which illustrates that optimal time-consistent investment strategy is increasing with respect to μ_y, namely, the bigger the expectation of each jump amplitude of the risky's price, the more amount the company invests in the risky asset, see Figure 1(e).

(6) $\frac{\partial l}{\partial \sigma_y^2} = -\frac{\lambda_2 l}{\left(\sigma^2+\lambda_2\sigma_y^2\right)} < 0$ which illustrates that optimal time-consistent investment strategy is decreasing with respect to σ_y^2, namely, the bigger the second-order moment of each jump amplitude of the risky's price, the less amount the company invests in the risky asset, see Figure 1(f).

(7) $\frac{\partial l}{\partial \lambda_2} = \frac{\mu_y\sigma^2-r\sigma_y^2}{\gamma e^{r_0(T-t)}\left(\sigma^2+\lambda_2\sigma_y^2\right)^2}$ which illustrates that optimal time-consistent investment strategy is decreasing (increasing) with respect to λ_2 when $\frac{\mu_y}{\sigma_y^2} < \frac{r}{\sigma^2}$ ($\frac{\mu_y}{\sigma_y^2} > \frac{r}{\sigma^2}$). For example, when $\frac{\mu_y}{\sigma_y^2} < \frac{r}{\sigma^2}$, the bigger the intensity of the jumps of the risky's price, the less amount the company invests in the risky asset, see Figure 1(g).

5.2 Analysis of the equilibrium value function

In this subsection, we will work on numerical analysis of the value function in the dual diffusion approximation model. Figure 2 shows that how the coefficients involved impact on the value function. For convenience but without loss of generality, assume $\rho > 0$ except the analysis of ρ and $\sigma_0 > 0$. By (54) and some simple calculations, we can have the following findings:

(1) $\frac{\partial V}{\partial \gamma} = -\frac{(\sigma_0^2+\rho\sigma_o\sqrt{\lambda_1\sigma_z^2}+\lambda_1\sigma_z^2)}{4r_0}\left(e^{2r_0(T-t)}-1\right) - \frac{\xi}{2\gamma^2}(T-t) < 0$ which illustrates that the value function is decreasing with respect to the coefficient risk aversion γ, namely, the larger risk aversion the company has, the smaller the optimal mean-variance utilities is, see Figure 2(a).

(2) $\frac{\partial V}{\partial r_0}$ has a complex expression with respect to r_0. Thus we only analyze its impact on the value function from numerical methods. When all the parameters except r_0 are fixed, our numerical example reveals when the risk-free rate increases, the value function increases, see Figure 2(b).

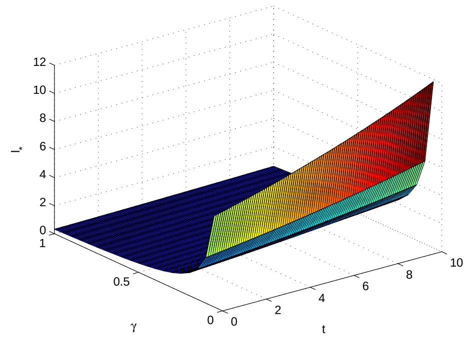
(3) $\frac{\partial V}{\partial r_1} = \frac{r+\lambda_2\mu_y}{\gamma(\sigma^2+\lambda_2\sigma_y^2)}(T-t) > 0$ which illustrates that the value function is increasing with respect to r_1, namely, the bigger the appreciation rate is, the bigger the optimal mean-variance utilities is, see Figure 2(c).

(4) $\frac{\partial V}{\partial \sigma^2} = -\frac{(r+\lambda_2\mu_y)^2(T-t)}{2\gamma\left(\sigma^2+\lambda_2\sigma_y^2\right)^2} < 0$ which reveals that the value function is decreasing with respect to σ^2, namely, the bigger the volatility of the market's risky asset is, the smaller the optimal mean-variance utilities is, see Figure 2(d).

