The Chebyshev Collection Method for Solving Fractional Order Klein-Gordon Equation

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Abstract: In this paper, we are implemented the Chebyshev spectral method for solving the non-linear fractional Klein-Gordon equation (FKGE). The fractional derivative is considered in the Caputo sense. We presented an approximate formula of the fractional derivative. The properties of the Chebyshev polynomials are used to reduce FKGE to the solution of system of ordinary differential equations which solved by using the finite difference method. Special attention is given to study the convergence analysis and estimate an upper bound of the error of the derived formula. The numerical results of applying this method to FKGE show the simplicity and the efficiency of the proposed method.

Key–Words: Fractional Klein-Gordon equation; Caputo derivative; Chebyshev spectral method; Convergence analysis.

1 Introduction

Fractional differential equations (FDEs) have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering and others. However, many researchers remain unaware of this field. FDEs have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [1]. Consequently, considerable attention has been given to the solutions of fractional differential equations of physical interest. Most FDEs do not have exact solutions, so approximate and numerical techniques ([2], [5], [25]-[28], [32]), must be used. Recently, several numerical methods to solve FDEs have been given such as, variational iteration method [7], homotopy perturbation method ([4], [5]), Adomian decomposition method ([8], [24]), homotopy analysis method [6], collocation method ([9]-[17], [29]-[31]).

The Klein-Gordon equation plays a significant role in mathematical physics and many scientific applications, such as solid-state physics, nonlinear optics, and quantum field theory [33]. The equation has attracted much attention in studying solitons ([20], [21]) and condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [3]. Wazwaz has obtained the various exact traveling wave solutions, such as compactons, solitons and periodic solutions by using the tanh method [33]. The study of numerical solutions of the Klein-Gordon equation has been investigated considerably in the last few years. In the previous studies, the most papers have carried out different spatial discretization of the equation ([3], [33], [34]).

Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and form the basis of the solution of differential equations ([9], [23]). Chebyshev polynomials are widely used in numerical computation. One of the advantages of using Chebyshev polynomials as a tool for expansion functions is the good representation of smooth functions by finite Chebyshev expansion provided that the function $x(t)$ is infinitely differentiable. The coefficients in Chebyshev expansion approach zero faster than any inverse power in n as n goes to infinity.

We describe some necessary definitions and mathematical preliminaries of the fractional calculus for our subsequent development.

Definition 1 *The Caputo fractional derivative operator D^{α} of order α is defined in the following form*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt,$$

where $\alpha > 0$, *and* $m-1 < \alpha \leq m,\ m \in \mathbb{N},\ x > 0$.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{\alpha}\left(\lambda\, f(x) + \mu\, g(x)\right) = \lambda\, D^{\alpha} f(x) + \mu\, D^{\alpha} g(x),$$

where λ and μ are constants. For the Caputo's derivative we have [19]

$$D^{\alpha}\, C = 0, \quad C \text{ is a constant}, \tag{1}$$

$$D^{\alpha}\, x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{and } n \geq \lceil \alpha \rceil. \end{cases} \tag{2}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0, 1, \cdots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see ([18], [19]).

The main aim of the presented paper is concerned with the application of the Chebyshev collocation method to obtain the numerical solution of the nonlinear fractional Klein-Gordon equation and study the convergence analysis of the proposed method. Khader [9] introduced a new approximate formula of the fractional derivative and used it to solve numerically the fractional diffusion equation. Also, Khader and Hendy in [13] used this formula to solve numerically the fractional delay differential equations. Chebyshev polynomials are well known family of orthogonal polynomials on the interval $[-1, 1]$ that have many applications ([13], [23]). They are widely used because of their good properties in the approximation of functions.

In this paper, we apply Chebyshev collocation method to obtain the numerical solution of the nonlinear fractional Klein-Gordon equation of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + a\, D^{\alpha} u(x,t) + b\, u(x,t) + c\, u^{\gamma}(x,t)$$
$$= f(x,t),\ \ x \in (0, L),\ \ t > 0,\ \ \alpha \in (1, 2], \tag{3}$$

where D^{α} denotes the Caputo fractional derivative of order α with respect to x, $u(x,t)$ is unknown function, and a, b, c and γ are known constants with $\gamma \in R$, $\gamma \neq \pm 1$.

