Improving the stability properties of sampling zeros of multivariable discrete model via Taylor method and multirate fast sampling

CHENG ZENG
College of Science
Guizhou Institute of Technology
No. 1 Caiguanlu Caijiaguan, 550003 Guiyang
CHINA
zengcheng1290@163.com

YINGYING SU
College of Automation
University of Chongqing
No. 174 Shazhengjie, 400044 Chongqing
CHINA

Abstract: It is well known that multirate input and hold, such as a generalized sample hold function (GSHF), can be used to shift the sampling zeros of a discrete-time model for a continuous-time system. This paper deals with the stability of sampling zeros, as the sampling period tends to zero, of discrete-time models that are composed of a GSHF, a continuous-time multivariable plant and a sampler in cascade. We propose a hold design that places the sampling zeros asymptotically to the stable region by deriving the approximate expressions of the sampling zeros as power series expansions with respect to a sampling period. The focus is on the evolution of the complex plant for the sampling zeros with the design parameters α_j of the GSHF. These parameters are only determined to propose the GSHF that obtains sampling zeros as stable as possible, or with improved stability properties even when unstable, for a given continuous-time multivariable system. The research is extended by obtaining the optimum value of α_j for sufficiently small sampling periods and a continuous-time plant, and meanwhile a new stability condition of the sampling zeros is obvious weaker than that of ZOH or FROH case.

Key–Words: Stability properties, sampled-data multivariable models, sampling zeros, generalized sample hold function, fast sampling, Taylor series expansions.

1 Introduction

It is well known that unstable zeros limited the achievable control performance, particular if zero cancellation techniques are used [1, 2]. When a continuous-time system is discretized, poles p_i are transformed as $p_i \rightarrow exp(p_i T)$, where T is the sampling period. The transformations of zeros, however, are much more complicated and the stability of zeros is not preserved in the discretization process in some cases [1]. Since it is generally impossible to derive a closed-form expression between the continuous-time zeros and the discrete-time ones, the efforts were devoted to the analysis of the discretization zeros for fast sampling rates (i.e. limiting zeros), especially sampling zeros, in the earlier research studies [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 18, 20, 21, 22, 23, 24, 25].

The key attempt to study this problem was considered by Åström et al. [1], who described the asymptotic behavior of the discrete-time zeros for small T as functions of their continuous-time counterparts and of the relative order of the system being discretized. Due to this theorem, a part of zeros called intrinsic zeros [3, 4, 26], go to unity while the remaining discretization zeros [3, 26], which are due to sampling and modulation, go toward zeros of certain polynomials called Euler [27, 28], normal [5] or reciprocal polynomials [3, 29], completely determined by the value of the relative order of the continuous-time system.

For single-input, single-output (SISO) systems, at least one of the limiting zeros lies strictly outside the unit circle if the relative degree of a continuous-time transfer function is greater than or equal to three [1, 3, 5, 12, 21]. This fact indicates that even though all the zeros of such a continuous-time system are stable, the corresponding discrete-time system has an unstable zero in the limiting case as T tends to zero. Next, when the relative degree is one or two, a part of limiting zeros are located just on the unit circle, i.e., in the marginal case of the stability. Thus, the asymptotic behavior of the limiting zeros is an interesting issue because the limiting zeros are stable for sufficiently small T if they approach the unit circle from inside as T tends to zero. Hagiwara et al. [3] and Ishitobi et al. [12] analyzed respectively the asymptotic behavior of the limiting zeros for the SISO models in the case of various holds and derived stability conditions of the limiting zeros for sufficiently small T.

For multivariable systems, the zeros of discrete-time system were firstly studied by Hayakawa et al. [6]. They explored the locations of the limiting zeros on the basis of the degrees of the infinite elementary divisors [30]. Weller [7] also demonstrated Åström et al.'s result [1] can be extended directly to decouplable multivariable systems. Furthermore, Ishitobi [8] and Liang et al. [11] gave a new stability condition for zeros of discrete-time systems when all the degrees of the infinite elementary divisors are two and three. In the process of research, the properties of zeros of sampled multivariable systems were mainly discussed in the case of a zero order hold (ZOH). In addition, Liang et al. showed the stability of zeros may be improved by using a fractional order hold (FROH) [17] or a generalized sample hold function (GSHF) [23] stead of ZOH.