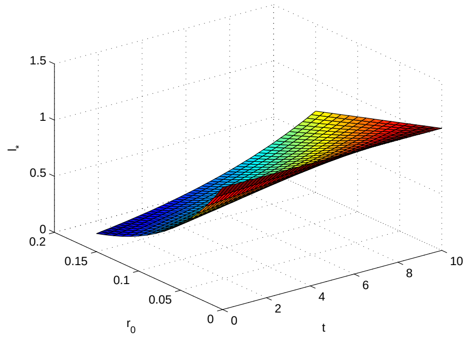
(5) $\frac{\partial V}{\partial \sigma_0^2} = -\frac{\gamma(1+\frac{\rho}{2}\sqrt{\frac{\lambda_1\sigma_z^2}{\sigma_0^2}})}{4r_0}\left(e^{2r_0(T-t)}-1\right) < 0$ which reveals that the value function is decreasing with respect to σ_0^2, namely, when the volatility of the risky asset increases, the optimal mean-variance utilities decrease, see Figure 2(e).

(6) $\frac{\partial V}{\partial \lambda_1} = \frac{\mu_z}{r_0}\left(e^{r_0(T-t)}-1\right) - \frac{\gamma(\frac{\rho\sigma_o}{2}\sqrt{\frac{\sigma_z^2}{\lambda_1}}+\sigma_z^2)}{4r_0} \times \left(e^{2r_0(T-t)}-1\right)$ which illustrates that the value function is decreasing (increasing) with respect to λ_1 when

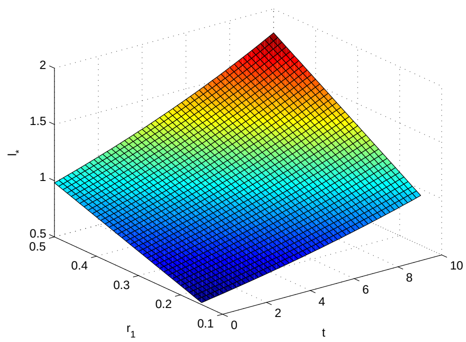
$$\frac{\mu_z}{r_0}\left(e^{r_0(T-t)}-1\right) < \frac{\gamma(\frac{\rho\sigma_o}{2}\sqrt{\frac{\sigma_z^2}{\lambda_1}}+\sigma_z^2)}{4r_0}\left(e^{2r_0(T-t)}-1\right)$$



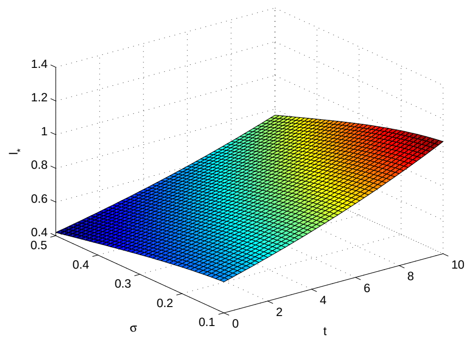
(a) The effect of γ on optimal time-consistent strategy



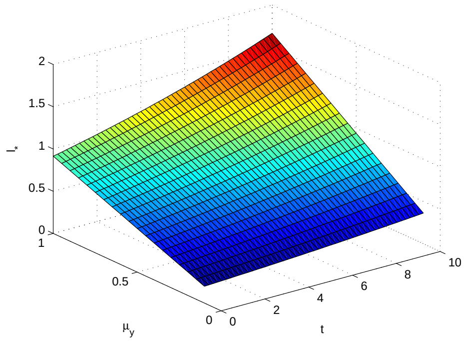
(b) The effect of r_0 on optimal time-consistent strategy



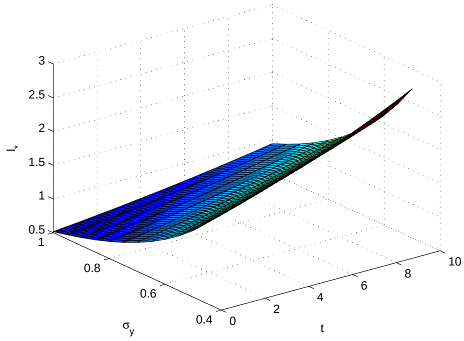
(c) The effect of r_1 on optimal time-consistent strategy



