Exponential Stabilization of 1-d Wave Equation with Distributed Disturbance

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Abstract: In this paper, we consider the stabilization problem of a one-dimensional wave equation with unknown disturbance. In order to stabilize the system with disturbance, we design a distributed feedback controller by employing the idea of sliding mode control technology. For the resulted nonlinear closed-loop system, we prove its solvability by using the maximal monotone operator. Further we prove the exponentially stable of the closed-loop system.

Key–Words: 1-d wave equation; distributed control; distributed disturbance; exponential stabilization.

1 Introduction

Many practical systems composed of certain flexible parts are governed by partial differential equations (PDEs), such as strings, plates, shells and so on. For these systems, one of the important tasks is to design various feedback control laws to stabilize the systems to their equilibriums as fast as possible. In the last decades, some important stabilization results have been obtained for the infinite-dimensional systems such as wave equation and flexible beam (e.g., see [1, 2, 3, 4, 5]), including stabilization of the systems with time delay [6, 7, 8, 9]. We observe that these systems are modeled in the ideal operational environment with exact mathematical model in which the internal and external disturbances are neglected. Since the uncertain disturbances from the internal and external of systems always exist in the real world, and these disturbances effect seriously the performance of the systems, which may distort the systems, we must take the disturbances into account when we study the stabilization problem of systems.

In the past years, there are many scholars making great efforts for the anti-disturbance problems, for example, [10, 11]. One of the more successful approaches of rejecting disturbance is the sliding mode control (briefly, SMC). For a long time, the sliding mode control has been recognized as a powerful control tool [12] for the finite-dimensional systems. Recent decades, scholars are trying to extend this method from finite-dimensional system to infinite-dimensional systems [13, 14, 15, 16]. There have been some successful examples, for instance, Drakunov *et al.* in [17] proposed a sliding mode control law for a heat equation with boundary control and disturbance; Cheng *et al.* in [18] studied a parabolic PDE system with parameter variations and boundary uncertainties by using boundary sliding mode control approach; Pisano *et al.* in [19] investigated tracking control problem of wave equation with distributed control and disturbance, and applying the sliding mode control they obtained the asymptotical stability of the closed-loop system. There exist other approaches of anti-disturbance, such as active disturbance rejection control (ADRC) in [20, 21], Lyapunov control in [22, 23], adaptive control in [24, 25], and LMI-based design [26] etc. These methods also can be extended to the distributed parameter systems. We observe that the resulted system is a time-variant and nonlinear system under the sliding mode control; the solvability and stability analysis of the closed-loop system are major difficulty. For some systems, with stronger sliding mode control, the closed-loop system might have no solution. So when we design the feedback controller, we must take account into two things of stabilization property of the system and solvability of the closed-loop system.

In the present paper, we study a 1-dimensional wave equation with unknown disturbance, whose dynamic behavior is governed by the following partial differential equation:

$$\begin{cases} w_{tt}(x,t)=w_{xx}(x,t)+u(x,t)+r(x,t),\ x\in(0,1), \\ w(0,t)=0, \quad w(1,t)=0, \end{cases} \tag{1}$$

with initial data

$$w(x,0) = w_0(x), \;\; w_t(x,0) = w_1(x)$$

and observation or system output

$$y(x,t) = \big(w(x,t), \;\; w_t(x,t)\big),$$

where $x \in [0,1]$ is the space variable; t is the time variable; $w(x,t)$ is the state of the system; $u(x,t)$ is the distributed control input and $r(x,t)$ is a source term which represents the uncertain disturbance; (w_0, w_1) is the initial state of the system and $y(x,t)$ is the measured output.

Usually the disturbance has finite energy, so we can suppose that $r(x,t)$ is a uniformly bounded and measurable function, that is, there exists a positive constant R such that $|r(x,t)| \le R$ for all $x \in (0,1)$ and $t > 0$. Throughout the paper, we use w_x or w' to denote the partial derivative of w with respect to x, and $\dot{w}$ or w_t denote the partial derivative with respect to t.

It is well-known that if there is no disturbance, then the system (1) becomes the following one:

$$\begin{cases} w_{tt}(x,t)=w_{xx}(x,t)+u(x,t), \;\; x \in (0,1), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \\ y(x,t) = \big(w(x,t), w_t(x,t)\big). \end{cases} \tag{2}$$

It has been proved that under the velocity feedback control

$$u(x,t) = -kw_t(x,t), \quad k > 0$$

the corresponding closed-loop system is exponentially stable [27, 28]. However, due to the presence of $r(x,t)$, this stabilizer is not robust to the external disturbance. To see this point, we consider the case that $r(x,t) = r$ is a constant. Under the feedback control law $u(x,t) = -kw_t(x,t)$, the system (1) has a nonzero solution

$$(w(x,t), w_t(x,t)) = (\frac{r}{2}x^2 - \frac{r}{2}x, 0).$$

Therefore, we need to redesign controller based on the aforementioned control.