We also assume the following initial conditions

$$u(x,0) = g_1(x),\ \ u_t(x,0) = g_2(x),\ x \in (0, L) \tag{4}$$

and the following boundary conditions

$$u(0,t) = u(L,t) = 0.$$

The organization of this paper is as follows. In the next section, derivation an approximate formula for fractional derivatives using Chebyshev series expansion. In section 3, the error analysis of the introduced approximate formula is given. In section 4, the procedure of the solution using Chebyshev collocation method is given. The conclusion is given in section 5.

2 Derivation an approximate formula for fractional derivatives using Chebyshev series expansion

The well known Chebyshev polynomials [23] are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula

$$T_{n+1}(z) = 2z T_n(z) - T_{n-1}(z),$$
$$T_0(z) = 1,$$
$$T_1(z) = z, \quad n = 1, 2, \cdots$$

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by

$$T_n(z) = n \sum_{i=0}^{[\frac{n}{2}]} (-1)^i\ 2^{n-2\,i-1}\ \frac{(n-i-1)!}{(i)!\,(n-2\,i)!}\ z^{n-2\,i} \tag{5}$$

where $[n/2]$ denotes the integer part of $n/2$. The orthogonality condition is

$$\int_{-1}^{1} \frac{T_i(z)\, T_j(z)}{\sqrt{1-z^2}}\, dz = \begin{cases} \pi, & \text{for } i = j = 0; \\ \frac{\pi}{2}, & \text{for } i = j \neq 0; \\ 0, & \text{for } i \neq j. \end{cases}$$

In order to use these polynomials on the interval $[0, L]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = \frac{2}{L}t - 1$. The shifted Chebyshev polynomials are defined as

$$T_n^*(t) = T_n(\frac{2}{L}t - 1) = T_{2n}(\sqrt{t/L}).$$

The analytic form of the shifted Chebyshev polynomials $T_n^*(t)$ of degree n is given by

$$T_n^*(t) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}\,(n+k-1)!}{L^k (2k)!\,(n-k)!}\, t^k \qquad (6)$$

for $n = 2, 3, \ldots$.

The function $x(t)$, which belongs to the space of square integrable functions in $[0, L]$, may be expressed in terms of shifted Chebyshev polynomials as

$$x(t) = \sum_{i=0}^{\infty} c_i\, T_i^*(t), \qquad (7)$$

where the coefficients c_i are given by

$$\begin{aligned} c_0 &= \frac{1}{\pi} \int_0^L \frac{x(t)\, T_0^*(t)}{\sqrt{Lt - t^2}}\, dt, \\ c_i &= \frac{2}{\pi} \int_0^L \frac{x(t)\, T_i^*(t)}{\sqrt{Lt - t^2}}\, dt, \quad i = 1, 2, \cdots \end{aligned} \qquad (8)$$

In practice, only the first $(m+1)$-terms of shifted Chebyshev polynomials are considered. Then we have

$$x_m(t) = \sum_{i=0}^{m} c_i\, T_i^*(t). \qquad (9)$$

The main approximate formula of the fractional derivative of $x_m(t)$ is given in the following theorem.

Theorem 2 *Let $x(t)$ be approximated by Chebyshev polynomials as (9) and also suppose $\alpha > 0$, then*

$$D^\alpha(x_m(t)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\alpha)}\, t^{k-\alpha}, \qquad (10)$$

where $w_{i,k}^{(\alpha)}$ is given by

$$w_{i,k}^{(\alpha)} = (-1)^{i-k} \frac{2^{2k}\, i\,(i+k-1)!\Gamma(k+1)}{L^k (i-k)!\,(2\,k)!\,\Gamma(k+1-\alpha)}. \qquad (11)$$

Proof. Since the Caputo's fractional differentiation is a linear operation we have

$$D^\alpha(x_m(t)) = \sum_{i=0}^{m} c_i\, D^\alpha(T_i^*(t)). \qquad (12)$$

Employing Eqs.(1) and (2) we have

$$D^\alpha\, T_i^*(t) = 0, \quad i = 0, 1, \cdots, \lceil\alpha\rceil - 1, \quad \alpha > 0. \qquad (13)$$

Also, for $i = \lceil\alpha\rceil, \lceil\alpha\rceil+1, ..., m$, and by using Eqs.(1) and (2), we get

$$\begin{aligned} D^\alpha\, T_i^*(t) &= i \sum_{k=\lceil\alpha\rceil}^{i} (-1)^{i-k} \frac{2^{2k}(i+k-1)!}{L^k (i-k)!\,(2\,k)!}\, D^\alpha t^k \\ &= i \sum_{k=\lceil\alpha\rceil}^{i} (-1)^{i-k} \frac{2^{2k}(i+k-1)!\Gamma(k+1)}{L^k (i-k)!\,(2\,k)!\,\Gamma(k+1-\alpha)} t^{k-\alpha}. \end{aligned} \qquad (14)$$

A combination of Eqs.(13), (14) and (11) leads to the desired result and then completes the proof of the theorem. □

3 Error analysis

In this section, special attention is given to study the convergence analysis and evaluate an upper bound of the error of the proposed formula.

Theorem 3 *(Chebyshev truncation theorem) [23] The error in approximating $x(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$x_m(t) = \sum_{k=0}^{m} c_k\, T_k(t), \qquad (15)$$

then

$$E_T(m) \equiv |x(t) - x_m(t)| \le \sum_{k=m+1}^{\infty} |c_k|, \qquad (16)$$

for all $x(t)$, all m, and all $t \in [-1, 1]$.

Theorem 4 *The Caputo fractional derivative of order α for the shifted Chebyshev polynomials can be expressed in terms of the shifted Chebyshev polynomials themselves in the following form*

$$D^\alpha(T_i^*(t)) = \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k}\, T_j^*(t) \qquad (17)$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i-k}\, 2i(i+k-1)!\; \Gamma(k-\alpha+\frac{1}{2})}{h_j \Gamma(k+\frac{1}{2})\,(i-k)!\,\Gamma(k-\alpha-j+1)\,\Gamma(k+j-\alpha+1) L^k}$$
$j = 0, 1, \cdots$.

The proof please see [2].

Theorem 5 *The error*

$$|E_T(m)| = |D^\alpha x(t) - D^\alpha x_m(t)|$$

in approximating $D^\alpha x(t)$ by $D^\alpha x_m(t)$ is bounded by

$$|E_T(m)| \leq \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \Big) \Big|. \quad (18)$$

Proof. A combination of Eqs. (7), (9) and (17) leads to

$$|E_T(m)| = \Big| D^\alpha x(t) - D^\alpha x_m(t) \Big|$$
$$= \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} T_j^*(t) \Big) \Big|,$$

but $|T_j^*(t)| \leq 1$, so, we can obtain

$$|E_T(m)| \leq \left| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \Big) \right|,$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, summing the bounds and hence completes the proof of the theorem. $\square$

4 Numerical implementation

In this section, we solve numerically the nonlinear fractional-order Klein-Gordon equation using the approach Chebyshev spectral method. Some numerical examples are presented to validate the solution scheme. Symbolic computations are carried out using Matlab 8. To achieve this propose we will consider the following two examples ([4], [5]).

Example 6 *Consider the fractional-order nonlinear Klein-Gordon problem*

$$\frac{\partial^2 u(x,t)}{\partial t^2} - D^\alpha u(x,t) + u^3(x,t) = f(x,t), \quad x \in (0,1),\ t > 0,\ \alpha \in (1,2], \quad (19)$$

with the following initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad (20)$$

and the following boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \quad (21)$$

where the source term $f(x,t)$ is given by

$$f(x,t) = 2x^\alpha - \Gamma(\alpha+1)\, t^2 + x^{3\alpha}\, t^6.$$

The exact solution of this problem is $u(x,t) = x^\alpha\, t^2$.

In order to use the Chebyshev collocation method, we approximate $u(x,t)$ with $m = 3$ as

$$u_3(x,t) = \sum_{i=0}^{3} u_i(t)\, T_i^*(x). \quad (22)$$

From Eq.(19) and Theorem 2 we have

$$\sum_{i=0}^{3} \ddot{u}_i(t)\, T_i^*(x) - \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_i(t)\, w_{i,k}^{(\alpha)} x^{k-\alpha} + \left(\sum_{i=0}^{3} u_i(t) T_i^*(x) \right)^3 = f(x,t). \quad (23)$$

We now collocate Eq.(23) at $(m+1-\lceil\alpha\rceil)$ points x_p, $p = 0, 1, \cdots, m - \lceil\alpha\rceil$ as

$$\sum_{i=0}^{3} \ddot{u}_i(t)\, T_i^*(x_p) - \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_i(t)\, w_{i,k}^{(\alpha)} x_p^{k-\alpha} + \left(\sum_{i=0}^{3} u_i(t) T_i^*(x_p) \right)^3 = f(x_p,t). \quad (24)$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $T^*_{m+1-\lceil\alpha\rceil}(x)$.

In this case, the roots x_p of shifted Chebyshev polynomial $T_2^*(x)$, i.e.

$$x_0 = 0.146447, \quad x_1 = 0.8872983.$$

Also, by substituting Eq.(22) in the boundary conditions (21) we can find

$$\sum_{i=0}^{3} (-1)^i\, u_i(t) = 0, \quad \sum_{i=0}^{3} u_i(t) = 0. \quad (25)$$

By using Eqs.(24) and (25) we obtain the following non-linear system of ODEs

$$\ddot{u}_0(t) + k_1\, \ddot{u}_1(t) + k_2\, \ddot{u}_3(t) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_i(t)\, w_{i,k}^{(\alpha)} x_0^{k-\alpha} - \left(\sum_{i=0}^{3} u_i(t) T_i^*(x_0) \right)^3 + f_0(t), \quad (26)$$

$$\ddot{u}_0(t) + k_{11}\, \ddot{u}_1(t) + k_{22}\, \ddot{u}_3(t) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_i(t)\, w_{i,k}^{(\alpha)} x_1^{k-\alpha}$$

$$-\left(\sum_{i=0}^{3} u_i(t) T_i^*(x_1)\right)^3 + f_1(t), \qquad (27)$$

$$u_0(t) - u_1(t) + u_2(t) - u_3(t) = 0, \qquad (28)$$

$$u_0(t) + u_1(t) + u_2(t) + u_3(t) = 0, \qquad (29)$$

where

$$k_1 = T_1^*(x_0), \quad k_2 = T_3^*(x_0),$$
$$k_{11} = T_1^*(x_1), \quad k_{22} = T_3^*(x_1).$$

Now, to use finite difference method [22] for solving the system (26)-(29), we use the notations $t_n = n\Delta t$ to be the integration time $0 \le t_n \le T$, $\Delta t = T/N$, for $n = 0, 1, ..., N$. Define $u_i^n = u_i(t_n)$, $f_i^n = f_i(t_n)$, $i = 1, 2$. Then the system (26)-(29), is discretized and takes the following form

$$\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\Delta t^2} + k_1 \frac{u_1^{n+1} - 2u_1^n + u_1^{n-1}}{\Delta t^2} + k_2 \frac{u_3^{n+1} - 2u_3^n + u_3^{n-1}}{\Delta t^2} = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_i^{n+1} w_{i,k}^{(\alpha)} x_0^{k-\alpha} - \left(\sum_{i=0}^{3} u_i^{n+1} T_i^*(x_0)\right)^3 + f_0^{n+1}, \qquad (30)$$

$$\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\Delta t^2} + k_{11} \frac{u_1^{n+1} - 2u_1^n + u_1^{n-1}}{\Delta t^2} + k_{22} \frac{u_3^{n+1} - 2u_3^n + u_3^{n-1}}{\Delta t^2} = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_i^{n+1} w_{i,k}^{(\alpha)} x_1^{k-\alpha} - \left(\sum_{i=0}^{3} u_i^{n+1} T_i^*(x_1)\right)^3 + f_1^{n+1}, \qquad (31)$$

$$u_0^{n+1} - u_1^{n+1} + u_2^{n+1} - u_3^{n+1} = 0, \qquad (32)$$

$$u_0^{n+1} + u_1^{n+1} + u_2^{n+1} + u_3^{n+1} = 0. \qquad (33)$$

This system presents the numerical scheme of the proposed problem (19) using the fractional finite difference method. Solving this system using the Newton iteration method yields the numerical solution of the non-linear fractional Klein-Gordon equation (19).

At $n = 1$, we will evaluate the values of $u^0 = (u_0^0, u_1^0, u_2^0, u_3^0)$ and $u^1 = (u_0^1, u_1^1, u_2^1, u_3^1)$ using the initial conditions (20). Therefore, we can obtain the solutions

$$u^n = (u_0^n, u_1^n, u_2^n, u_3^n), \ n = 2, 3, ..., N$$

using the numerical scheme (30)-(33).

The obtained numerical results by means of the proposed method are shown in table 1 and figures 1 and 2. In the table 1, the absolute errors between the exact solution u_{ex} and the approximate solution u_{approx}, at $m = 3$, $m = 5$ and $m = 7$ with the final time $T = 1$ are given. But, in the figure 1 we presented a comparison between the solution and the approximate solution using the proposed method at $\alpha = 1.8$ for different values of the final time $T_f = 0.25$ and 0.5 at time step $\Delta t = 0.005$ with $m = 3$. Also, in figure 2 we presented the behavior of approximate solution at $t = 2$ for different values of $\alpha = 1.1, 1.4, 1.7$ and 2 at time step $\Delta t = 0.005$ with $m = 3$.

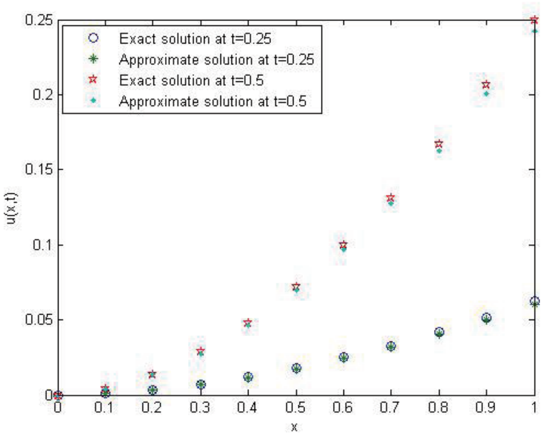

Figure 1. Comparison between the exact solution and the approximate solution at $\alpha = 1.8$ for different values of time t.

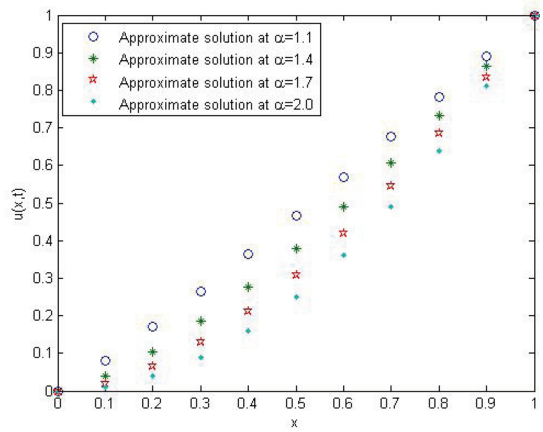

Figure 2. The behavior of the approximate solution at the final time $T_f = 2$ for different values of α.

The absolute error between the exact solution and the approximate solution at $m = 3,\ m = 5$ and $m = 7$ with $T = 1$.

x	$\lvert u_{ex} - u_{approx}\rvert$ at $m = 3$	$\lvert u_{ex} - u_{approx}\rvert$ at $m = 5$	$\lvert u_{ex} - u_{approx}\rvert$ at $m = 7$
0.0	0.120849 e-03	0.274260 e-04	0.300445 e-05
0.1	0.021894 e-03	0.423794 e-04	0.417436 e-05
0.2	0.126809 e-03	0.373716 e-04	0.544455 e-05
0.3	0.321820 e-03	0.843125 e-04	0.617464 e-05
0.4	0.424838 e-03	0.323010 e-04	0.648473 e-05
0.5	0.429844 e-03	0.363133 e-04	0.639412 e-05
0.6	0.523805 e-03	0.193954 e-04	0.595429 e-05
0.7	0.623867 e-03	0.293780 e-04	0.531430 e-05
0.8	0.725877 e-03	0.493488 e-04	0.459438 e-05
0.9	0.726879 e-03	0.283224 e-04	0.379345 e-05
1.0	0.822821 e-03	0.773238 e-04	0.300045 e-05

Example 7 *Consider the fractional non-linear Klein-Gordon problem*

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2} &= D^{\alpha}u(x,t) - \tfrac{3}{4}\,u(x,t) \\ &\quad + \tfrac{3}{2}\,u^3(x,t) + f(x,t),\end{aligned}$$

$$x \in (0,5), t > 0, \alpha \in (1,2], \qquad (34)$$

with the following initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad (35)$$

and the following boundary conditions

$$u(0,t) = 0, \qquad u(5,t) = 0, \qquad (36)$$

where the source term $f(x,t)$ is given by

$$\begin{aligned} f(x,t) &= 6x^{\alpha}(5-x)t + x^{\alpha}(5-x)t^3 \\ &\quad -1.5x^{3\alpha}(5-x)^3 t^9 \\ &\quad -(5\alpha! - (\alpha+1)!x). \end{aligned}$$

The exact solution of this problem is

$$u(x,t) = x^{\alpha}(5-x)\,t^3.$$

The obtained numerical results by means of the proposed method are shown in figures 3 and 4. In the figure 3, we presented a comparison between the exact solution and the approximate solution using the proposed method at $\alpha = 1.6$ for different values of the final time $T_f = 1.75$ and 1.5 at time step $\Delta t = 0.005$ with $m = 3$. Also, in figure 4 we presented the behavior of the approximate solution at $T_f = 2$ for different values of $\alpha = 1.2, 1.5, 1.8$ and 2 at time step $\Delta t = 0.002$ with $m = 3$.

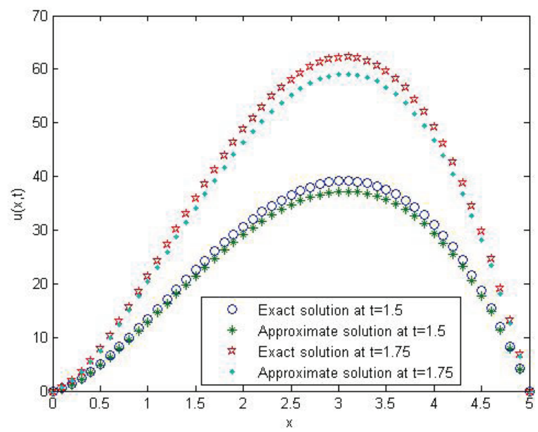

Figure 3. Comparison between the exact solution and the approximate solution at $\alpha = 1.6$ for different values of time t.

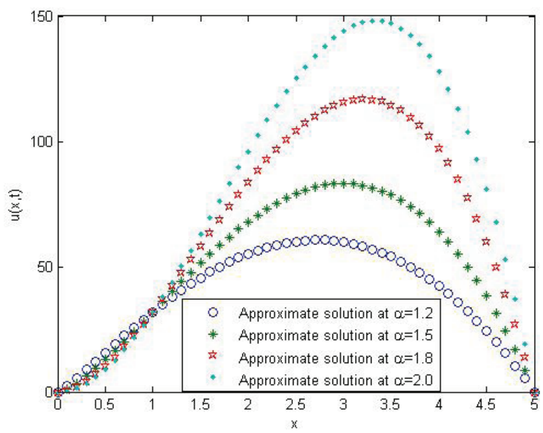

Figure 4. The behavior of the approximate solution at the final time $T_f = 2$ for different values of α.

5 Conclusion and remarks

In this article, we are implemented Chebyshev spectral method for solving the non-linear fractional Klein-Gordon equation. The fractional derivative is considered in the Caputo sense. The properties of the Chebyshev polynomials are used to reduce the proposed problem to the solution of system of ordinary differential equations which solved by using finite difference method. Special attention is given to study the convergence analysis and estimate an upper bound of the error of the derived formula. The solution obtained using the suggested method is in excellent agreement with the already existing ones and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (22). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the technique. All computations in this paper are done using Matlab 8.

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