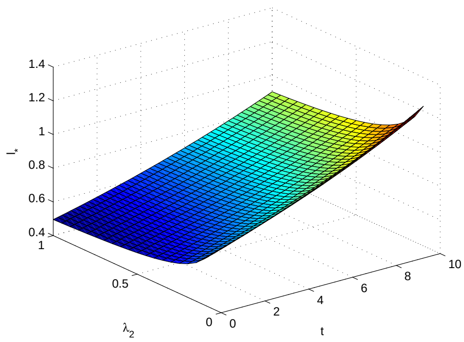
(d) The effect of σ^2 on optimal time-consistent strategy



(e) The effect of μ_y on optimal time-consistent strategy



(f) The effect of σ_y^2 on optimal time-consistent strategy



(g) The effect of λ_2 on optimal time-consistent strategy

Figure 1: The effect of parameters on optimal time-consistent strategy

$\left(\frac{\mu_z}{r_0}\left(e^{r_0(T-t)}-1\right) > \frac{\gamma(\frac{\rho\sigma_o}{2}\sqrt{\frac{\sigma_z^2}{\lambda_1}}+\sigma_z^2)}{4r_0}\left(e^{2r_0(T-t)}-1\right)\right)$. For example, when the value function is increasing with respect to λ_1, the bigger the intensity of the jumps of the risky's price becomes, the bigger the optimal mean-variance utilities become, see Figure 2(f).

(7) $\frac{\partial V}{\partial \sigma_z^2} = -\frac{\gamma(\frac{\rho\sigma_o}{2}\sqrt{\frac{\lambda_1}{\sigma_z^2}}+\lambda_1)}{4r_0}\left(e^{2r_0(T-t)}-1\right) < 0$ which illustrates that the value function is decreasing with respect to σ_z^2, namely, the smaller the second moment of the positive income is, the bigger the optimal mean-variance utilities become, see Figure 2(g).

(8) $\frac{\partial V}{\partial \mu_z} = \frac{\lambda_1}{r_0}\left(e^{r_0(T-t)}-1\right) > 0$ which shows that the value function is increasing with respect to μ_z, namely, the bigger the expectation of the positive income, the bigger the optimal mean-variance utilities become, see Figure 2(h).

(9) $\frac{\partial V}{\partial \lambda_2} = \frac{(2\mu_y\sigma^2 - r\sigma_y^2 + \lambda_2\mu_y\sigma_y^2)(r+\lambda_2\mu_y)(T-t)}{2\gamma\left(\sigma^2+\lambda_2\sigma_y^2\right)^2}$ which illustrates that the value function is decreasing (increasing) with respect to λ_2 when $\lambda_2 > \frac{r\sigma_y^2-2\mu_y\sigma^2}{\mu_y\sigma_y^2}$ ($\lambda_2 < \frac{r\sigma_y^2-2\mu_y\sigma^2}{\mu_y\sigma_y^2}$). For example, when $\lambda_2 > \frac{r\sigma_y^2-2\mu_y\sigma^2}{\mu_y\sigma_y^2}$, the bigger the intensity of the jumps of the risky's price, the bigger the optimal mean-variance utilities become, see Figure 2(i).

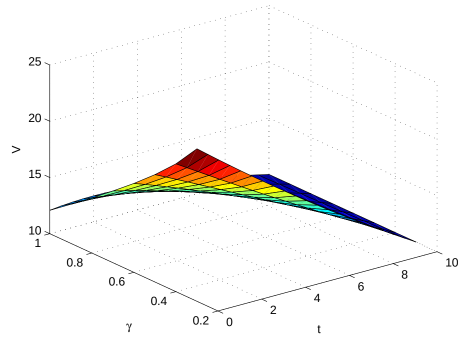
(10) $\frac{\partial V}{\partial \mu_y} = \frac{\lambda_2(r+\lambda_2\mu_y)(T-t)}{\gamma\left(\sigma^2+\lambda_2\sigma_y^2\right)} > 0$ which illustrates the value function is increasing with respect to μ_y, namely, the bigger the expectation of each jump amplitude of the risky's price is, the bigger the optimal mean-variance utilities become, see Figure 2(j).

(11) $\frac{\partial V}{\partial \sigma_y^2} = -\frac{\lambda_2(r+\lambda_2\mu_y)^2(T-t)}{2\gamma\left(\sigma^2+\lambda_2\sigma_y^2\right)^2} < 0$ which shows that the value function is increasing with respect to σ_y^2, namely, the bigger the second-order moment of each jump amplitude of the risky's price , the smaller the optimal mean-variance utilities become, see Figure 2(k).

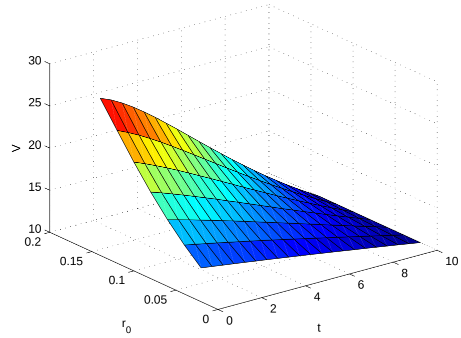
(12) $\frac{\partial V}{\partial \rho} = -\frac{\gamma\sigma_0\sqrt{\lambda_1\sigma_z^2}}{4r_0}\left(e^{2r_0(T-t)}-1\right) < 0$ which reveals that the value function is decreasing with respect to ρ, namely, when ρ increases, the optimal mean-variance utilities decrease, see Figure 2(l).

6 Conclusion

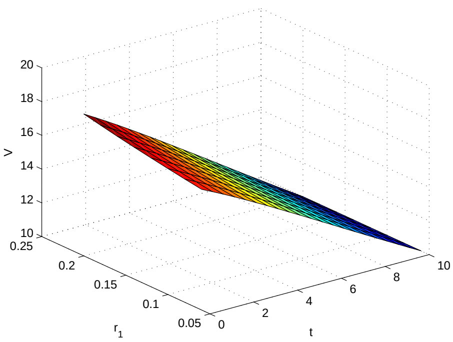
In this paper, we investigate the dual model with diffusion including the classic dual model and the dual approximation model. We are concerned on optimal time-consistent investment strategy under mean-variance criterion. We assume that companies can in-



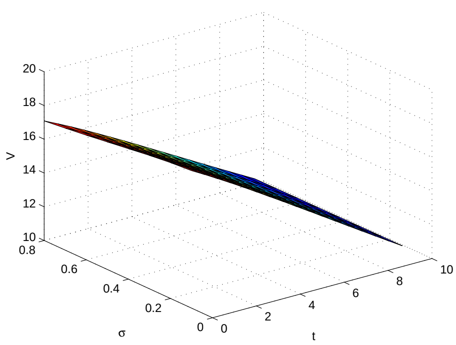
(a) The effect of γ on the equilibrium value function



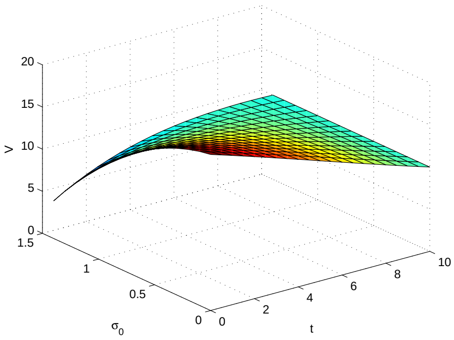
(b) The effect of r_0 on the equilibrium value function



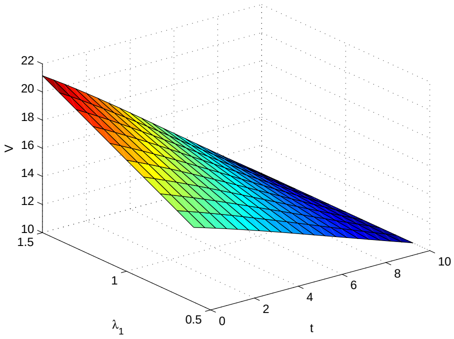
(c) The effect of r_1 on the equilibrium value function



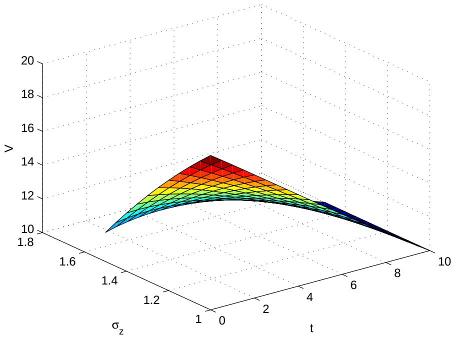
(d) The effect of σ^2 on the equilibrium value function