Our idea is that we divide the control into two parts: one ensures the exponentially stability of the non-disturbance system and the other is used to reject disturbance. Employing the idea of the sliding mode control, we take the feedback control law as

$$u(x,t) = -kw_t(x,t) - M\mathrm{sign}[\rho w(x,t) + w_t(x,t)], \tag{3}$$

where

$$\mathrm{sign}(x) = \begin{cases} -1, & x < 0, \\ [-1,1], & x = 0, \\ 1, & x > 0, \end{cases}$$

the parameters ρ, k and M are positive constants with $\rho \in (0,k)$ and $M \ge R$. We will determine the relation between parameters k and ρ later in the following analysis.

Under the feedback control law (3), the closed-loop system corresponding to (1) is

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - kw_t(x,t) + r(x,t) \\ \qquad -M\mathrm{sign}[\rho w(x,t) + w_t(x,t)], \quad x \in (0,1), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{cases} \tag{4}$$

Note that the controller includes a sign function, the closed-loop system (4) is a nonlinear system with discontinuous nonlinear term. Moreover, it is a nonautonomous system due to the existence of disturbance.

The design of the second term of controller is very novel, its different form will lead to different effect. At the same time, it will yield different difficulty in latter analysis. In our design, it seems to be a simple extension of sliding mode control from the finite dimensional space to infinite-dimensional space. Indeed, this form has many advantages that will be seen in solvability and stability analysis. Although so, there are still many challenge problems to face. One of the challenges is the well-posedness of the closed-loop system. Since the resulted closed-loop system is a semi-linear system with discontinuous nonlinear term, which in fact is a differential inclusion equation. In finite-dimensional problem, most of literatures have to use the Ritz method or approximate approach to obtain the existence of the Filippov solution, and then, by limit process, obtain the solvability of the closed-loop system [29, 30]. Maybe it is an approach to solve Filippov-type equation but not the effective one. The second challenge comes from the stability analysis of the closed-loop system. Since the disturbance is time-varying, the classical LaSalle invariance set principle [31] does not give the asymptotic behavior of the closed-loop system. The contributions of this paper are that we find a simple way to prove the existence and uniqueness of solution to the closed-loop system; and by the multiplier method, we establish the exponentially stability of the system (4).

The rest of the paper is organized as follows. In section 2, we deal with the existence and uniqueness of the solution of the closed-loop system by maximal monotone theory. In section 3, we discuss the stability of the system (4). By multiplier method, we obtain the exponential stability of the system (4). In Section 4, we present some numerical simulations to check our theoretical results. Finally, in section 5, we give a concluding remark.

2 Well-posedness of the Closed-loop System

In this section, we will discuss the existence and uniqueness of a solution to the closed-loop system (4).

For the convenience, we rewrite the system (4) as follows:

$$\left\{\begin{array}{l} w_{tt}(x,t) = w_{xx}(x,t) + r(x,t) - kw_t(x,t) \\ \quad - M\mathrm{sign}[\rho w(x,t) + w_t(x,t)], \quad x \in (0,1), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{array}\right. \tag{5}$$

Remark 1. *Since the sign function is a set-valued function, the system (5) should understand as the differential inclusion, i.e.,*

$$\left\{\begin{array}{l} -w_{tt}(x,t) + w_{xx}(x,t) - kw_t(x,t) + r(x,t) \\ \quad \in M\mathrm{sign}[\rho w(x,t) + w_t(x,t)], \quad x \in (0,1), \\ w(0,t) = 0, \quad w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x). \end{array}\right. \tag{6}$$

Since only at the origin, the sign function takes its value in a set [-1,1], we still use the form (5) for convenience.

We begin with recalling concept of the maximal monotone operator.

Definition 2. *[32, Definition 2.1, pp.28] Let $\mathbb{X}$ be a real Banach space, and $\mathbb{X}^*$ be its dual. The set $A \subset \mathbb{X}\times\mathbb{X}^*$(equivalently the operator $A : \mathbb{X} \to \mathbb{X}^*$) is said to be monotone if*

$$\langle x_1 - x_2, y_1 - y_2\rangle_{\mathbb{X},\mathbb{X}^*} \geq 0, \quad \forall\ [x_i, y_i] \in A,\ i = 1, 2.$$

A monotone set $A \subset \mathbb{X} \times \mathbb{X}^*$ is said to be maximal monotone if it is not properly obtained in any other monotone subset of $\mathbb{X} \times \mathbb{X}^*$.

For a maximal monotone operator, we have the following results.

Lemma 3. *[32, Corollary 2.2, pp.36] Let $\mathbb{X}$ be a reflexive Banach space and let A be a coercive maximal monotone subset of $\mathbb{X}\times\mathbb{X}^*$. Then A is surjective, that is, $\mathcal{R}(A) = \mathbb{X}^*$.*

The following lemma gives the solvability of abstract evolutionary equation.

Lemma 4. *[32, Corollary 4.1, pp.131] Let $\mathbb{X}$ be a real Banach space with the norm $||\cdot||$ and let $A \subset \mathbb{X}\times\mathbb{X}$ be a quasi-m-accretive set of $\mathbb{X}\times\mathbb{X}$. Consider the Cauchy problem*

$$\left\{\begin{array}{ll} \frac{dy}{dt}(t) + Ay(t) \ni f(t), & t \in [0,T], \\ y(0) = y_0, & \end{array}\right. \tag{7}$$

where $y_0 \in \mathbb{X}$ and $f \in L^1(0,T;\mathbb{X})$. Then, for each $y_0 \in \overline{\mathcal{D}(A)}$ and $f \in L^1(0,T;\mathbb{X})$ there is a unique mild solution y to (7).

To applying the maximal monotone operator theory, we choose the state space as follows:

$$\mathcal{H} = H^1_E(0,1) \times L^2(0,1)$$

where $H^k_E(0,1) = \{f \in H^k(0,1) | f(0) = f(1) = 0\}$, and $H^k(0,1)$ is the usual Sobolev space of the order $k(k = 1, 2)$.

In the state space $\mathcal{H}$, the inner product is defined as follows, for $Y_1 = (w_1, z_1)^\top, Y_2 = (w_2, z_2)^\top \in \mathcal{H}$,

$$\langle Y_1, Y_2\rangle_{\mathcal{H}} = \int_0^1 w_1'(x)w_2'(x)dx + \int_0^1 z_1(x)z_2(x)dx. \tag{8}$$

It is easy to check that $(\mathcal{H}, ||\cdot||_{\mathcal{H}})$ is a Hilbert space.

We define the system operator $\mathcal{A}$ in $\mathcal{H}$:

$$\mathcal{A}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} z - \rho w \\ w_{xx} + \rho(k-\rho)w - (k-\rho)z - M\mathrm{sign}[z] \end{pmatrix}$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \left\{(w,z) \in \mathcal{H} \;\middle|\; \begin{array}{l} w \in H^2_E(0,1), \\ z \in H^1_E(0,1). \end{array}\right\}$$

Then the closed loop system (5) can be written as a nonlinear evolutionary equation in $\mathcal{H}$

$$\left\{\begin{array}{l} \dot{Y}(t) = \mathcal{A}Y(t) + R(t), \\ Y(0) = Y_0, \end{array}\right. \tag{9}$$

where $Y(t) = (w(\cdot,t), \dot{w}(\cdot,t) + \rho w(\cdot,t))^\top$, $R(t) = (0, r(\cdot,t))^\top$ and $Y_0 = (w_0(x), w_1(x) + \rho w_0(x))^\top$.

Remark 5. *According to Remark 1, $\mathcal{A}$ defined above is a multi-valued operator. Thus, (9) should also be written as a differential inclusion: $\dot{Y}(t) - R(t) \in \mathcal{A}Y(t)$.*

The operator $\mathcal{A}$ satisfies the following proposition.

Proposition 6. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. If the parameters ρ and k satisfy the following conditions:*

1) when $k^2 \leq 32$, $\rho \in (0, k)$;

2) when $k^2 \geq 32$, ρ is in the subset

$$\rho \in \left(0, \frac{k-\sqrt{k^2-32}}{2}\right) \bigcup \left(\frac{k+\sqrt{k^2-32}}{2}, k\right),$$

then $-\mathcal{A}$ is a maximal monotone operator.

Proof: We complete the proof of Proposition 6 by two steps.

First step: monotony of $-\mathcal{A}$

For any $Y_1, Y_2 \in D(\mathcal{A})$, by the definition of the inner product of $\mathcal{H}$, we have

$$\begin{aligned}
&\langle \mathcal{A}Y_1 - \mathcal{A}Y_2, Y_1 - Y_2\rangle_{\mathcal{H}} \\
= & \int_0^1 (z_1' - z_2')(w_1' - w_2')dx \\
& -\rho \int_0^1 (w_1' - w_2')(w_1' - w_2')dx \\
& + \int_0^1 (w_1'' - w_2'')(z_1 - z_2)dx \\
& + \rho(k-\rho) \int_0^1 (w_1 - w_2)(z_1 - z_2)dx \\
& -(k-\rho) \int_0^1 (z_1 - z_2)(z_1 - z_2)dx \\
& -M \int_0^1 \Big(\text{sign}(z_1) - \text{sign}(z_2)\Big)(z_1 - z_2)dx \\
= & -\rho \int_0^1 |w_1' - w_2'|^2 dx \\
& + \rho(k-\rho) \int_0^1 (w_1 - w_2)(z_1 - z_2)dx \\
& -(k-\rho) \int_0^1 |z_1 - z_2|^2 dx \\
& -M \int_0^1 \Big(\text{sign}(z_1) - \text{sign}(z_2)\Big)(z_1 - z_2)dx \\
= & -\rho \int_0^1 |w_1' - w_2'|^2 dx \\
& -(k-\rho) \int_0^1 \left[(z_1 - z_2) - \frac{\rho}{2}(w_1 - w_2)\right]^2 dx \\
& + \frac{\rho^2}{4}(k-\rho) \int_0^1 (w_1 - w_2)^2 dx \\
& -M \int_0^1 \Big(\text{sign}(z_1) - \text{sign}(z_2)\Big)(z_1 - z_2)dx.
\end{aligned}$$

Since

$$\int_0^1 (w_1 - w_2)^2 dx \leq \frac{1}{2} \int_0^1 (w_1' - w_2')^2 dx$$

and for any $y_1, y_2 \in \mathbb{R}$

$$[\text{sign}(y_1) - \text{sign}(y_2)](y_1 - y_2) \geq 0,$$

so for $\rho \in (0, k)$, we have

$$\begin{aligned}
&\langle \mathcal{A}Y_1 - \mathcal{A}Y_2, Y_1 - Y_2\rangle_{\mathcal{H}} \\
\leq & -\rho(1 - \frac{\rho}{8}(k-\rho)) \int_0^1 (w_1' - w_2')^2 dx
\end{aligned}$$

We request $\rho \in (0, k)$ satisfies the following equality

$$\frac{\rho}{8}(k-\rho) \leq 1,$$

which is equivalent to

$$-\rho^2 + k\rho = -(\rho - \frac{k}{2})^2 + \frac{k^2}{4} \leq 8.$$

Clearly, if $k^2 - 32 \leq 0$, then for all $\rho \in (0, k)$, the above inequality holds true. If $k^2 - 32 > 0$, then we have

$$\rho \in \left(0, \frac{k-\sqrt{k^2-32}}{2}\right) \bigcup \left(\frac{k+\sqrt{k^2-32}}{2}, k\right).$$

Therefore, when ρ and k satisfy the request, we have

$$\langle \mathcal{A}Y_1 - \mathcal{A}Y_2, Y_1 - Y_2\rangle_{\mathcal{H}} \leq 0,$$

which claims that $-\mathcal{A}$ is monotone.

Second step: maximality

According to the definition of the maximal operator, we only need to show that $\mathcal{R}(I - \mathcal{A}) = \mathcal{H}$.

For $\forall (f, g)^\top \in \mathcal{H}$ given, we consider the following equation

$$(I - \mathcal{A})(w, z)^\top = (f, g)^\top,$$

namely,

$$\begin{aligned}
& w(x) - z(x) + \rho w(x) = f(x), \\
& z(x) - w''(x) - \rho(k-\rho)w(x) + (k-\rho)z(x) \\
& + M\text{sign}(z(x)) = g(x).
\end{aligned}$$

Equivalently, $z(x) = (1+\rho)w(x) - f(x)$ and $w(x)$ satisfies the following differential equation

$$\begin{aligned}
& -w''(x) + (k+1)w(x) \\
& + M\text{sign}((1+\rho)w(x) - f(x)) \\
& = g(x) + (k - \rho + 1)f(x)
\end{aligned}$$

and the boundary conditions $w(0) = w(1) = 0$.

Let $\lambda = \sqrt{k+1}$. Then the differential equation can be rewritten as

$$\begin{aligned}
& w''(x) = \lambda^2 w(x) \\
& + M\text{sign}((1+\rho)w(x) - f(x)) \\
& -[g(x) + (\lambda^2 - \rho)f(x)]. \qquad (10)
\end{aligned}$$

Set

$$\begin{cases} d = \frac{\int_0^1 [(\lambda^2-\rho)f(s)+g(s)]\sinh(\lambda(1-s))ds}{\lambda\sinh\lambda}, \\ p = \frac{\cosh\lambda-1}{\lambda^2\sinh\lambda}, \\ \phi(x) = \frac{\int_0^x [(\lambda^2-\rho)f(s)+g(s)]\sinh(\lambda(x-s))ds}{\lambda}, \\ \psi(x) = \frac{\cosh\lambda x-1}{\lambda^2}. \end{cases}$$

Lemma 7. *Let p and $\psi(x)$ be defined as before. Then we have*

$$-Mp\sinh\lambda x + M\psi(x) \le Mp\sinh\lambda x - M\psi(x), \quad (11)$$

i.e.,

$$Mp\sinh\lambda x - M\psi(x) \ge 0.$$

Proof. Set

$$S(x) = p\sinh\lambda x - \psi(x).$$

By the definition, we have

$$\begin{aligned} S(x) &= \frac{\cosh\lambda-1}{\lambda^2\sinh\lambda}\sinh\lambda x - \frac{\cosh\lambda x-1}{\lambda^2} \\ &= \frac{(\cosh\lambda-1)\sinh\lambda x - (\cosh\lambda x-1)\sinh\lambda}{\lambda^2\sinh\lambda} \\ &= \frac{\sinh\lambda - \sinh\lambda x - \sinh(\lambda-\lambda x)}{\lambda^2\sinh\lambda}. \end{aligned}$$

Obviously, $S(0) = S(1) = 0$ and

$$S(\frac{1}{2}) = \frac{\sinh\lambda - 2\sinh\frac{\lambda}{2}}{\lambda^2\sinh\lambda}.$$

$S(x)$ is increasing for $x \in [0, \frac{1}{2}]$, $S(x)$ is decreasing for $x \in [\frac{1}{2}, 1]$. So

$$0 \le S(x) \le S(\frac{1}{2}) = \frac{\sinh\lambda - 2\sinh\frac{\lambda}{2}}{\lambda^2\sinh\lambda},$$

therefore, the inequality (11) is true. □

Using the result of Lemma 7, we can solve the differential equation (10). When

$$\begin{aligned} & d\sinh\lambda x - \frac{f(x)}{\rho+1} - \phi(x) \\ > \ & Mp\sinh\lambda x - M\psi(x), \end{aligned}$$

w takes the form $w(x) = c_1\sinh\lambda x - \phi(x) + M\psi(x)$ where $c_1 = d - Mp$;

When

$$\begin{aligned} & d\sinh\lambda x - \frac{f(x)}{\rho+1} - \phi(x) \\ \in \ & [-Mp\sinh\lambda x + M\psi(x), Mp\sinh\lambda x - M\psi(x)], \end{aligned}$$

w takes $w(x) = \frac{f(x)}{\rho+1}$;

When

$$\begin{aligned} & d\sinh\lambda x - \frac{f(x)}{\rho+1} - \phi(x) \\ < \ & -Mp\sinh\lambda x + M\psi(x), \end{aligned}$$

it takes $w(x) = c_2\sinh\lambda x - \phi(x) - M\psi(x)$ where $c_2 = d + Mp$.

Thus, we have proved that $\mathcal{R}(I - \mathcal{A}) = \mathcal{H}$. This finishes the maximality proof.

The two steps give the proof of Proposition 6. □

According to the Proposition 6 and Lemma 4, we can show that the well-posedness of the nonlinear evolutionary equation (9), i.e., the closed-loop system (5).

Theorem 8. *Let $\mathcal{A}$ be defined as before. If the parameters ρ and k satisfy the following condition:*

1) when $k^2 \le 32$, $\rho \in (0, k)$;

2) when $k^2 \ge 32$, ρ is in the subset

$$\rho \in \left(0, \frac{k - \sqrt{k^2-32}}{2}\right) \bigcup \left(\frac{k + \sqrt{k^2-32}}{2}, k\right),$$

then, for each $Y_0 \in \overline{D(\mathcal{A})}$, there is a unique mild solution to the equation (9).

3 Exponential Stability of the Closed-Loop System

In this section, we will discuss the exponential stability of the closed-loop system (4). In (4), if $r(x,t) = 0$ and $M = 0$, from the spectral analysis we see that for suitably small k the system has bigger exponential decay rate. So in this section, we suppose that k and ρ are small. More precisely, we suppose that $\rho \in (0, 1)$.

Note that the energy functional of system (4) is

$$E(t) = \frac{1}{2}\int_0^1 |w_x(x,t)|^2 dx + \frac{1}{2}\int_0^1 |w_t(x,t)|^2 dx. \quad (12)$$

Differentiating (12) with respect to t, using the boundary conditions and integrating by parts, we obtain

$$\begin{aligned} \dot{E}(t) &= \int_0^1 [w_x(x,t)w_{xt}(x,t) + w_t(x,t)w_{tt}(x,t)]dx \\ &= -k\int_0^1 w_t^2(x,t)dx \\ &\quad -M\int_0^1 \text{sign}[\rho w(x,t) + w_t(x,t)]w_t(x,t)dx \\ &\quad +\int_0^1 r(x,t)w_t(x,t)dx. \quad (13) \end{aligned}$$

To further estimate inequality, we need the following lemma which is a direct result of Cauchy-Schwartz inequality.

Lemma 9. *Let $\phi_1(x,t), \phi_2(x,t)$ be the continuous functions defined on $[0,1]\times\mathbb{R}_+$. Then the following inequalities hold*

$$|\phi_1(x,t)\phi_2(x,t)| \le \phi_1^2(x,t)+\phi_2^2(x,t), \tag{14}$$

and

$$|\phi_1(x,t)\phi_2(x,t)| \le |\Big(\frac{1}{\sqrt{\delta}}\phi_1(x,t)\Big)\Big(\sqrt{\delta}\phi_2(x,t)\Big)| \le \frac{1}{\delta}\phi_1^2(x,t)+\delta\phi_2^2(x,t), \tag{15}$$

where $\delta > 0$.

To study the stability of (4), we consider the functional

$$V_\rho(t) = E(t) + \rho\int_0^1 w(x,t)w_t(x,t)dx, \tag{16}$$

where $\rho \in (0,1)$ is the same as in (4).

Since

$$|\int_0^1 w(x,t)w_t(x,t)dx| \le \frac{1}{2}\int_0^1 [w_t^2(x,t)+w_x^2(x,t)]dx = E(t),$$

so, $V_\rho(t)$ has the following property:

$$(1-\rho)E(t) \le V_\rho(t) \le (1+\rho)E(t). \tag{17}$$

Lemma 10. *If the parameters ρ and k in (16) satisfy the inequality $0<\rho<\frac{k}{1+k^2}$, then there exists a positive constant η such that*

$$\dot V_\rho(t) \le -\eta V_\rho(t).$$

Proof: Differentiating (16) with respect to t and integrating by parts, using equality (13) and the boundary conditions, we obtain

$$\begin{aligned}
\dot V_\rho(t) &= \dot E(t) + \rho\int_0^1 w_t^2(x,t)dx + \rho\int_0^1 w(x,t)w_{tt}(x,t)dx\\
&= -k\int_0^1 w_t^2(x,t)dx - M\int_0^1 \mathrm{sign}[\rho w(x,t)+w_t(x,t)]w_t(x,t)dx\\
&\quad + \int_0^1 r(x,t)w_t(x,t)dx + \rho\int_0^1 w_t^2(x,t)dx\\
&\quad -\rho\int_0^1 w_x^2(x,t)dx - k\rho\int_0^1 w(x,t)w_t(x,t)dx\\
&\quad -M\rho\int_0^1 \mathrm{sign}[\rho w(x,t)+w_t(x,t)]w(x,t)dx\\
&\quad +\rho\int_0^1 w(x,t)r(x,t)dx\\
&= -(k-\rho)\int_0^1 w_t^2(x,t)dx - \rho\int_0^1 w_x^2(x,t)dx\\
&\quad -\int_0^1 (M-r(x,t))|\rho w(x,t)+w_t(x,t)|dx\\
&\quad -k\rho\int_0^1 w(x,t)w_t(x,t)dx\\
&\le -(k-\rho)\int_0^1 w_t^2(x,t)dx - \rho\int_0^1 w_x^2(x,t)dx\\
&\quad -k\rho\int_0^1 w(x,t)w_t(x,t)dx.
\end{aligned}$$

Using inequalities (15), we get

$$\begin{aligned}
\dot V_\rho(t) &\le -(k-\rho)\int_0^1 w_t^2(x,t)dx - \rho\int_0^1 w_x^2(x,t)dx\\
&\quad +k\rho\int_0^1 [\frac{w_t^2(x,t)}{\delta_2} + \delta_2 w^2(x,t)]dx\\
&\le -(k-\rho-\frac{k\rho}{\delta_2})\int_0^1 w_t^2(x,t)dx\\
&\quad -(\rho-k\rho\delta_2)\int_0^1 w_x^2(x,t)dx.
\end{aligned}$$

Obviously, when $k-\rho-\frac{k\rho}{\delta_2}>0$, $\rho-k\rho\delta_2>0$, we have

$$\begin{aligned}
\dot V_\rho(t) &\le -(k-\rho-\frac{k\rho}{\delta_2})\int_0^1 w_t^2(x,t)dx\\
&\quad -(\rho-k\rho\delta_2)\int_0^1 w_x^2(x,t)dx\\
&\le -2\min\{(k-\rho-\frac{k\rho}{\delta_2}),(\rho-k\rho\delta_2)\}E(t)\\
&\le -\frac{2\min\{(k-\rho-\frac{k\rho}{\delta_2}),(\rho-k\rho\delta_2)\}}{1+\rho}V_\rho(t)\\
&= -\eta V_\rho(t),
\end{aligned}$$

where

$$\eta = \frac{2\min\{(k-\rho-k\rho\delta_2),(\rho-\frac{k\rho}{\delta_2})\}}{1+\rho}.$$

In what follows, we show solvability of the inequality equations

$$k-\rho-\frac{k\rho}{\delta_2}>0, \quad \rho-k\rho\delta_2>0$$

where k , ρ and δ_2 are positive constants.

From $\rho-k\rho\delta_2>0$ we get $\delta_2<\frac{1}{k}$. From $k-\rho-\frac{k\rho}{\delta_2}>0$, we get $\delta_2>\frac{k\rho}{k-\rho}$. So we must request k and ρ satisfy $\frac{k\rho}{k-\rho}<\frac{1}{k}$. Clearly, ρ and k satisfy the above inequality provided that $0<\rho<\frac{k}{1+k^2}$.

We can choose δ_2, for example,

$$\delta_2=\frac{(k-2\rho)+\sqrt{(k-2\rho)^2+4k^2\rho^2}}{2k\rho}$$

such that $V_\rho(t)\le e^{-\eta t}V_\rho(0)$. In this case, we have

$$\eta=\frac{k-\sqrt{(k-2\rho)^2+4k^2\rho^2}}{1+\rho}.$$

According to above discussion, we have proved the following theorem.

Theorem 11. *Suppose that $|r(x,t)|\le R$. Let the control law be given as (3), where k and ρ satisfy relation $0<\rho<\frac{k}{1+k^2}$ and $M\ge R$. Then for any initial data $(w_0,w_1)\in\mathcal{H}$, the closed-loop system (4) is exponentially stable.*

Remark 12. *We present that the controller given by (3) is robust for any $|r(x,t)|\le R$. But it is necessary that $\rho\ne 0$.*

If $\rho=0$, the feedback controller becomes

$$u(x,t)=-kw_t(x,t)-M\mathrm{sign}[w_t(x,t)], \tag{18}$$

then the closed-loop system (1) with (18) is

$$\begin{cases} w_{tt}(x,t)=w_{xx}(x,t)-kw_t(x,t)+r(x,t) \\ \qquad -M\mathrm{sign}[w_t(x,t)], \quad x\in(0,1), \\ w(0,t)=0, \quad w(1,t)=0, \\ w(x,0)=w_0(x), \quad w_t(x,0)=w_1(x). \end{cases} \tag{19}$$

It is easy to see that the energy function defined by (12) along the trajectory of (19) holds

$$\dot{E}(t) \quad \le \quad -k\int_0^1 w_t^2(x,t)dx\le 0, \tag{20}$$

that implies that the energy of system (19) is decreasing. But system (19) is not asymptotically stable to zero. Indeed, if $r(x,t)=0$, we take $M=1$ and $\mathrm{sign}(0)=0.6$. Clearly $w(x,t)=0.3x^2-0.3x$ satisfies the system (19). Hence, $\lim_{t\to\infty}w(x,t)\nrightarrow 0$ as $x\in(0,1)$.

4 Numerical Simulation

In this section, we consider a numerical example for the system (4) and system (19). The purpose of numerical simulation is to check the result of this paper.

We choose model parameters of the system (4) as follows:

the initial data:

$$w(x,0)=20\cos(3\pi x+\frac{\pi}{2}),$$

$$w_t(x,0)=20\cos(3\pi x+\frac{\pi}{2})$$

and the disturbance

$$r(x,t)=-50x\cos(2\pi t).$$

The controller parameters are

$$k=2, \quad M=60, \quad \rho=0.5.$$

We used the Backward Euler Method in time and Chebyshev spectral method in space and programmed the code in Matlab (see [33]). The spatial grid size $N=40$ and time step $dt=0.0001$.

1) We compare displacement change of the systems between the uncontrolled systems and controlled system (4).

(a) the pictures for uncontrolled system are given by Figures 1 and 2.

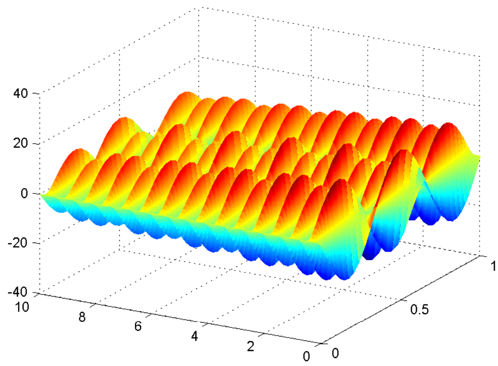

Figure 1: Displacement of the system (4) without control input.

Figure 1 shows the displacement of the system (4) under disturbances without control input, i.e., $u(x,t)=0$;

Figure 2 shows the dynamic behavior of the cross-section of same system at $x=0.5$.

From both figures we see that it results in the instability.

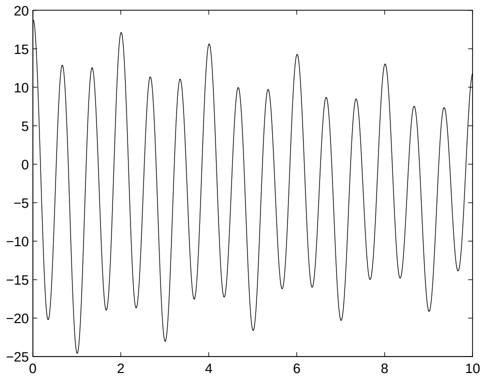

Figure 2: Displacement of the system (4) without control input at $x = 0.5$.

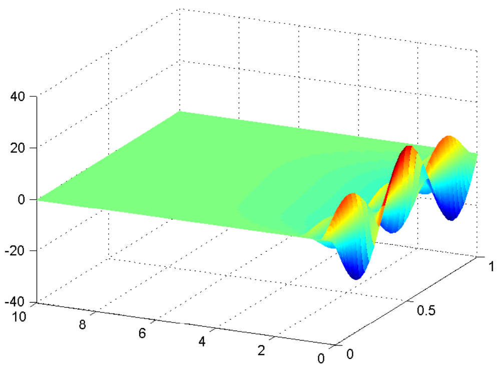

Figure 3: Displacement of the closed-loop system (4).

(b) the pictures for controlled system are given by Figures 3 and 4.

Figure 3 exhibits the displacement the change of the closed-loop system (4); And Figure 4 shows the dynamic behavior of the cross-section of the controlled system at $x = 0.5$.

As we see, the displacement of the system converges to zero quickly.

2) We compare displacement change of the systems between the controlled system (4) and controlled system (19).

Here we main simulate the dynamic behavior of (19), in this case, $\rho = 0$. The simulation results are shown by pictures 5 and 6.

Figure 5 and 6 illustrate the displacement of system (19) and the behavior of the cross-section of the system at $x = 0.5$, with all the same parameters as that in Figure 3 except $\rho = 0$.

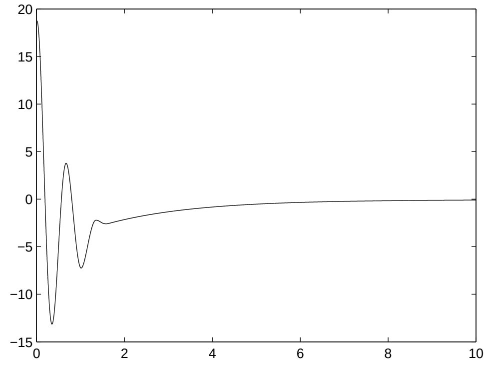

Figure 4: Displacement of the closed-loop system (4) at $x = 0.5$.

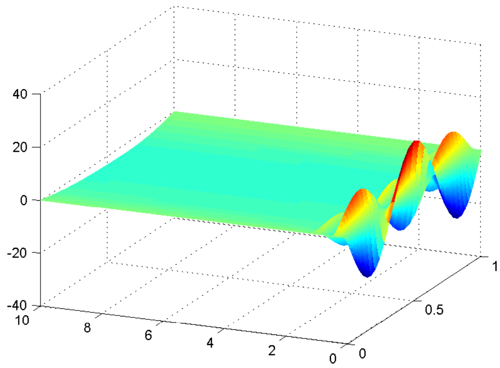

Figure 5: Displacement of the closed-loop system (19) with $\rho = 0$.

From Figure 5 and 6, we see that the control law

$$u(x,t) = -kw_t(x,t) - M\text{sign}[w_t(x,t)]$$

cannot stabilize the solution of the system (19) to zero. However, it might achieve the bounded stability in the sense of Lyapunov functional, as shown in both figures.

5 Concluding Remark

In this paper, we studied the stabilization problem of a wave equation with unknown disturbance. Employing the idea of sliding mode control, we designed a nonlinear feedback control law. By using a trick and theory of the monotone operators, we proved the solvability of the corresponding closed-loop system. Furthermore, we construct a multiplier term add to the energy functional which is equivalent to the energy

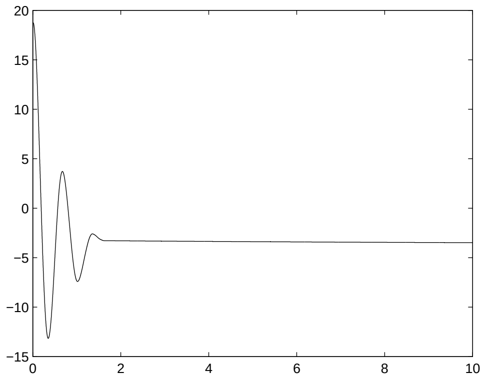

Figure 6: Displacement of the closed-loop system (19) at $x = 0.5$.

functional, and proved the exponential stability of the system.

We need to point out that the control strategy used in this paper is simple and valid, it can avoid the difficulty aroused the complex control design, including the solvability and stability analysis of the closed-loop system. As a test of theoretical result, we give some Numerical simulations. The simulation results show that the control law is effective, it has faster decay rate.

In addition, we emphases that the control method used in this paper also can be applied to other systems with distributed uncertain disturbances, such as beams, schrödinger equation. In future, we will extend this controller design method from the interior control to the boundary control.

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