Recently, Ishitobi et al. [31, 32] have analyzed the asymptotic properties of the zeros of multivariable sampled-data systems on the basis of the normal form representation of the continuous-time system and have also given an approximate expression of zeros of a sampled-data system in the case of a ZOH as power series with respect to a sampling period when all the relative degrees of the continuous-time system are less than or equal to two. This is a new approach which using to the Taylor series expansions and relative degrees. Moreover, a new method is better than a old way in terms of the simplicity and accuracy. However, from the result of Ishitobi et al. [31, 32], the limiting zeros lie just on the unit circle when all the relative degrees are one or two. Therefore, attention should be directed to the improvement of stability of limiting zeros for discrete-time multivariable systems.

In this paper, we continue the work of discretization zeros by studying a GSHF as the hold function instead of a ZOH. By using a GSHF, the asymptotic properties and stability condition of the limiting zeros of the continuous-time systems have been greatly strengthened, and further the control performance has been also improved. Our main results of the paper analyze the asymptotic properties of the limiting zeros of the sampled-data multivariable systems with a GSHF by Taylor expansion with respect to a sampling period when all the relative degrees of a continuous-time transfer function are one and two. Furthermore, a condition that assures stability of zeros for sufficiently small sampling periods is given. Besides the obvious differences in terms of techniques in studying the stability of discretization zeros in the case of a ZOH and a GSHF, we feel that this study is important, especially for system analysis and design.

2 System description

Consider a square time-invariant multivariable system expressed by

$$S_C : \begin{cases} \dot{\overline{x}} = A_0\overline{x} + B_0 u \\ y = C_0\overline{x}, \quad C_0 = [c_1, c_2, \cdots, c_m]^T \end{cases} \tag{1}$$

with a state vector $\overline{x} \in R^n$, an input vector $u \in R^m$, an output vector $y \in R^m$. It is assumed in this paper that the system S_C is minimal and decouplable by static state feedback, and that the transfer function matrix $G_s = C_0(sI - A_0)^{-1}B_0$ has full rank.

Zeros of multivariable systems are defined in several ways. Multivariable zeros can be termed system zeros, invariant zeros, transmission zeros and so on. In spite of many differences and ambiguities, all those definitions of multivariable zeros refer or claim to be extensions to those for SISO system. Then, the definition of invariant zeros, transmission zeros and system zeros for the system all coincide and some of the properties of zeros in SISO systems are inherited in the discretization process [33]. Thus, these zeros are simply called the zeros throughout this article. We denote the relative degree corresponding to the ith output y_i by r_i, i.e., $c_i^T A_0^{r_i - 1} B_0 \neq \mathbf{0^T}$ and $c_i^T A_0^j B_0 = \mathbf{0^T}, j = 0, 1, \cdots, r_i - 2$. Then, the decoupling matrix D is represented as

$$D = \begin{bmatrix} c_1^T A_0^{r_1-1} B_0 \\ c_2^T A_0^{r_2-1} B_0 \\ \vdots \\ c_m^T A_0^{r_m-1} B_0 \end{bmatrix} \tag{2}$$

and nonsingular from the assumption.

3 Zeros of approximate sampled-data models with a GSHF

Let S_D be a sampled-data system of a series connection of a GSHF, the continuous-time system S_C and a sampler with a sampling period T.