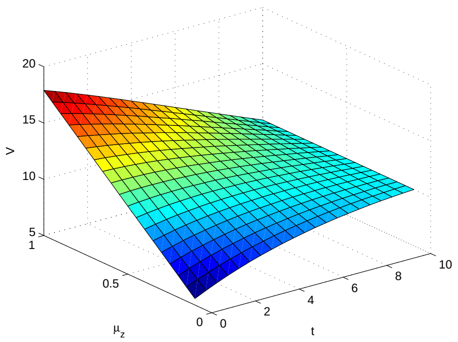
(e) The effect of σ_0^2 on the equilibrium value function



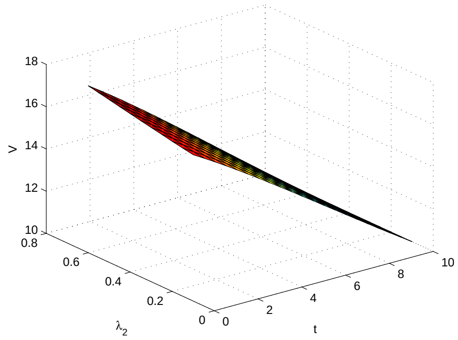
(f) The effect of λ_1 on the equilibrium value function



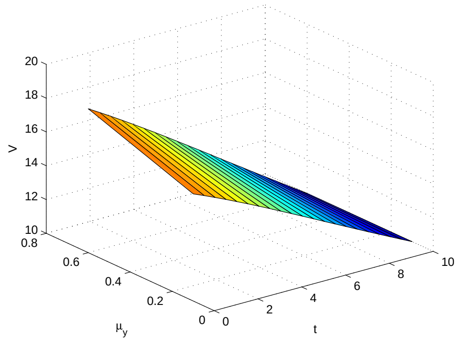
(g) The effect of σ_z^2 on the equilibrium value function



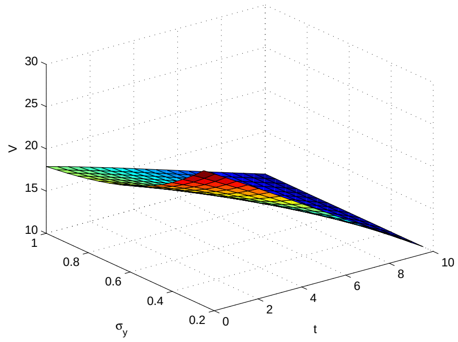
(h) The effect of μ_z on the equilibrium value function



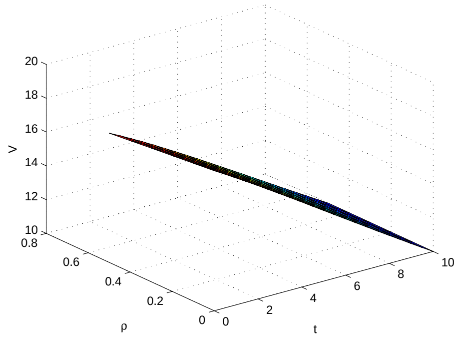
(i) The effect of λ_2 on the equilibrium value function



(j) The effect of μ_y on the equilibrium value function



(k) The effect of σ_y^2 on the equilibrium value function



(l) The effect of ρ on the equilibrium value function

Figure 2: The effect of parameters on the equilibrium value function

vest into a financial market which has a risk-free asset and n risky assets. Short-selling and borrowing money are allowed. Surprisingly optimal time-consistent investment strategy and optimal value function have the same expressions in both cases when $\rho = 0$. However, in practice, the company can not be allowed to borrow at the risk-free rate to invest in risky asset. An interesting further research topic is to investigate the precommitted strategy where Short-selling and borrowing money are not allowed. It is necessary to compare the precommitted strategy with time-consistent strategy. Another interesting further research topic is to investigate the optimal strategy for the dual model with regime switching. This is a challenge to derive the optimal strategy and optimal value function.

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