Exact Solution of Time-Fractional Partial Differential Equations Using Sumudu Transform

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Abstract: In this study, we propose a new algorithm to find exact solution of nonlinear time- fractional partial differential equations. The new algorithm basically illustrates how two powerful algorithms, the homotopy perturbation method and the Sumudu transform method can be combined and used to get exact solutions of fractional partial differential equations. We also present four different examples to illustrate the preciseness and effectiveness of this algorithm.

Key–Words: Mitting-Leffler functions; biological population model; Caputo derivative; Sumudu transform; Fokker-Plank equation.

1 Introduction

Nonlinear partial differential equations (NPDE) with integer or fractional order have played a very important role in various fields of science and engineering, such as mechanics, electricity, chemistry, biology, control theory, signal processing and image processing. In all these scientific fields, it is important to obtain exact or approximate solutions of nonlinear fractional partial differential equations (NFPDE). But in general, there exists no method that gives an exact solution for NFPDEs and most of the obtained solutions are only approximations. Searching of exacts solutions of NFPDEs in mathematical and other scientific applications is still quite challenging and needs new methods. Computing the exact solution of these equations is of considerable importance, because the exact solutions can help to understand the mechanism and complexity of phenomena that have been modeled by NPDEs with integer or fractional order.

Numerous analytical and numerical methods have been presented in recent years. Some of these analytical methods are Fourier transform method [28], the fractional Greens function method [27], the popular Laplace transform method [32, 33], the Sumudu transform method [11], the Iteration method [33], the Mellin transform method and the method of orthogonal polynomials [32].

Some numerical methods are also popular, such us the Homotopy Perturbation Method (HPM) [12, 13, 14], the Modified Homotopy Perturbation Method (MHPM) [20] , the Differential Transform Method (DTM) [29], the Variational Iteration Method (VIM) [9], the New Iterative Method (NIM) [5, 6], the Homotopy Analysis Method (HAM) [1, 15], the Sumudu decomposition method [8], the Adomian Decomposition Method (ADM) [2, 7], and the Iterative Laplace Transform Method (ILTM)[19].

Among these methods, the HPM is a universal approach which can be used to solve FODEs and FPDEs. on the other hand, various methods are combined with the homotopy perturbation method, such as the Variational Homotopy Perturbation Method which is a combination of the variational iteration method and the homotopy perturbation method [31]. Another such combination is the Homotopy Perturbation Transformation Method which was constructed by combining two powerful methods; namely, the homotopy perturbation method and the Laplace transform method [26]. A third such approach is the Homo-Separation of Variables which was constructed by combining two well-established mathematical methods, namely, the homotopy perturbation method and the separation of variables method [21, 22, 23].

There are numerous integral transforms such as the Laplace, Sumudu, Fourier, Mellin, and Hankel to solve PDEs. Of these, the Laplace transformation and Sumudu transformation are the most widely used. The Sumudu transformation method is one of the most important transform methods introduced in the early 1990s by Gamage K. Watugala. It is a powerful tool for solving many kinds of PDEs in various fields of science and engineering [30, 34]. And also various methods are combined with the Sumudu transformation method such as the Homotopy Analysis Sumudu Transform Method (HASTM) [35] which is a combination of the homotopy analysis method and the Sumudu transformation method. Another example is the Sumudu Decomposition Method (SDM) [25], which is a combination of the Sumudu transform method and the adomian decomposition method.

In this paper, a new approach is proposed to use the homotopy perturbation Sumudu transform method (HPSTM) to derive the exact solution of various types of fractional partial differential equations. This method is a combination of the homotopy perturbation method and the Sumudu transform method. However, Singh [36] used the homotopy perturbation Sumudu transform method to obtain the exact solution of nonlinear equations which are PDEs of integer order.

The paper is structured in six sections. In section 2, we begin with an introduction to some necessary definitions of fractional calculus theory. In section 3 we describe the basic idea of the homotopy perturbation method. In section 4 we describe the homotopy perturbation Sumudu transform method. In section 5, we present four examples to show the efficiency of using the HPSTM to solve FPDEs and also to compare our result with those obtained by other existing methods. Finally, relevant conclusions are drawn in section 6.