We are interested in the sampled-data model for the continuous-time multivariable system (1) with a GSHF. However, it is difficult to make a GSHF in practice because it is generally composed of exponential and sinusoidal functions. Thus, we consider a piecewise constant GSHF (PC GSHF) defined by piecewise constant impulse responses [20, 21, 22, 23]

$$h(t) = \begin{cases} \alpha_1, & t \in [0, \frac{T}{N}), \\ \alpha_2, & t \in [\frac{T}{N}, \frac{2T}{N}), \\ \cdots & \cdots \\ \alpha_N, & t \in [\frac{(N-1)T}{N}, T). \end{cases} \quad (3)$$

Clearly, PC GSHF keep a regular partition in time of sampling interval $[0, T)$ as in the case of the ZOH (see Fig. 1). When multiplicity output of PC GSHF showed in Fig. 2 is considered, each sampling period T is equally divided into N subperiods of length $\Delta = \frac{T}{N}$ and the control input over the subinterval $[kT, \Delta]$ is described by

$$u(kT + \Delta) = u_j(kT), \quad \frac{(j-1)T}{N} \leq \Delta < \frac{jT}{N} \quad (4)$$

From (3),(4) can be rewritten as

$$u_j(kT) = \alpha_j u(kT), \quad j = 1, \ldots, N \quad (5)$$

where α_j is a real constant.

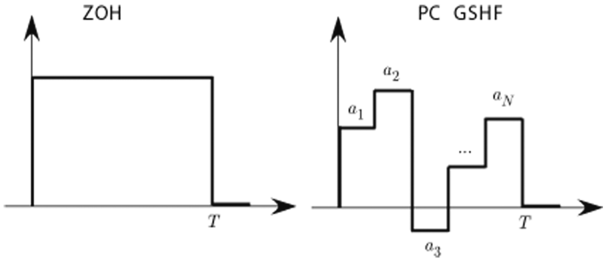

Figure 1: Pulse response of a ZOH and a PC GSHF

This paper derives approximate models of the sampled-data system S_D from the normal form representation of the continuous-time system S_C, and gives approximate expressions of the zeros of S_D as power series with respect to a sampling period T. Here, attention is restricted to the case that some of the relative degree of S_C are one and the rest are two; i.e., $r_1 = \cdots = r_p = 1$, $r_{p+1} = \cdots = r_m = 2$. More precisely, our focus is that the stability performance of the discretization zeros can be greatly enhanced for this situation.

It is assumed that the relative degrees $r_1, \cdots, r_p, 0 < p < m$ for the outputs $y_1, \cdots, y_p$ are one and the rest $r_{p+1}, \cdots, r_m$ for the outputs $y_{p+1}, \cdots, y_m$ are two without loss of generality.

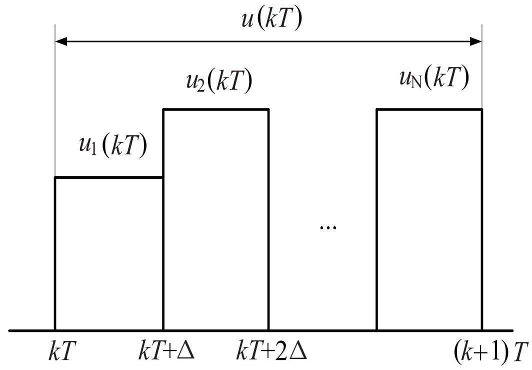

Figure 2: Multiplicity output of a PC GSHF

Then, the system matrix C_0 and the decoupling matrix have the following form.

$$C_0 = \begin{bmatrix} C_{10} \\ C_{20} \end{bmatrix}, D = \begin{bmatrix} C_{10}B_0 \\ C_{20}A_0B_0 \end{bmatrix},$$
$$|D| \neq 0, \quad C_{20}B_0 = O \quad (6)$$

where C_{10} and C_{20} are $p \times n$ and $(m-p) \times n$ matrices, respectively.

First, a sampled-data model is derived from the normal form of S_C. The normal form of (1) with $r_1 = \cdots = r_p = 1$, $r_{p+1} = \cdots = r_m = 2$ is expressed by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (7)$$

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ O & O & I_{m-p} & O \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}, \\ B &= \begin{bmatrix} B_1 & O & B_2 & O \end{bmatrix}^T, \\ C &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} I_p & O & O & O \\ O & I_{m-p} & O & O \end{bmatrix} \end{aligned}$$

where

$$x = \left[y_1^T, y_2^T, y_3^T, \upsilon^T\right]^T, y_1 = [y_1, \cdots, y_p]^T,$$
$$y_2 = [y_{p+1}, \cdots, y_m]^T,$$
$$y_3 = \dot{y}_2 = [\dot{y}_{p+1}, \cdots, \dot{y}_m]^T$$
$$\upsilon = [x_{2m-p+1}, \cdots, x_n]^T, x = S_b\overline{x},$$
$$S_b = \begin{bmatrix} C_{10}^T & C_{20}^T & (C_{20}A_0)^T & W^T \end{bmatrix}^T,$$
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \end{bmatrix} = C_{10}A_0S_b^{-1},$$
$$\begin{bmatrix} A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix} = C_{20}A_0^2S_b^{-1},$$
$$A = S_bA_0S_b^{-1}, B = S_bB_0, C = C_0S_b^{-1}$$

and matrix $W, A_{31}, A_{32}, A_{33}, A_{34}$ are necessarily obtained by $WB_0 = 0$ and $WA_0 = A_{31}C_{10} + A_{32}C_{20} + A_{33}C_{20}A_0 + A_{34}W$. Further, the roots of the equation $|sI - A_{34}| = 0$ are zeros of S_C and the following relations hold.

$$B_1 = C_1B = C_{10}B_0, C_2B = C_{20}B_0 = O \tag{8}$$

$$B_2 = C_2AB = C_{20}A_0B_0 \tag{9}$$

The following theorem holds.

Theorem 1 *The zeros of a sampled-data system corresponding to a continuous-time transfer function (1) with $r_1 = \cdots = r_p = 1$, $r_{p+1} = \cdots = r_m = 2$ are given approximately in the case of a PC GSHF for small sampling periods T by the roots of*

$$|F_I(z)F_s(z)| = 0 \tag{10}$$

where

$$\begin{aligned} F_I(z) &= (z-1)I - TA_{34} \\ F_s(z) &= (z+1)I + \frac{2}{c_N^2(\alpha)}\left[c_N^1(\alpha) - c_N^2(\alpha)\right]c_{20}A_0B_0D_R \\ &\quad + \frac{T}{3c_N^2(\alpha)}\left[c_N^2(\alpha) - c_N^3(\alpha)\right]c_{20}A_0^2B_0D_R \end{aligned}$$

where D_R is an $m \times (m-p)$ matrix which are defined by $\begin{bmatrix} D_L & D_R \end{bmatrix} = D^{-1}$

Proof: First, the approximate sampled-data model for multivariable decouplable continuous-time system with a PC GSHF is derived. When multiplicity output of PC GSHF showed in Fig. 2 is considered, each sampling period T is equally divided into N subperiods of length $\Delta = \dfrac{T}{N}$ and the control input over the subinterval $[kT, \Delta]$ is described by

$$u_j(kT) = \alpha_j u(kT), \quad j = 1, \ldots, N$$

where α_j is a real constant.

Furthermore, It can be seen from Fig. 1 that a PC GSHF keeps a regular partition in time of sampling interval $[0, T)$ as in the case of the ZOH. Therefore, we present an approximate sampled-data model for multivariable decouplable continuous-time system with PC GSHF by means of the Taylor's expansion formula in the sampling subperiods when the input is a $u_{1,k}$, $u_{2,k}$, $\cdots$, $u_{N,k}$.

When the input is a $u_{1,k}(kT \le u_k < (k + \frac{1}{N})T)$, one can obtain for sufficiently small sampling periods that

$$\mathbf{y}_{k+\frac{1}{N}} = A_{k,\frac{1}{N}}\mathbf{y}_k + B_{k,\frac{1}{N}}y_k^{(r)} \tag{11}$$

$$y_k^{(r)} = b_k + a_k\alpha_1u_k \tag{12}$$

where

$$A_{k,\frac{1}{N}} = \begin{bmatrix} 1 & \frac{T}{N} & \cdots & \frac{1}{(r-1)!}(\frac{T}{N})^{r-1} \\ 0 & 1 & \cdots & \frac{1}{(r-2)!}(\frac{T}{N})^{r-2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{T}{N} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$B_{k,\frac{1}{N}} = \left[\frac{1}{r!}\left(\frac{T}{N}\right)^r \ \frac{1}{(r-1)!}\left(\frac{T}{N}\right)^{r-1} \cdots \frac{T}{N} \right]^T$$

Similarly, we give approximate asymptotic expressions of the output $\mathbf{y}_{k+\frac{2}{N}}, \cdots, \mathbf{y}_{k+1}$ as power series expansions with respect to a sufficiently small sampling period when the input is the $u_{2,k}, \cdots, u_{N,k}$, respectively. Deleting $\mathbf{y}_{k+\frac{1}{N}}, \cdots, \mathbf{y}_{k+\frac{N-1}{N}}$ to lead to the approximate expressions of $\mathbf{y}_{k+1}$:

$$\begin{aligned} \mathbf{y}_{k+1} &= A_k^N\mathbf{y}_k + \left(A_k^{N-1} + A_k^{N-2} + \cdots + A_k + I_r\right)B_kb_k \\ &\quad + \left(\alpha_1A_k^{N-1} + \cdots + \alpha_{N-1}A_k + \alpha_NI_r\right)B_ka_ku_k \end{aligned} \tag{13}$$

where

$$A_k = A_{k,1} = \cdots = A_{k,\frac{1}{N}}$$

$$B_k = B_{k,1} = \cdots = B_{k,\frac{1}{N}}$$

$$(A_k^{N-1} + \cdots + A_k + I_r)B_k = e$$

$$(\alpha_1A_k^{N-1} + \cdots + \alpha_{N-1}A_k + \alpha_NI_r)B_k = d \tag{14}$$

and then

$$A_k^N = \begin{bmatrix} 1 & T & \cdots & \frac{T^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{T^{r-2}}{(r-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{15}$$

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, d = \begin{bmatrix} d \\ d' \end{bmatrix}$$

$$d' = \begin{bmatrix} d'_1 & d'_2 & \cdots & d'_{r-1} \end{bmatrix}^T$$

$$e_1 = \frac{T^r}{r!}, e_2 = \begin{bmatrix} \frac{T^{r-1}}{(r-1)!} & \cdots & \frac{T^2}{2!} & T \end{bmatrix}^T$$

$$d = \frac{T^r}{r!}c_N^r(\alpha), d'_k = \frac{T^{r-k}}{(r-k)!}c_N^{r-k}(\alpha), \tag{16}$$

$$k = 1, 2, \cdots, r-1$$

where

$$c_N^r(\alpha) = \sum_{j=1}^{N}(\Lambda_{1j}^r - \Lambda_{2j}^r)\alpha_j$$

$$\Lambda_{1j} = 1 - \frac{j-1}{N}, \quad \Lambda_{2j} = 1 - \frac{j}{N}$$

It is assumed that the relative degrees $r_1, \cdots, r_p, 0 < p < m$ for the outputs $y_1, \cdots, y_p$ are one and the rest $r_{p+1}, \cdots, r_m$ for the outputs $y_{p+1}, \cdots, y_m$ are two. When a PC GSHF is used, note that the relations $\dot{u} = \ddot{u} = \cdots = 0$. It leads to the derivatives of the output vector and the vector υ represented by x and u as

$$\begin{aligned} \dot{y}_1 &= C_1Ax + C_1Bu \\ \ddot{y}_1 &= C_1A^2x + C_1ABu \qquad (17) \\ &\vdots \end{aligned}$$

$$\begin{aligned} \dot{y}_2 &= C_2Ax + C_2Bu \\ \ddot{y}_2 &= C_2A^2x + C_2ABu \\ y_2^{(3)} &= C_2A^3x + C_2A^2Bu \qquad (18) \\ &\vdots \end{aligned}$$

$$\begin{aligned} \dot{\upsilon} &= \begin{bmatrix} A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} x \qquad (19) \\ &\vdots \end{aligned}$$

In addition, a sampled-data state vector is defined by $x_k^T = [y_{1,k}^T, y_{2,k}^T, y_{3,k}^T, \upsilon_k^T]^T$ where the subscript k denotes the sampling instant $t = kT, k = 0, 1, 2, \cdots$, and relations (17) and (18) are substituting into the right hand side of (13). Meanwhile, Substituting (19) into the right hand side of

$$\upsilon_{k+1} = \sum_{i=0}^{\infty} \frac{T^i}{i!}\upsilon_k^i \qquad (20)$$

In the following, an approximate model of the above sampled-data system is considered by neglecting the higher order terms, and the approximate expression of the zeros is calculated. Here, the following approximate expressions are treated.

$$\begin{aligned} y_{1,k+1} = {} & y_{1,k} + T\left[C_1Ax_k + c_N^1(\alpha)C_1Bu_k\right] \\ & + \frac{T^2}{2!}\left[C_1A^2x_k + c_N^2(\alpha)C_1ABu_k\right] \quad (21) \end{aligned}$$

$$\begin{aligned} y_{2,k+1} = {} & y_{2,k} + T\dot{y}_{2,k} \\ & + \frac{T^2}{2!}\left[C_2A^2x_k + c_N^2(\alpha)C_2ABu_k\right] \\ & + \frac{T^3}{3!}\left[C_2A^3x_k + c_N^3(\alpha)C_2A^2Bu_k\right] \quad (22) \end{aligned}$$

$$\begin{aligned} \dot{y}_{2,k+1} = {} & \dot{y}_{2,k} + T\left[C_2A^2x_k + c_N^1(\alpha)C_2ABu_k\right] \\ & + \frac{T^2}{2!}\left[C_2A^3x_k + c_N^2(\alpha)C_2A^2Bu_k\right] \quad (23) \end{aligned}$$

$$\upsilon_{k+1} = \upsilon_k + T\begin{bmatrix} A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} x_k \quad (24)$$

Setting $y_{1,k} = y_{1,k+1} = y_{2,k} = y_{2,k+1} = 0$ yields the zeros of the sampled-data system such that

$$M_0 \begin{bmatrix} Z[\dot{y}_{2,k}] \\ Z[u_k] \\ Z[\upsilon_k] \end{bmatrix} \qquad (25)$$

where $Z[.]$ denotes the z-transform and

$$M_0 = \begin{bmatrix} M_{011} & M_{012} & M_{013} \\ M_{021} & M_{022} & M_{023} \\ M_{031} & M_{032} & M_{033} \\ M_{041} & M_{042} & M_{043} \end{bmatrix} \qquad (26)$$

where

$$\begin{aligned} M_{011} &= (-z+1)I + TA_{23}, \\ M_{012} &= Tc_N^1(\alpha)C_2AB + \frac{T^2}{2!}c_N^2(\alpha)C_2A^2B, \\ M_{013} &= TA_{24}, M_{021} = A_{13}, \\ M_{022} &= c_N^1(\alpha)C_1B + \frac{T}{2!}c_N^2(\alpha)C_1AB, \\ M_{023} &= A_{14}, M_{031} = I + \frac{T}{2}A_{23}, \\ M_{032} &= \frac{T}{2}c_N^2(\alpha)C_2AB + \frac{T^2}{3!}c_N^3(\alpha)C_2A^2B, \\ M_{033} &= \frac{T}{2}A_{24}, \\ M_{041} &= TA_{33}, M_{042} = O, \\ M_{043} &= (-z+1)I + TA_{34} \end{aligned}$$

Hence, the zeros of sampled-data system is obtained by $det(M_0) = 0$.

From the relationship

$$M = \begin{bmatrix} I_{m-p} & O & -2I_{m-p} & O \\ O & I_m & O & O \\ O & O & I_{m-p} & O \\ O & O & O & I_{n-2m+p} \end{bmatrix} M_0 \quad (27)$$

it is obvious that the condition $det(M) = 0$ is equivalent to $det(M_0) = 0$. The matrix multiplication (27) gives

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \qquad (28)$$

where

$$M_{11}=\begin{bmatrix}-(z+1)I & \Delta_1\\ A_{13} & c_N^1(\alpha)C_1B+\frac{T}{2}c_N^2(\alpha)C_1AB\\ I+\dfrac{T}{2}A_{23} & \frac{T}{2}c_N^2(\alpha)C_2AB+\frac{T^2}{3!}c_N^3(\alpha)C_2A^2B\end{bmatrix}$$

$$M_{12}=\begin{bmatrix}O(T^2)\\ A_{14}\\ \dfrac{T}{2}A_{24}\end{bmatrix},\quad M_{21}=\begin{bmatrix}TA_{33} & O\end{bmatrix}$$

$$M_{22}=[(-z+1)I+TA_{34}]$$

$$\Delta_1=T[c_N^1(\alpha)-c_N^2(\alpha)]C_2AB+\frac{T^2}{6}[c_N^2(\alpha)-c_N^3(\alpha)]C_2A^2B$$

Simple calculation yields

$$|M|=|M_{22}||M_{11}-M_{12}M_{22}^{-1}M_{21}| \tag{29}$$

and

$$\begin{aligned}M_{12}M_{22}^{-1}M_{21}&\approx\begin{bmatrix}O(T^2)\\ A_{14}\\ \dfrac{T}{2}A_{24}\end{bmatrix}\frac{1}{(-z+1)}\\ &\times\left(I-\frac{1}{(-z+1)}TA_{34}\right)\begin{bmatrix}TA_{33} & O\end{bmatrix}\\ &=\begin{bmatrix}O(T^3) & \\ O(T) & O\\ O(T^2) & \end{bmatrix}\end{aligned} \tag{30}$$

Note here that the order of each block matrix of the left hand side in M_{11} is lower than that in $M_{12}M_{22}^{-1}M_{21}$, then we have

$$|M_{11}-M_{12}M_{22}^{-1}M_{21}|\approx|M_{11}|$$

Determinant calculation leads to

$$\begin{aligned}|M_{11}|=&\begin{vmatrix}c_N^1(\alpha)C_1B+\dfrac{T}{2!}c_N^2(\alpha)C_1AB\\ \dfrac{T}{2!}c_N^2(\alpha)C_2AB+\dfrac{T^2}{3!}c_N^3(\alpha)C_2A^2B\end{vmatrix}\\ &\times\Big|-(z+1)I-\Big\{T[c_N^1(\alpha)-c_N^2(\alpha)]C_2AB\\ &+\frac{T^2}{6}[c_N^2(\alpha)-c_N^3(\alpha)]C_2A^2B\Big\}\\ &\times\begin{bmatrix}c_N^1(\alpha)C_1B+\dfrac{T}{2!}c_N^2(\alpha)C_1AB\\ \dfrac{T}{2!}c_N^2(\alpha)C_2AB+\dfrac{T^2}{3!}c_N^3(\alpha)C_2A^2B\end{bmatrix}^{-1}\\ &\times\begin{bmatrix}A_{13}\\ I+\dfrac{T}{2!}A_{23}\end{bmatrix}\Big|\end{aligned}$$

Further, we have

$$\begin{aligned}&\begin{bmatrix}c_N^1(\alpha)C_1B+\dfrac{T}{2!}c_N^2(\alpha)C_1AB\\ \dfrac{T}{2!}c_N^2(\alpha)C_2AB+\dfrac{T^2}{3!}c_N^3(\alpha)C_2A^2B\end{bmatrix}^{-1}\\ \times&\begin{bmatrix}A_{13}\\ I+\dfrac{T}{2!}A_{23}\end{bmatrix}\\ =&\left\{\begin{bmatrix}c_N^1(\alpha)I & O\\ O & \dfrac{T}{2}c_N^2(\alpha)I\end{bmatrix}D\Big(I+D^{-1}\right.\\ \times&\left.\left.\begin{bmatrix}\dfrac{1}{2}\dfrac{c_N^2(\alpha)}{c_N^1(\alpha)}C_1AB\\ \dfrac{1}{3}\dfrac{c_N^3(\alpha)}{c_N^2(\alpha)}C_2A^2B\end{bmatrix}T\right)\right\}^{-1}\begin{bmatrix}A_{13}\\ I+\dfrac{T}{2!}A_{23}\end{bmatrix}\\ \approx&\left(I-D^{-1}\begin{bmatrix}\dfrac{1}{2}\dfrac{c_N^2(\alpha)}{c_N^1(\alpha)}C_1AB\\ \dfrac{1}{3}\dfrac{c_N^3(\alpha)}{c_N^2(\alpha)}C_2A^2B\end{bmatrix}T\right)D^{-1}\\ \times&\begin{bmatrix}\dfrac{1}{c_N^1(\alpha)}I & O\\ O & \dfrac{2}{Tc_N^2(\alpha)}I\end{bmatrix}\times\begin{bmatrix}A_{13}\\ I+\dfrac{T}{2!}A_{23}\end{bmatrix}\\ \approx&\begin{bmatrix}D_L & D_R\end{bmatrix}\begin{bmatrix}\dfrac{1}{c_N^1(\alpha)}A_{13}\\ \dfrac{2}{Tc_N^2(\alpha)}I\end{bmatrix}\\ \approx&\frac{2}{Tc_N^2(\alpha)}D_R\end{aligned}$$

As a result, it holds with a constant K_0 that

$$\begin{aligned}|M|\approx&K_0|(z-1)I-TA_{34}|\times\Big|(z+1)I\\ &+\frac{2}{c_N^2(\alpha)}\left[c_N^1(\alpha)-c_N^2(\alpha)\right]c_{20}A_0B_0D_R\\ &+\frac{T}{3c_N^2(\alpha)}\left[c_N^2(\alpha)-c_N^3(\alpha)\right]c_{20}A_0^2B_0D_R\Big|\\ =&K_0|F_I(z)F_s(z)|\end{aligned} \tag{31}$$

The proof is completed. Q.E.D.

Remark 2 *An approximate value of the sampling zero is expressed as*

$$\begin{aligned}&\Big|(z+1)I+\tfrac{2}{c_N^2(\alpha)}[c_N^1(\alpha)-c_N^2(\alpha)]c_{20}A_0B_0D_R\\ &+\tfrac{T}{3c_N^2(\alpha)}[c_N^2(\alpha)-c_N^3(\alpha)]c_{20}A_0^2B_0D_R\Big|=0\end{aligned} \tag{32}$$

and the approximate values of the intrinsic zero are derived from

$$|(z-1)I-TA_{34}|=0 \tag{33}$$

Remark 3 *It is found from Theorem 1 that sampling zeros with a PC GSHF for a small sampling period T are expressed approximately by the eigenvalues of γ, and that all of the zeros are stable, i.e., located strictly inside the unit circle when all the erigenvalues of γ have negative real parts, where*

$$\begin{aligned}\gamma &= \frac{2}{c_N^2(\alpha)}[c_N^1(\alpha) - c_N^2(\alpha)]c_{20}A_0B_0D_R \\ &+ \frac{T}{3c_N^2(\alpha)}[c_N^2(\alpha) - c_N^3(\alpha)]c_{20}A_0^2B_0D_R\end{aligned}$$

It is obvious to see the matrix γ is a simpler expression than the previous result [23].

Remark 4 *Theorem 1 is applicable to also a continuous-time system with multiple zeros by using a similar approach, and furthermore, it is obvious that (32) can definitely provide a more accurate approximate values for the sampling zeros in the case of the relative degrees being one and two.*

Next, we consider the case of $N = 3$ for the sake of simplicity of the hold. