Existence and nonexstence of positive solutions for a singular higher-order nonlinear fractional differential equation

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Abstract: In this paper, we establish some sufficient conditions for the existence and nonexstence of positive solutions to a general class of higher-order nonlinear fractional differential equation. The results are established by converting the problem into an equivalent integral equation and applying Banach fixed point theorem, nonlinear differentiation of Leray-Schauder type and the fixed point theorems of cone expansion and compression of norm type. As applications, some examples are also provided to illustrate our main results.

Key–Words: Fractional differential equations; Positive solutions; Boundary value problems; Fixed point theorem.

1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so on. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. For example, in physics, the traditional way to deal with the behavior of certain materials under the influence of external forces in mechanics uses the laws of Hooke and Newton. If we are dealing with viscous liquids, then we can use Newton's law $\eta\varepsilon'(t) = \sigma(t)$, where $\sigma(t)$ and $\varepsilon(t)$ denote stress and strain at time t respectively, η is the so-called viscosity of the material. In view of all some possible interpolation properties, it is natural for us to design the classical Newton's law according to

$$\eta D^{k}_{0+}\varepsilon(t) = \sigma(t),\ k \in (n-1, n),\ n \in \mathbb{N},$$

which is called Nutting's law [1]. In consequence, the subject of fractional differential equations is gaining much importance and attention. There are a large number of papers dealing with the existence, nonexistence, multiplicity of solutions and positive solutions with initial or boundary value problem for some nonlinear fractional differential equations. For details and examples, see [2–25] and the references therein. In [14], Wang have discussed the existence of positive solutions to the fractional boundary value problem with changing sign nonlinearity as follows:

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0. \end{cases}$$

In [9], Zhang have investigated the existence of multiple positive solutions for the fractional differential equation with a negatively perturbed term Dirichlet-type boundary value problem as follows:

$$\begin{cases} -D^{\alpha}_{0+}u(t) = p(t)f(t, u(t)) - q(t), \quad 0 < t < 1, \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville derivative, $2 < \alpha \le 3$ is a real number, $q : (0, 1) \to [0, +\infty)$ is Lebesgue integrable and does not vanish identically on any subinterval of (0,1).

However, to the best our knowledge, there is rare paper dealing with the singular higher-order nonlinear fractional differential equation. Motivated by the above mentioned discussions, in this paper, we will study the existence and nonexistence of positive solutions for the following higher-order singular nonlinear fractional differential equation (BVP for short):

$$\begin{cases} -D^{\alpha}_{0+}u(t) = p(t)f(t, u(t)) + q(t), \\ u(0) = u^{(k)}(0) = u(1) = 0, \end{cases} \tag{1}$$

where $0 < t < 1$, $k = 1, 2, \ldots, n-2$. D^{α}_{0+} is the standard Riemann-Liouville derivative of order $n - 1 < \alpha \le n, n \ge 3$. $f \in C([0, 1] \times$

$[0, +\infty)$, $[0, +\infty))$, f may be singular at $u = 0$. $p(t)$, $q(t) \in C([0, 1], [0, +\infty))$, $p(t)$ may be singular at $t = 0$ or / and at $t = 1$, and $q(t)$ is Lebesgue integrable and does not vanish identically on any subinterval of $(0, 1)$. By applying Banach fixed point theorem, nonlinear differentiation of Leray-Schauder type and the fixed point theorems of cone expansion and compression of norm type, sufficient conditions for the existence and nonexistence of positive solutions to a general class of boundary value problems for a higher-order singular nonlinear fractional differential equation are obtained.

The rest of this paper is organized as follows. In Section 2, we recall some useful denitions and properties, and present the properties of the Green's functions. In Section 3, we give some sufficient conditions for the existence or nonexistence of positive solutions for BVP (1). In Section 4, some examples are also provided to illustrate the validity of our main results. Finally, the conclusion is made to simply recall the methods, skills and applications of this paper in Section 5.

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions and lemmas on the fractional calculus theory.

Definition 1. *([26, 27]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by*

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2. *([26, 27]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by*

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 3. *([18]) Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

for some $C_i \in \mathbb{R}, i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 4. *([28]) Let E be a Banach space with $C \subseteq E$ closed and convex. Assume U is a relatively open subset of C with $\theta \in U$ and $T : \overline{U} \to C$ is a continuous compact map. Then either*

(a) *T has a fixed point in $\overline{U}$; or*

(b) *there exists $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda T u$.*

Lemma 5. ([29]) *Let E be a Banach space, $P \subseteq E$ a cone, and Ω_1, Ω_2 are two bounded open balls of E centered at the origin with $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) *$\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, or*

(ii) *$\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$*

holds. Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now we present the Green's function and its relative properties for the boundary value problem (1).

Lemma 6. *Given $h \in C[0, 1]$, and $n - 1 < \alpha \leq n$, the unique solution of* (2)

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0,\ 0 < t < 1, \\ u(0) = \cdots = u^{(k)}(0) = u(1) = 0. \end{cases} \tag{2}$$

is

$$u(t) = \int_0^1 G(t, s) h(s) \mathrm{d}s, \tag{3}$$

where $k = 1, 2, \ldots, n - 2$,

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, 0 \leq t \leq s \leq 1. \end{cases} \tag{4}$$

Proof: Applying Lemma 3, we get

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1} \\ &\quad + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \end{aligned}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. From $u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0$, we can obtain that $c_2 = c_3 = \cdots = c_{n-2} = 0$. Then

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1}.$$

By $u(1) = 0$, we have

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds.$$

Therefore, the unique solution of boundary value problem (2) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &\quad + t^{\alpha-1} \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds \right]. \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t t^{\alpha-1} (1-s)^{\alpha-1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1} (1-s)^{\alpha-1} h(s) ds \\ &= \int_0^1 G(t,s) h(s) ds, \end{aligned}$$

where $G(t,s)$ is defined by (4). Now, we will prove the uniqueness of solution for BVP (2). In fact, let $u_1(t)$, $u_2(t)$ are any two solutions of (2). Denote $w(t) = u_1(t) - u_2(t)$, then (2) be changed into the following system:

$$\begin{cases} D_{0^+}^{\alpha} w(t) = 0, \ t \in [0,1], \ n-1 < \alpha \le n, \\ w(0) = w'(0) = \cdots = w^{(n-2)}(0) = w(1) = 0. \end{cases}$$

Similar to above argument, we get $w(t) = 0$, that is $u_1(t) = u_2(t)$, which mean that the solution for BVP (2) is unique.The proof is complete. □

Lemma 7. *The function $G(t,s)$ is defined by* (4) *satisfies*

(i) $G(t,s) \ge 0$ *is continuous for all t,* $s \in [0,1]$, $G(t,s) > 0$ *for all t,* $s \in (0,1)$;

(ii) $G(t,s) \le G(\tau(s),s)$ *for each t,* $s \in [0,1]$, *and* $\min_{t\in[\theta,1-\theta]} G(t,s) \ge \gamma G(\tau(s),s), \forall s \in [0,1]$, *where* $\theta \in (0,1/2)$, γ *is a constant with* $0 < \gamma \le \theta^{\alpha-1}$ *and*

$$\tau(s) = \frac{s}{1-(1-s)^{(\alpha-1)/(\alpha-2)}} > s.$$

Similar to the methods of the Proposition 2.2 and Proposition 2.9 in the paper [13], we can prove Lemma 7. So we omit it.

3 Existence and nonexistence of positive solutions

In this section, we will discuss the existence and nonexistence of positive solutions for BVP (1).

Let $J_\theta = [\theta, 1-\theta]$ for $\theta \in (0,1/2)$ and $E = \{u(t) : u(t) \in C[0,1]\}$ denote a real Banach space with the norm $\|\cdot\|$ defined by $\|u\| = \max_{0\le t\le 1} \left|u(t)\right|$. Define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \ge 0\}.$$

Let

$$K = \left\{ u \in P : u \ge 0, \min_{t\in J_\theta} u(t) \ge \gamma \|u\| \right\}, \quad (5)$$

$$K_r = \{u \in K : \|u\| < r\}, \quad (6)$$

with $\partial K_r = \{u \in K : \|u\| = r\}$.

From Lemma 6, we can obtain the following important Lemma 8, which indicate that BVP (1) and (7) have the same solutions.

Lemma 8. *Suppose that $f(t,u)$ is continuous, then $u \in E$ is a solution of BVP* (1) *if and only if $u \in E$ is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s) \left[p(s) f(s,u(s)) + q(s)\right] ds. \quad (7)$$

Define the operator $T : E \to E$ as follows

$$(Tu)(t) = \int_0^1 G(t,s) \left[p(s) f(s,u(s)) + q(s)\right] ds. \quad (8)$$

Then by Lemma 8, the fixed point of operator T coincides with the solution of system (7).

Lemma 9. *Let $f(t,u)$ be continuous on $(0,1) \times [0,+\infty) \to [0,+\infty)$, then $T : P \to P$ and $T : K \to K$ defined by* (8) *is completely continuous.*

Proof: Firstly, we will show that $T : P \to P$ is uniformly bounded and equicontinuous. In fact, Let $u \in P$, in view of nonnegativeness and continuity of functions $G(t,s), p(t), q(t)$ and $f(t,u(t))$, we conclude that $T : P \to P$ is continuous.

Let $\Omega \in P$ be bounded, that is, there exists a positive constant $h > 0$ such that $\|u\| \le h$ for all $u \in \Omega$.

Let

$$M = \max_{0\le s\le 1, 0\le u\le h} \{\left|p(s) f(s,u(s)) + q(s)\right| + 1\}.$$

Then we have

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}(Tu)(t)\\ &\le \int_0^1 G(\tau(s),s)\left[p(s)f(s,u(s))+q(s)\right]ds\\ &\le M\int_0^1 G(\tau(s),s)ds.\end{aligned}$$

Hence, $T(\Omega)$ is uniformly bounded.

Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for fixed $s\in[0,1]$ and for any $\varepsilon>0$, there exists a constant $\delta>0$ such that any $t_1,t_2\in[0,1]$ and $|t_1-t_2|<\delta$,

$$|G(t_1,s)-G(t_2,s)|<\frac{\varepsilon}{M}.$$

Then

$$\begin{aligned}&|Tu(t_2)-Tu(t_1)|\\ \le\ & M\int_0^1 |G(t_1,s)-G(t_2,s)|\,ds<\varepsilon,\end{aligned}$$

that is to say, $T(P)$ is equicontinuous. By the means of the Arzela-Ascoli theorem, we have $T:P\to P$ is completely continuous.

Now we show that $T:K\to K$. In fact, For any $u\in K$, Lemma 7 implies that

$$\begin{aligned}(Tu)(t)\ \ \ge\ \ &\gamma\int_0^1 G(\tau(s),s)[p(s)f(s,u(s))\\ &+q(s)]ds,\ \ t\in J_\theta.\end{aligned}$$

On the other hand

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}(Tu)(t)\\ &\le \int_0^1 G(\tau(s),s)[p(s)f(s,u(s))+q(s)]ds.\end{aligned}$$

Then $(Tu)(t)\ge\gamma\|Tu\|$, which implies $T:K\to K$.

According to the Ascoli-Arzela theorem, we have proved that $T:K\to K$ is completely continuous operator. The proof is complete. □

Theorem 10. *Assume that $p(t)$ is continuous on $(0,1)\to[0,+\infty)$, and $f(t,u)$ is continuous on $(0,1)\times[0,+\infty)\to[0,+\infty)$, and there exists a positive functions $m(t)$ that satisfies*

$$\big|f(t,u_1)-f(t,u_2)\big|\le m(t)\big|u_1-u_2\big|,$$

for all $t\in(0,1),u_1,u_2\in[0,+\infty)$. Then BVP (1) has a unique positive solution if

$$\rho=\int_0^1 G(\tau(s),s)p(s)m(s)ds<1. \qquad (9)$$

Proof: For all $u\in P$, by the nonnegativeness of $G(t,s),p(t),q(t)$ and $f(t,u)$, we have $(Tu)(t)\ge 0$. Hence, $T(P)\subset P$. From Lemma 7, we obtain

$$\begin{aligned}&\|Tu_2-Tu_1\|=\max_{t\in[0,1]}\big|Tu_2-Tu_1\big|\\ &=\max_{t\in[0,1]}\bigg|\int_0^1 G(t,s)[p(s)f(s,u_1(s))+q(s)\\ &\qquad -p(s)f(s,u_2(s))-q(s)]ds\bigg|\\ &\le\int_0^1 |G(\tau(s),s)p(s)[f(s,u_1(s))-f(s,u_2(s))]|\,ds\\ &\le\int_0^1 G(\tau(s),s)p(s)m(s)ds\|u_1-u_2\|\\ &=\rho\|u_1-u_2\|.\end{aligned}$$

From Lemma 9, T is completely continuous. By (9) and the Banach contraction mapping principle, the operator T has a unique fixed point in P, which is the unique positive solution of BVP (1). This completes the proof. □

Theorem 11. *Assume that $p(t)$ and $q(t)$ are continuous on $(0,1)$, and $f(t,u)$ is continuous on $(0,1)\times[0,+\infty)$. For all $t\in[0,1]$, if there exist $c_1(t)\ge 0,c_2(t)\ge 0$ such that the following conditions (H_1) and (H_2) hold.*

(H_1) $\big|f(t,u(t))\big|\le c_1(t)+c_2(t)\big|u(t)\big|$.

(H_2) $C_1=\int_0^1 G(\tau(s),s)p(s)c_2(s)ds<1,$
$C_2=\int_0^1 G(\tau(s),s)p(s)c_1(s)ds<\infty,$
$C_3=\int_0^1 G(\tau(s),s)q(s)ds<\infty.$

Then the BVP (1) has at least one positive solution $u(t)$ in

$$Q=\left\{u\in P:\ \|u\|<\frac{C_2+C_3}{1-C_1}\right\}.$$

Proof: For convenience, we denote $r=\frac{C_2+C_3}{1-C_1}$. Define the operator $T:Q\to P$ as (8). Let $u\in Q$, that is, $\|u\|<r$. From Lemma 7, we obtain

$$\begin{aligned}\|Tu\|&=\max_{t\in[0,1]}\big|Tu(t)\big|\\ &=\max_{t\in[0,1]}\left|\int_0^1 G(t,s)[p(s)f(s,u(s))+q(s)]ds\right|\\ &\le\int_0^1 G(\tau(s),s)p(s)(c_1(s)+c_2(s)\big|u(s)\big|)ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 G(\tau(s),s)q(s)ds \\
&\le \int_0^1 G(\tau(s),s)p(s)c_1(s)ds \\
&\quad+\int_0^1 G(\tau(s),s)p(s)c_2(s)ds\|u\| \\
&\quad+\int_0^1 G(\tau(s),s)q(s)ds \\
&= C_2 + C_1\|u\| + C_3 < r.
\end{aligned}
$$

Thus $Tu \in \overline{Q}$. By Lemma 9, we have $T : Q \to \overline{Q}$ is completely continuous.

Consider the eigenvalue problem

$$u = \lambda Tu, \lambda \in (0,1). \tag{10}$$

Under the assumption that $u \in \partial Q$, that is, u is a solution of (10) for a $\lambda \in (0,1)$, one obtains

$$
\begin{aligned}
&\|u\| = \|\lambda Tu\| \\
&= \lambda \max_{t\in[0,1]} \left|\int_0^1 G(t,s)\left[p(s)f(s,u(s)) + q(s)\right]ds\right| \\
&\le \int_0^1 G(\tau(s),s)p(s)(c_1(s) + c_2(s)\left|u(s)\right|)ds \\
&\quad+\int_0^1 G(\tau(s),s)q(s)ds \\
&\le \int_0^1 G(\tau(s),s)p(s)c_1(s)ds \\
&\quad+\int_0^1 G(\tau(s),s)p(s)c_2(s)ds\|u\| \\
&\quad+\int_0^1 G(\tau(s),s)q(s)ds \\
&= C_2 + C_1\|u\| + C_3 < r.
\end{aligned}
$$

So $\|u\| \neq r$, which is contradiction with $u \in \partial Q$. According to Lemma 4, T has a fixed point $u \in \overline{Q}$, Therefore, BVP (1) has at least one positive solution. This completes the proof. □

In the rest of the paper, for the convenience of presentation, we always suppose that the following hypotheses hold.

(B_1) $p, q \in C((0,1),[0,+\infty)), p(t) \not\equiv 0$ and $q(t) \not\equiv 0$ on any subinterval of $(0,1)$.

(B_2) $f \in C((0,1)\times[0,+\infty),[0,+\infty))$, and $f(t,0) = 0$ uniformly with respect to t on $(0,1)$.

In addition, we introduce some notations as follows:

$$f^\delta = \limsup_{u\to\delta} \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$f_\delta = \liminf_{u\to\delta} \min_{t\in[0,1]} \frac{f(t,u)}{u},$$

where δ denotes 0 or $+\infty$, and

$$\sigma_1 = \int_0^1 G(\tau(s),s)p(s)ds,$$

$$\sigma_2 = \int_0^1 G(\tau(s),s)q(s)ds.$$

Theorem 12. *Assume that (B_1)–(B_2) hold. And suppose that one of the following conditions is satisfied:*

(H_3) *$f_0 > 1/(\gamma^2\sigma_1)$ and $f^\infty < 1/\sigma_1$ (particularly, $f_0 = \infty$ and $f^\infty = 0$).*

(H_4) *there exist two constants r_2, R_2 with $0 < r_2 \le R_2$ and $R_2 > \sigma_2$ such that $f(t,\cdot)$ is nondecreasing on $[0,R_2]$ for all $t \in [0,1]$, $f(t,\gamma r_2) \ge r_2/(\gamma\sigma_1)$, and $f(t,R_2) \le (R_2-\sigma_2)/\sigma_1$ for all $t \in [0,1]$.*

Then BVP (1) has at least one positive solution.

Proof: Let T be cone preserving completely continuous that is defined by (8).

Case 1. When the condition (H_3) holds. Considering $f_0 > 1/(\gamma^2\int_0^1 G(\tau(s),s)p(s)ds)$, there exists $r_1 > 0$ such that $f(t,u) \ge (f_0-\varepsilon_1)u$, for all $t \in [0,1], u \in [0,r_1]$, where $\varepsilon_1 > 0$, satisfies $(f_0-\varepsilon_1)\gamma^2\int_0^1 G(\tau(s),s)p(s)ds \ge 1$. Then, for $t \in [0,1], u \in \partial K_{r_1}$, from Lemma 7 and (5) we get

$$
\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}(Tu)(t) \\
&= \max_{t\in[0,1]} \int_0^1 G(t,s)\left[p(s)f(s,u(s)) + q(s)\right]ds \\
&\ge \max_{t\in J_\theta} \int_0^1 G(t,s)[p(s)f(s,u(s)) + q(s)]ds \\
&\ge \min_{t\in J_\theta} \int_0^1 G(t,s)p(s)f(s,u(s))ds \\
&\ge \gamma\int_0^1 G(\tau(s),s)p(s)f(s,u(s))ds \\
&\ge \gamma\int_0^1 G(\tau(s),s)p(s)(f_0-\varepsilon_1)u(s)ds \\
&\ge (f_0-\varepsilon_1)\gamma^2\int_0^1 G(\tau(s),s)p(s)ds\|u\| \ge \|u\|.
\end{aligned}
$$

Therefore,

$$\|Tu\| \ge \|u\|, \quad u \in \partial K_{r_1}. \tag{11}$$

On the other hand, for $f^\infty < 1/\sigma_1$, there exists $\overline{R}_1 > 0$ such that $f(t,u) \le (f^\infty+\varepsilon_2)u$ for all $t \in [0,1], u \in$

$(R_1, +\infty)$, where $\varepsilon_2 > 0$, satisfies $\sigma_1(f^\infty + \varepsilon_2) \le 1$. Set $M = \max_{t\in[0,1],u\in[0,R_1]} f(t,u)$, Then, $f(t,u) \le M + (f^\infty + \varepsilon_2)u$.

Choose $R_1 \ge \max\{r_1, R_1, (M\sigma_1 + \sigma_2)(1 - \sigma_1(f^\infty + \varepsilon_2))^{-1}\}$. Then for $t \in [0,1], u \in \partial K_{R_1}$, from Lemma 7 and $\|u\| = \max_{0\le t\le 1} \left|u(t)\right| = R_1$, we also get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} (Tu)(t) \\
&= \max_{t\in[0,1]} \int_0^1 G(t,s)\left[p(s)f(s,u(s)) + q(s)\right] ds \\
&\le \int_0^1 G(\tau(s),s)p(s)(M + (f^\infty + \varepsilon_2)u(s))ds \\
&\quad + \int_0^1 G(\tau(s),s)q(s)ds \\
&\le M \int_0^1 G(\tau(s),s)p(s)ds \\
&\quad + (f^\infty + \varepsilon_2) \int_0^1 G(\tau(s),s)p(s)ds\|u\| \\
&\quad + \int_0^1 G(\tau(s),s)q(s)ds \\
&= M\sigma_1 + \sigma_2 + (f^\infty + \varepsilon_2)\sigma_1 R_1 \\
&\le [1 - \sigma_1(f^\infty + \varepsilon_2)]R_1 + (f^\infty + \varepsilon_2)\sigma_1 R_1 \\
&= R_1 = \|u\|.
\end{aligned}$$

So, we have

$$\|Tu\| \le \|u\|, \quad u \in \partial K_{R_1}. \tag{12}$$

Case 2. When the condition (H_4) holds. From (5) and (6), for $u \in \partial K_{r_2}$, we have $\|u\| = r_2$ for $t \in J_\theta$. Therefore, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} (Tu)(t) \\
&= \max_{t\in[0,1]} \int_0^1 G(t,s)\left[p(s)f(s,u(s)) + q(s)\right] ds \\
&\ge \min_{t\in J_\theta} \int_0^1 G(t,s)p(s)f(s,u(s))ds \\
&\ge \gamma \int_0^1 G(\tau(s),s)p(s)f(s,u(s))ds \\
&\ge \gamma \frac{r_2}{\gamma \int_0^1 G(\tau(s),s)p(s)ds} \int_0^1 G(\tau(s),s)p(s)ds \\
&= r_2 = \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| \ge \|u\|, \quad u \in \partial K_{r_2}. \tag{13}$$

On the other hand, for $u \in \partial K_{R_2}$, we have $\|u\| = R_2$ for $t \in [0,1]$, from Lemma 7 and (H_4), we obtain

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} (Tu)(t) \\
&= \max_{t\in[0,1]} \int_0^1 G(t,s)\left[p(s)f(s,u(s)) + q(s)\right] ds \\
&= \int_0^1 G(\tau(s),s)\left[p(s)f(s,u(s)) + q(s)\right] ds \\
&\le \frac{R_2 - \sigma_2}{\sigma_1} \int_0^1 G(\tau(s),s)p(s)ds \\
&\quad + \int_0^1 G(\tau(s),s)q(s)ds \\
&= R_2 = \|u\|.
\end{aligned}$$

So, we have

$$\|Tu\| \le \|u\|, \quad u \in \partial K_{R_2}. \tag{14}$$

Applying Lemma 5 to (11) and (12), or (13) and (14), one yields that T has a fixed point $u \in P \cap (K_{R_1} \setminus K_{r_1})$ or $u \in P \cap (K_{R_2} \setminus K_{r_2})$ with $u(t) \ge \gamma\|u\| > 0$, $t \in [0,1]$. Thus it follows that BVP (1) has a positive solution u. We complete the proof of Theorem 12. □

Similarly, we have the following result.

Theorem 13. *Assume that (B_1)–(B_2) hold. And supposes that the following condition is satisfied.*

(H_5) $f^0 < 1/\sigma_1$ *and* $f_\infty > 1/(\gamma^2\sigma_1)$*(particularly,* $f^0 = 0$ *and* $f_\infty = \infty$*).*

Then BVP (1) *has at least one positive solution.*

Theorem 14. *Assume that (B_1)–(B_3) hold. And supposes that the following two conditions are satisfied.*

(H_6) $f_0 > 1/(\gamma^2\sigma_1)$ *and* $f_\infty > 1/(\gamma^2\sigma_1)$ *(particularly,* $f_0 = f_\infty = \infty$*.)*

(H_7) *there exist a constant* $c > \sigma_2 > 0$ *and a closed interval* $[a,b] \subset [0,1]$*, such that* $\max_{t\in[a,b],u\in\partial K_c} f(t,u) < (c - \sigma_2)/\sigma_1$.

Then BVP (1) *has at least two positive solutions* u_1, u_2*, which satisfy*

$$0 < \|u_1\| < c < \|u_2\|. \tag{15}$$

Proof: On the one hand, we consider condition (H_6). Choose r, R with $0 < r < 1 < R$. If $f_0 > 1/(\gamma^2\sigma_1)$, then similar to the proof of (11), we have

$$\|Tu\| \ge \|u\|, \quad u \in \partial K_r. \tag{16}$$

If $f_\infty > 1/(\gamma^2\sigma_1)$, then similar to the proof of (11), we also have

$$\|Tu\| \ge \|u\|, \quad u \in \partial K_R. \tag{17}$$

On the other hand, together with $(H_7), u \in \partial K_c$, we have

$$
\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}(Tu)(t) \\
&= \max_{t\in[0,1]}\int_0^1 G(t,s)\left[p(s)f(s,u(s))+q(s)\right]ds \\
&= \int_0^1 G(\tau(s),s)\left[p(s)f(s,u(s))+q(s)\right]ds \\
&< \frac{c-\sigma_2}{\sigma_1}\int_0^1 G(\tau(s),s)p(s)ds \\
&\quad + \int_0^1 G(\tau(s),s)q(s)ds = c = \|u\|.
\end{aligned}
$$

Therefore,

$$\|Tu\| < \|u\|, \quad u \in \partial K_c. \tag{18}$$

Applying Lemma 5 to (16)-(18), one yields that T has a fixed point $u_1 \in P\cap(\overline{K}_c \setminus K_r)$, and a fixed point $u_2 \in P \cap (\overline{K}_R \setminus K_c)$ with $u_i(t) \geq \gamma\|u\| > 0$, $t \in [0,1], i = 1,2$. Thus it follows that BVP (1) has at least two positive solutions u_1 and u_2. Noticing (18), we have $\|u_1\| \neq c, \|u_2\| \neq c$. Therefore (15) holds, We complete the proof of Theorem 14. □

Similarly, we have the following results.

Theorem 15. *Assume that (B_1)–(B_2) hold. And supposes that the following two conditions are satisfied:*

(H_8) $f^0 < 1/\sigma_1$ *and* $f^\infty < 1/\sigma_1$(*particularly,* $f^0 = f^\infty = 0$).

(H_9) *there exist a constant $B > 0$ and a closed interval $[a,b] \subset [0,1]$, such that*

$$\max_{t\in[a,b],u\in\partial K_B} f(t,u) > B/(\gamma\sigma_1).$$

Then BVP (1) has at least two positive solutions u_1, u_2, which satisfy

$$0 < \|u_1\| < B < \|u_2\|.$$

In the following arguments, we focus on the results of nonexistence of positive solutions for BVP (1).

Theorem 16. *Assume that (B_1)–(B_2) hold. And supposes that one of the following conditions is satisfied.*

(H_{10}) $f(t,u) < (u-\sigma_2)/\sigma_1, \forall\, t \in [0,1], u > \sigma_2$.

(H_{11}) $f(t,u) > u/\gamma^2\sigma_1, \forall\, t \in [0,1], u > 0$.

Then BVP (1) has no positive solution.

Proof: Assume to the contrary that $u(t)$ is a positive solution of the BVP (1). Then, $u \in K, u(t) > 0$ for $t \in (0,1)$.

Case 1. For $u > \sigma_2$, from Lemma 7 and $\|u\| = \max_{0\leq t\leq 1}\left|u(t)\right|$ we get

$$
\begin{aligned}
&\|u\| = \max_{t\in[0,1]}|u(t)| \\
&= \max_{t\in[0,1]}\int_0^1 G(t,s)\left[p(s)f(s,u(s))+q(s)\right]ds \\
&< \int_0^1 G(\tau(s),s)p(s)\frac{u-\sigma_2}{\sigma_1}ds \\
&\quad + \int_0^1 G(\tau(s),s)q(s)ds \\
&\leq \int_0^1 G(\tau(s),s)p(s)\frac{\|u\|}{\sigma_1}ds \\
&\quad - \frac{\sigma_2}{\sigma_1}\int_0^1 G(\tau(s),s)p(s)ds + \int_0^1 G(\tau(s),s)q(s)ds \\
&= \frac{1}{\sigma_1}\int_0^1 G(\tau(s),s)p(s)ds\|u\| = \|u\|.
\end{aligned}
$$

which is a contradiction. Thus BVP (1) has no positive solution.

Case 2. For $u > 0$, from Lemma 7 and (5) we also get

$$
\begin{aligned}
\|u\| &= \max_{t\in[0,1]}|u(t)| \\
&= \max_{t\in[0,1]}\int_0^1 G(t,s)\left[p(s)f(s,u(s))+q(s)\right]ds \\
&\geq \min_{t\in J_\theta}\int_0^1 G(t,s)p(s)f(s,u(s))ds \\
&\geq \gamma\int_0^1 G(\tau(s),s)p(s)f(s,u(s))ds \\
&> \gamma\int_0^1 G(\tau(s),s)p(s)\frac{u(s)}{\gamma^2\sigma_1}ds \\
&\geq \frac{\gamma^2}{\gamma^2\sigma_1}\int_0^1 G(\tau(s),s)p(s)ds\|u\| = \|u\|.
\end{aligned}
$$

which is a contradiction. Therefore, BVP (1) has no positive solution. The proof is complete. □

4 Illustrative example

Example 17. *Consider the following BVP of nonlinear fractional differential equations:*

$$
\begin{cases}
-D_{0^+}^{\frac{5}{2}}u = \dfrac{|\sin u|\log_{\frac{1}{2}} t(1-t)}{1+\log_{\frac{1}{2}} t(1-t)} + \dfrac{1}{2^t}, \\
u(0) = u'(0) = u''(0) = u(1) = 0,
\end{cases} \tag{19}
$$

where, $p(t) = 1$, $q(t) = \frac{1}{2^t}$, $f(t,u) = \frac{|\sin u| \log_{\frac{1}{2}} t(1-t)}{1+\log_{\frac{1}{2}} t(1-t)}$.

For all $u_1, u_2 \in [0, +\infty)$, $t \in [0,1]$, we have

$$|f(t,u_1) - f(t,u_2)| \leq \frac{\log_{\frac{1}{2}} t(1-t)}{1+\log_{\frac{1}{2}} t(1-t)} |u_1 - u_2|.$$

So, taking $m(t) = \frac{\log_{\frac{1}{2}} t(1-t)}{1+\log_{\frac{1}{2}} t(1-t)} \geq \frac{2}{3} > 0$, we obtain

$$\begin{aligned}\rho &= \int_0^1 G(\tau(s), s) p(s) m(s) ds \\ &\leq \int_0^1 G(\tau(s), s) ds < \frac{1}{\Gamma(\frac{5}{2})} \int_0^1 (1-s)^{\frac{3}{2}} ds \\ &= \frac{1}{\Gamma(\frac{7}{2})} \approx 0.3009 < 1,\end{aligned}$$

$$C_1 = \rho < 1, \quad C_2 = \int_0^1 G(\tau(s), s) ds < \infty,$$

$$C_3 = \int_0^1 G(\tau(s), s) \frac{1}{2^s} ds < \int_0^1 G(\tau(s), s) ds < \infty.$$

According to Theorem 10 or Theorem 11, BVP (19) has a unique positive solution.

Example 18. *Consider the following BVP of nonlinear fractional differential equations:*

$$\begin{cases} -D_{0^+}^{\frac{5}{2}} u = \frac{1}{t(1-t)} [u]^a + \frac{1}{2^t}, \\ u(0) = u'(0) = u''(0) = u(1) = 0. \end{cases} \tag{20}$$

Let $f(t,u) = \frac{1}{t(1-t)} u^a, p(t) = 1, q(t) = \frac{1}{2^t}$.

(i) when $0 < a < 1$. It is easy to see that (B_1)–(B_2) hold. By simple computation, we have $f_0 = \infty, f^\infty = 0$, which satisfies the condition (H_3). Thus it follows that BVP (20) has a positive solution by Theorem 12.

(ii) when $1 < a < \infty$. It is easy to see that (B_1)–(B_2) hold. By simple computation, we have $f^0 = 0, f_\infty = \infty$, which satisfies the condition (H_5). Thus it follows that BVP (20) has a positive solution by Theorem 13.

Example 19. *Consider the following BVP of nonlinear fractional differential equations:*

$$\begin{cases} -D_{0^+}^{\frac{5}{2}} u = \left| \frac{u \ln u}{t(1-t)} \right| + \frac{1}{2^t}, \\ u(0) = u'(0) = u''(0) = u(1) = 0. \end{cases} \tag{21}$$

Let $[\frac{1}{4}, \frac{3}{4}] \subset [0,1]$, $c = 1$, $f(t,u) = \left| \frac{u \ln u}{t(1-t)} \right|$, $p(t) = 1$, $q(t) = \frac{1}{2^t} \in (\frac{1}{2}, 1)$, so $\frac{\sigma_2}{\sigma_1} \in (\frac{1}{2}, 1)$. It is easy to see that $(B_1) - (B_2)$ hold. By simple computation,

$$\begin{aligned} f_0 &= \liminf_{u\to 0} \min_{t\in[0,1]} \left| \frac{u \ln u}{t(1-t)u} \right| \\ &= \liminf_{u\to 0} 4|\ln u| = \infty, \end{aligned}$$

$$\begin{aligned} f_\infty &= \liminf_{u\to \infty} \min_{t\in[0,1]} \left| \frac{u \ln u}{t(1-t)u} \right| \\ &= \liminf_{u\to \infty} 4|\ln u| = \infty. \end{aligned}$$

By $u \in \partial K_c$, we have $\|u\| = \max_{t\in[0,1]} |u(t)| = 1$, so $0 < u(t) \leq 1, f(t,u) = \frac{-u \ln u}{t(1-t)}$. Noting that $f(t,u)$ arrive at maximum at $u = 1/e, t = 1/4$ or $t = 3/4$, we get

$$\max_{t\in[1/4,3/4], u\in\partial K_c} f(t,u) \leq \frac{16}{3e} \approx 1.9620,$$

and

$$\begin{aligned} \frac{c - \sigma_2}{\sigma_1} &> \frac{1}{\int_0^1 G(\tau(s), s) ds} - 1 \\ &\approx \frac{1}{0.0636} - 1 \approx 14.7233. \end{aligned}$$

So $\max_{t\in[1/4,3/4], u\in\partial K_c} f(t,u) < (c - \sigma_2)/\sigma_1, c > \sigma_1 > \sigma_2 > 0$. Thus it follows that BVP (21) has at least two positive solutions u_1, u_2, with $0 < \|u_1\| < 1 < \|u_2\|$ by Theorem 14.

5 Conclusions

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so on, and involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. There are a large number of papers dealing with the existence, nonexistence, multiplicity of solutions and positive solutions with initial or boundary value problem for some nonlinear fractional differential equations. In this paper, we have studied the existence and nonexistence of positive solutions for a class of boundary value problems involving in higher-order singular nonlinear fractional differential equation. By applying Banach fixed point theorem, nonlinear differentiation of Leray-Schauder type and the fixed point theorems of cone expansion and compression of norm type, sufficient conditions for the existence and nonexistence of positive solutions have been obtained. The our results obtained are new and interesting and the our methods can be used to study the existence of positive solutions for other types of boundary value problems of nonlinear fractional differential equation.

Acknowledgements: The author thanks the referees for a number of suggestions which have improved many aspects of this article. This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant (No. 11161025), Yunnan Province natural scientific research fund project (No. 2011FZ058).

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