2 Basic Definitions of Fractional Calculus

In this section, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 1 *A real function $f(t)$, $t > 0$, is said to be in the space $C_\sigma, \sigma \in \mathbb{R}$, if there exists a real number $p > \sigma$ such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_σ^m if $f^m \in C_\sigma$, $m \in \mathbb{N}$.*

Definition 2 *The left sided Riemann–Liouville fractional integral of order $\alpha \geq 0$, of a function $f \in C_\sigma$, $\sigma \geq -1$ is defined as:*

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta)\, d\zeta, \tag{1}$$

where $\alpha > 0$, $t > 0$ and $\Gamma(\alpha)$ is the gamma function.

Definition 3 *Let $f \in C_\mu^n$, $n \in \mathbb{N} \cup \{0\}$. The left sided Caputo fractional derivative of f in the Caputo sense is defined by [31] as follows:*

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} f^{(n)}(\zeta)\, d\,\zeta, \\ \qquad\qquad n-1 < \alpha \leq n, \\ D_n^t f(t) \qquad\qquad \alpha = n, \end{cases} \tag{2}$$

Note that according to [1], Eqs.(1) and (2) become:

$$J_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(x,\zeta) d\zeta \tag{3}$$

for $\alpha > 0, t > 0$, and

$$D_t^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\zeta)^{n-\alpha-1} f^{(n)}(x,\zeta) d\zeta \tag{4}$$

for $n-1 < \alpha \leq n$.

Definition 4 *The single parameter and the two parameters variants of the Mitting-leffler function are denoted by $E_\alpha(t)$ and $E_{\alpha,\beta}(t)$, respectively, which are relevant for their connection with fractional calculus, and are defined as:*

$$E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + 1)},\ \alpha > 0,\ t \in \mathbb{C}, \tag{5}$$

$$E_{\alpha,\beta}(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(\alpha j + \beta)},\ \ \alpha,\beta > 0,\ t \in \mathbb{C} \tag{6}$$

Some special cases of the Mitting-Leffler function are as follows:

1. $E_1(t) = e^t$
2. $E_{\alpha,1}(t) = E_\alpha(t)$
3. $\frac{d^k}{dt^k}\left[t^{\beta-1} E_{\alpha,\beta}(at^\alpha)\right] = t^{\beta-k-1} E_{\alpha,\beta-k}(at^\alpha)$

Other properties of the Mitting-leffler functions can be found in [24]. These functions are generalizations of the exponential function, because, most linear differential equations of fractional order have solutions that are expressed in terms of these functions.

Definition 5 *Sumudu transform over the following set of functions,*

$$A = \left\{ f(t) \;\middle|\; \begin{array}{l} \exists M,\ \tau_1, \tau_2 > 0,\ |f(t)| < Me^{\frac{|t|}{\tau_j}} \\ if\ t \in (-1)^j \times [0, \infty) \end{array} \right\}, \tag{7}$$

is defined by

$$\mathbb{S}[f(t)] = G(u) = \int_0^\infty f(ut)e^{-t}dt, \tag{8}$$

where $u \in (\tau_1, \tau_2)$*,*

Some special properties of the Sumudu transform are as follows:

1. $\mathbb{S}\left[1\right] = 1;$
2. $\mathbb{S}\left[\frac{t^n}{\Gamma(n+1)}\right] = u^n, \quad n > 0;$
3. $\mathbb{S}\left[e^{at}\right] = \frac{1}{1-au};$
4. $\mathbb{S}\left[\alpha f\left(x\right) + \beta g\left(x\right)\right] = \alpha \mathbb{S}\left[f\left(x\right)\right] + \mathbb{S}\left[g\left(x\right)\right]$

Other properties of the Sumudu transform can be found in [3].

Definition 6 $G(u)$ *is the Sumudu transform of* $f(t)$*. And according to ref. [3] we have:*

1. $G(1/s)/s$*, is a meromorphic function, with singularities having* $Re\left(s\right) < \gamma$*, and*
2. *there exists a circular region* Γ *with radius R and positive constants, M and k, with*

$$\left|\frac{G(1/s)}{s}\right| < MR^{-k},$$

then the function $f(t)$ *is given by*

$$\begin{aligned} \mathbb{S}^{-1}[G(s)] &= \tfrac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}G\left(\tfrac{1}{s}\right)\tfrac{ds}{s} \\ &= \sum residuse\left[e^{st}\tfrac{G(1/s)}{s}\right] \end{aligned} \tag{9}$$

Definition 7 *The Sumudu transform,* $\mathbb{S}\left[f\left(t\right)\right]$*, of the Caputo fractional integral is defined as [11]*

$$\mathbb{S}\left[D_t^\alpha f\left(t\right)\right] = \frac{G\left(u\right)}{u^\alpha} - \sum_{k=0}^{n-1}\frac{f^{(k)}\left(0\right)}{u^{\alpha-k}}, \tag{10}$$

then it can be easily understood that

$$\mathbb{S}\left[D_t^\alpha f\left(x,t\right)\right] = \frac{\mathbb{S}\left[f\left(x,t\right)\right]}{u^\alpha} - \sum_{k=0}^{n-1}\frac{f^{(k)}\left(x,0\right)}{u^{\alpha-k}}, \quad n-1 < \alpha \leq n, \tag{11}$$

3 The Basic Idea of the Homotopy Perturbation Method

In this section, we will briefly present the algorithm of this method. At first, the following nonlinear differential equation is considered:

$$\mathcal{A}\left(u\right) - f\left(x\right) = 0, \quad x \in \Omega, \tag{12}$$

with the boundary conditions

$$B\left(u, \partial u/\partial n\right) = 0, \quad x \in \Gamma, \tag{13}$$

where $\mathcal{A}$, B, $f\left(x\right)$ and Γ are a general differential function operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

The operator $\mathcal{A}$ can be decomposed into a linear operator, denoted by $\mathbf{L}$, and a nonlinear operator, denoted by $\mathbf{N}$. Therefore, Eq.(12) can be written as follows

$$\mathbf{L}\left(u\right) + \mathbf{N}\left(u\right) - f\left(x\right) = 0, \tag{14}$$

Now we construct a homotopy $v(x,p) : \Omega \times [0,1] \rightarrow \mathbb{R}$ which satisfies:

$$\begin{aligned} \mathbf{H}\left(v,p\right) = {} & (1-p)\left[\mathbf{L}\left(v\right) - \mathbf{L}\left(u_0\right)\right] + \\ & p[\mathcal{A}\left(u\right) - f\left(x\right)] = 0, \quad 0 \leq p \leq 1, \end{aligned} \tag{15}$$

which is equivalent to

$$\begin{aligned} \mathbf{H}\left(v,p\right) = {} & \mathbf{L}\left(v\right) - \mathbf{L}\left(u_0\right) + p\mathbf{L}\left(u_0\right) + \\ & p\left[\mathbf{N}\left(v\right) - f\left(x\right)\right] = 0, \quad 0 \leq p \leq 1, \end{aligned} \tag{16}$$

where u_0 is the initial approximation of Eq.(12) that satisfies the boundary conditions and p is an embedding parameter.

When the value of p is changed from zero to unity, we can easily see that

$$\mathbf{H}\left(v,0\right) = \mathbf{L}\left(v\right) - \mathbf{L}\left(u_0\right) = 0, \tag{17}$$

$$\begin{aligned} \mathbf{H}\left(v,1\right) &= \mathbf{L}\left(v\right) + \mathbf{N}\left(v\right) - f\left(x\right) \\ &= \mathcal{A}\left(u\right) - f\left(x\right) = 0, \end{aligned} \tag{18}$$

In topology, this changing process is called deformation, and Eqs. (17) and (18) are called homotopic.

If the p-parameter is considered as small, then the solution of Eqs. (14) and (15) can be expressed as a power series in p as follows

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \tag{19}$$

The best approximation for the solution of Eq. (12) is

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{20}$$

4 The Homotopy Perturbation Sumudu Transform Method

In order to elucidate the solution procedure of this method, we consider a general fractional nonlinear partial differential equation of the form:

$$D_t^{\alpha} w(x,t) = \mathbf{L}w(x,t) + \mathbf{N}w(x,t) + q(x,t), \quad (21)$$

with $n-1 < \alpha \leq n$, and subject to the initial condition

$$\frac{\partial^{(r)} w(x,0)}{\partial t^r} = w^{(r)}(x,0) = f_r(x), \quad r = 0,1,\ldots,n-1, \quad (22)$$

where $D_t^{\alpha} w(x,t)$ is the Caputo fractional derivative, $q(x,t)$ is the source term, $\mathbf{L}$ is the linear operator and $\mathbf{N}$ is the general nonlinear operator.

Taking the Sumudu transform (denoted throughout this paper by $\mathbb{S}$) on both sides of Eq. (21) we get

$$\mathbb{S}[D_t^{\alpha} w(x,t)] = \mathbb{S}[\mathbf{L}w(x,t) + \mathbf{N}w(x,t) + q(x,t)], \quad (23)$$

Using the property of the Sumudu transform and the initial conditions in Eq.(22), we have

$$u^{-\alpha}\mathbb{S}[w(x,t)] - \sum_{k=0}^{n-1} u^{-(\alpha-k)} w^{(k)}(x,0) = \mathbb{S}[\mathbf{L}w(x,t) + \mathbf{N}w(x,t) + q(x,t)], \quad (24)$$

and

$$\mathbb{S}[w(x,t)] = \sum_{k=0}^{n-1} u^k f_k(x) + u^{\alpha}\mathbb{S}[\mathbf{L}w(x,t) + \mathbf{N}w(x,t) + q(x,t)], \quad (25)$$

Operating with the Sumudu inverse on both sides of Eq.(25) we get:

$$w(x,t) = \mathbb{S}^{-1}[\sum_{k=0}^{n-1} u^k f_k(x)] + \mathbb{S}^{-1}[u^{\alpha}\mathbb{S}[\mathbf{L}w(x,t) + \mathbf{N}w(x,t) + q(x,t)]], \quad (26)$$

Now, applying the classical perturbation technique. And assuming that the solution of Eq. (26) is in the form

$$w(x,t) = \sum_{n=0}^{\infty} p^n w_n(x,t), \quad (27)$$

where $p \in [0,1]$ is the homotopy parameter.

The nonlinear term of Eq.(26) can be decomposed as

$$\mathbf{N}w(x,t) = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(w), \quad (28)$$

where $\mathcal{H}_i$ are He's polynomials, which can be calculated with the formula [10]

$$\mathcal{H}_n(w_0, w_1, w_2, \ldots w_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\mathbf{N}\left(\sum_{i=0}^{\infty} p^i w_i\right)\right]_{p=0}, \quad n = 0,1,2,\ldots, \quad (29)$$

Substituting Eqs. (27) and (28) in Eq. (26), we get

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = \mathbb{S}^{-1}\left[\sum_{k=0}^{n-1} u^k f_k(x)\right] + p\mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[\mathbf{L}\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right) + \sum_{n=0}^{\infty} p^n \mathcal{H}_n(w) + q(x,t)\right]\right], \quad (30)$$

Equating the terms with identical powers of p, we can obtain a series of equations as the follows:

$$\begin{aligned} p^0: &\quad w_0(x,t) = \mathbb{S}^{-1}\left[\sum_{k=0}^{n-1} u^k f_k(x)\right], \\ \ldots &\quad \ldots \\ p^n: &\quad w_n(x,t) = \mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[\mathbf{L}\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right) + \sum_{n=0}^{\infty} p^n \mathcal{H}_n(w) + q(x,t)\right]\right], \end{aligned} \quad (31)$$

By utilizing the results in Eq. (31), and substituting them into Eq. (27) then the solution of Eq. (21) can be expressed as a power series in p.

The best approximation for the solution of Eq. (21) is:

$$\begin{aligned} w(x,t) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n w_n(x,t) \\ &= w_0 + w_1 + w_2 + \cdots \end{aligned} \quad (32)$$

The solutions of Eq.(32) generally converge very rapidly (see [4, 17]).

5 Applications

In this section, in order to assess the applicability and the accuracy of the fractional homotopy Sumudu transform method in the last section, we consider the following four examples. In example 8, a case of the one variable time-fractional Fokker-Plank equation is solved. In example 9, we have generalized biological population model. In example 10, a two-dimensional fractional wave equation is solved and in example 11, a systems of linear FPDEs is solved.

Example 8 *Consider the linear time-fractional Fokker-Plank equation with the initial conditions of the form:*

$$D_t^\alpha w(x,t) = -\frac{\partial}{\partial x}\left(\frac{4}{x}w(x,t) - \frac{x}{3}\right)w(x,t) + \frac{\partial^2}{\partial x^2}w^2(x,t), \tag{33}$$

where $t > 0,\ x \in \mathbb{R},\ 0 < \alpha \le 1$, *subject to the initial condition*

$$w(x,0) = x^2 \tag{34}$$

Taking the Sumudu transform on both sides of Eq.(33), thus we get

$$\mathbb{S}[D_t^\alpha w(x,t)] = \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}w(x,t) - \frac{x}{3}\right)w(x,t) + \frac{\partial^2}{\partial x^2}w^2(x,t)\right]. \tag{35}$$

Using the property of the Sumudu transform and the initial condition in Eq.(34), we have

$$\mathbb{S}[w(x,t)] = x^2 + u^\alpha \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}w(x,t) - \frac{x}{3}\right)w(x,t) + \frac{\partial^2}{\partial x^2}w^2(x,t)\right], \tag{36}$$

Operating with the Sumudu inverse on both sides of Eq.(36) we get

$$w(x,t) = x^2 + \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}w(x,t) - \frac{x}{3}\right)w(x,t) + \frac{\partial^2}{\partial x^2}w^2(x,t)\right]\right]. \tag{37}$$

By applying the homotopy perturbation method, and substituting Eq.(27) in Eq.(37) we have

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = x^2 + p\mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}\sum_{n=0}^{\infty} p^n w_n(x,t) - \frac{x}{3}\right) \times \sum_{n=0}^{\infty} p^n w_n(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right)^2\right]\right]. \tag{38}$$

Equating the terms with identical powers of p, we get

$$\begin{aligned}
p^0 :&\quad w_0(x,t) = x^2,\\
p^1 :&\quad w_1(x,t) = \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}w_0(x,t) - \frac{x}{3}\right)\times w_0(x,t) + \frac{\partial^2}{\partial x^2}{w_0}^2(x,t)\right]\right]\\
&\quad = x^2\frac{t^\alpha}{\Gamma(\alpha+1)},\\
p^2 :&\quad w_2(x,t) = \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}w_2(x,t) - \frac{x}{3}\right)\times w_2(x,t) + \frac{\partial^2}{\partial x^2}{w_2}^2(x,t)\right]\right]\\
&\quad = x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad \cdots \quad \cdots\\
p^n :&\quad w_n(x,t) = \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[-\frac{\partial}{\partial x}\left(\frac{4}{x}w_n(x,t) - \frac{x}{3}\right)\times w_n(x,t) + \frac{\partial^2}{\partial x^2}{w_n}^2(x,t)\right]\right]\\
&\quad = x^2\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}
\end{aligned} \tag{39}$$

Thus the solution of Eq.(33) is given by

$$\begin{aligned}
w(x,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n w_n(x,t)\\
&= x^2 + x^2\frac{t^\alpha}{\Gamma(\alpha+1)} + x^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\\
&= x^2 E_\alpha(t^\alpha).
\end{aligned} \tag{40}$$

If we put $\alpha \to 1$ in Eq.(40) or solve Eq. (33) and (34) with $\alpha = 1$, we obtain the exact solution

$$w(x,t) = x^2 e^t$$

which is in full agreement with the result in Ref. [21, 40].

Example 9 *Consider the generalized biological population model of the form*

$$D_t^\alpha w(x,y,t) = \left(\frac{\partial^2 w^2(x,y,t)}{\partial x^2} + \frac{\partial^2 w^2(x,y,t)}{\partial y^2}\right) + w(x,y,t)(1 - rw(x,y,t)) \quad (41)$$

where $0 < \alpha \le 1$, *subject to the initial condition*

$$w(x,y,0) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}, \quad x,y,r \in \mathbb{R} \qquad (42)$$

Taking the Sumudu transform on both sides of Eq.(41), thus we get

$$\mathbb{S}[D_t^\alpha w(x,y,t)] = \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w^2(x,y,t) + w\left(x,y,t\right)\left(1 - rw\left(x,y,t\right)\right)\right].$$

Using the property of the Sumudu transform and the initial condition in Eq.(43), we have

$$u^{-\alpha}\mathbb{S}[w(x,y,t)] - u^{-\alpha} w(x,y,0) = \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w^2(x,y,t) + w(x,y,t)(1 - rw(x,y,t))\right],$$

and

$$\mathbb{S}[w(x,y,t)] = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} + u^\alpha \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w^2(x,y,t) + w\left(x,y,t\right)\left(1 - rw\left(x,y,t\right)\right)\right], \quad (43)$$

Operating with the Sumudu inverse on both sides of Eq.(42) we get

$$w\left(x,y,t\right) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} + \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w^2 + w\left(1 - rw\right)\right]\right], \quad (44)$$

By applying the homotopy perturbation method, and substituting Eq.(27) in Eq.(44) we have

$$\sum_{n=0}^{\infty} p^n w_n(x,y,t) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} + p\mathbb{S}^{-1}\left[u^\alpha \mathbb{S}\left[2\left(D_x^2 + D_y^2\right)\left(\sum_{n=0}^{\infty} p^n w_n\left(x,y,t\right)\right)^2 + \left(1 - r\sum_{n=0}^{\infty} p^n w_n\left(x,y,t\right)\right)\sum_{n=0}^{\infty} p^n w_n\left(x,y,t\right)\right]\right] \quad (45)$$

Equating the terms with identical powers of p, we get

$$p^0: \quad w_0(x,y,t) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)}$$

$$\begin{aligned}
p^1: \quad & w_1(x,y,t) = \\
& = \mathbb{S}^{-1}[u^\alpha \mathbb{S}\left[2(D_x^2 + D_y^2) w_1^2 + w_1(1 - rw_1)\right] \\
& = \frac{t^\alpha}{\Gamma(\alpha+1)} e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} \\
p^2: \quad & w_2(x,y,t) = \\
& = \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}[2(D_x^2 + D_y^2) w_2^2 + w_2(1 - rw_2)]\right] \\
& = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} \\
& \cdots \quad \cdots \\
p^n: \quad & w_n(x,y,t) = \\
& = \mathbb{S}^{-1}\left[u^\alpha \mathbb{S}[2(D_x^2 + D_y^2) w_n^2 + w_n(1 - rw_n)]\right] \\
& = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} \qquad (46)
\end{aligned}$$

Thus the solution of Eq.(41) is given by

$$\begin{aligned}
w(x,y,t) &= \lim_{p\to 1} \sum_{n=0}^{\infty} p^n w_n(x,y,t) \\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right) \\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)} E_\alpha\left(t^\alpha\right) \qquad (47)
\end{aligned}$$

If we put $\alpha \to 1$ in Eq.(47) or solve Eq. (41) and (42) with $\alpha = 1$, we obtain the exact solution

$$w\left(x,y,t\right) = e^{\left(\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\right)} e^t = e^{\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)+t}$$

which is in full agreement with the result in Ref. [1].

Example 10 *Consider the two-dimensional fractional wave equation of the form*

$$D_t^\alpha w\left(x,y,t\right) = 2\left(\frac{\partial^2 w\left(x,y,t\right)}{\partial x^2} + \frac{\partial^2 w\left(x,y,t\right)}{\partial y^2}\right), \quad (48)$$

where $1 < \alpha < 2, -\infty < x,y < \infty$, *subject to the initial condition*

$$w(x,y,0) = \sin(x)\sin(y), \quad \frac{\partial w(x,y,0)}{\partial t} = 0. \quad (49)$$

Taking the Sumudu transform on both sides of Eq.(48), thus we get

$$\mathbb{S}\left[D_t^\alpha w(x,y,t)\right] = \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w\right]$$

and

$$u^{-\alpha}\mathbb{S}\left[w\left(x,y,t\right)\right] - \left(u^{-\alpha} w\left(x,y,0\right) + u^{1-\alpha}\frac{\partial w\left(x,y,0\right)}{\partial t}\right) = \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w\right]$$

Using the property of the Sumudu transform and the initial condition in Eq.(49), we have

$$\mathbb{S}\left[w\left(x,y,t\right)\right] = \sin\left(x\right)\sin\left(y\right) + u^\alpha \mathbb{S}\left[2\left(D_x^2 + D_y^2\right) w\right]. \quad (50)$$

Operating with the Sumudu inverse on both sides of Eq.(50) we get

$$w(x,y,t) = \sin(x)\sin(y) + \mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[2\left(D_x^2 + D_y^2\right)w\right]\right]. \quad (51)$$

By applying the homotopy perturbation method, and substituting Eq.(27) in Eq.(51) we have

$$\sum_{n=0}^{\infty} p^n w_n(x,y,t) = \sin(x)\sin(y) + p\mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[2\left(D_x^2 + D_y^2\right)\left(\sum_{n=0}^{\infty} p^n w_n(x,y,t)\right)\right]\right] \quad (52)$$

Equating the terms with identical powers of p, we get

$$\begin{aligned}
p^0: &\quad w_0(x,y,t) = \sin(x)\sin(y),\\
p^1: &\quad w_1(x,y,t) = \frac{-4t^{\alpha}}{\Gamma(\alpha+1)}\sin(x)\sin(y),\\
p^2: &\quad w_2(x,y,t) = \frac{4^2t^{2\alpha}}{\Gamma(2\alpha+1)}\sin(xt)\sin(y),\\
&\cdots \quad \cdots\\
p^n: &\quad w_n(x,y,t) = \frac{(-1)^n 4^n t^{n\alpha}}{\Gamma(n\alpha+1)} sin(x)\sin(y).
\end{aligned} \quad (53)$$

Thus the solution of Eq.(48) is given by

$$\begin{aligned}
w(x,y,t) &= \lim_{p\to 1}\sum_{n=0}^{\infty} p^n w_n(x,y,t)\\
&= \sin(x)\sin(y)\times \left(1 - \frac{4t^{\alpha}}{\Gamma(\alpha+1)} + \frac{4^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right)\\
&= \sin(x)\sin(y)\sum_{n=0}^{\infty}\frac{(-1)^n 4^n t^{n\alpha}}{\Gamma(n\alpha+1)}\\
&= \sin(x)\sin(y)E_{\alpha}(-4t^{\alpha})
\end{aligned} \quad (54)$$

If we put $\alpha \to 2$ in Eq.(44) or solve Eq. (38) and (39) with $\alpha = 2$, we obtain the exact solution

$$w(x,y,t) = \sin(x)\sin(y)\sum_{n=0}^{\infty}\frac{(-1)^n(2t)^n}{\Gamma(2n+1)} = \sin(x)\sin(y)\cos(2t)$$

which is in full agreement with the result in Ref. [16, 38].

Example 11 *Consider the following system of linear FPDEs*

$$\begin{cases} D_t^{\alpha}w - v_x + v + w = 0,\\ D_t^{\beta}v - w_x + v + w = 0,\end{cases} \quad (55)$$

where $0 < \alpha, \beta < 1$, *subject to the initial condition*

$$w(x,0) = \sinh(x),\ v(x,0) = \cosh(x), \quad (56)$$

Taking the Sumudu transform on both sides of Eq.(55), thus we get

$$\begin{cases} \mathbb{S}\left[D_t^{\alpha}w\right] = \mathbb{S}\left[v_x - v - w\right],\\ \mathbb{S}\left[D_t^{\alpha}v\right] = \mathbb{S}\left[w_x - v - w\right],\end{cases} \quad (57)$$

Using the property of the Sumudu transform and the initial condition in Eq.(56), we have

$$\begin{cases} u^{-\alpha}\mathbb{S}\left[w(x,t)\right] - u^{-\alpha}w(x,0) = \mathbb{S}\left[v_x - v - w\right],\\ u^{-\alpha}\mathbb{S}\left[v(x,t)\right] - u^{-\alpha}v(x,0) = \mathbb{S}\left[w_x - v - w\right],\end{cases}$$

and

$$\begin{cases} u^{-\alpha}\mathbb{S}\left[w(x,t)\right] = u^{-\alpha}\sinh(x) + \mathbb{S}\left[v_x - v - w\right],\\ u^{-\alpha}\mathbb{S}\left[v(x,t)\right] = u^{-\alpha}\cosh(x) + \mathbb{S}\left[w_x - v - w\right],\end{cases} \quad (58)$$

Operating with the Sumudu inverse on both sides of Eq.(58) we get

$$\begin{cases} w(x,t) = \sinh(x) + \mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[v_x - v - w\right]\right],\\ v(x,t) = \cosh(x) + \mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[w_x - v - w\right]\right],\end{cases} \quad (59)$$

By applying the homotopy perturbation method, and substituting Eq.(27) in Eq.(59) we have

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = \sinh(x) + p\mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} q^n v_n(x,t)\right) - \sum_{n=0}^{\infty} q^n v_n(x,t) - \sum_{n=0}^{\infty} p^n w_n(x,t)\right]\right],$$

and

$$\sum_{n=0}^{\infty} q^n v_n(x,t) = \cosh(x) + p\mathbb{S}^{-1}\left[u^{\alpha}\mathbb{S}\left[\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} p^n w_n(x,t)\right) - \sum_{n=0}^{\infty} q^n v_n(x,t) - \sum_{n=0}^{\infty} p^n w_n(x,t)\right]\right]. \quad (60)$$

Equating the terms with identical powers of p, we get

$$\begin{aligned}
p^0: &\quad \begin{cases} w_0(x,t) = \sinh(x)\\ v_0(x,t) = \cosh(x)\end{cases}\\
p^1: &\quad \begin{cases} w_1(x,t) = \cosh(x)\frac{-t^{\alpha}}{\Gamma(\alpha+1)}\\ v_1(x,t) = \sinh(x)\frac{-t^{\beta}}{\Gamma(\beta+1)}\end{cases}
\end{aligned}$$

$$p^2: \quad \begin{cases} w_2(x,t) = \frac{-t^{\alpha+\beta}\cosh(x)}{\Gamma(\alpha+\beta+1)} + \frac{t^{\alpha+\beta}\sinh(x)}{\Gamma(\alpha+\beta+1)} \\ \qquad + \frac{t^{2\alpha}\cosh(x)}{\Gamma(2\alpha+1)} \\ v_2(x,t) = \frac{-t^{\alpha+\beta}\sinh(x)}{\Gamma(\alpha+\beta+1)} + \frac{t^{\alpha+\beta}\cosh(x)}{\Gamma(\alpha+\beta+1)} \\ \qquad + \frac{t^{2\beta}\sinh(x)}{\Gamma(2\beta+1)} \end{cases} \tag{61}$$

Thus the solution of Eq.(55) is given by

$$\begin{aligned} w(x,t) &= \sinh(x)\left(1+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\cdots\right) \\ &\quad -\cosh(x)\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right. \\ &\quad \left. -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right) \\ v(x,t) &= \cosh(x)\left(1+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\cdots\right) \\ &\quad -\sinh(x)\left(\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right. \\ &\quad \left. -\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\cdots\right) \end{aligned} \tag{62}$$

Setting $\alpha=\beta$ in Eqs. (62), we reproduce the solution of [18] as follows:

$$\begin{aligned} w(x,t) &= \sinh(x)\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right) \\ &- \cosh(x)\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right) \\ v(x,t) &= \cosh(x)\left(1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots\right) \\ &- \sinh(x)\left(\frac{t^{\alpha}}{\Gamma(\beta+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right) \end{aligned} \tag{63}$$

If we put $\alpha \to 1$ in Eq.(63) or solve Eq. (55) and (56) with $\alpha=\beta=1$, we obtain the exact solution

$$\begin{aligned} w(x,t) &= \sinh(x)\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right) \\ &\quad -\cosh(x)\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right) \\ &= \sinh(x-t) \\ v(x,t) &= \cosh(x)\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right) \\ &\quad -\sinh(x)\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\cdots\right) \\ &= \cosh(x-t) \end{aligned} \tag{64}$$

which is in full agreement with the result in Ref. [19].

6 Conclusion

Recently, a new transform technique known as Sumudu transform has been introduced by Watugala [37]. This transform technique has a great potential of applicability to solve FPDEs. In addition, the Chinese mathematician Ji-Huan He [12, 14] introduced the homotopy perturbation method. This method was a powerful tool to solve various kinds of nonlinear problems.

In this paper, we have introduced a combination of the homotopy perturbation method and the Sumudu transform method for time fractional problems. This combination builds a strong method called the HPST method. This method has been successfully applied to one- and two-dimensional fractional equations and also for systems of more than two linear and nonlinear partial differential equations. The HPSTM is an analytical method and runs by using the initial conditions only. Thus, it can be used to solve equations with fractional and integer order with respect to time. An important advantage of the new approach is its low computational load.

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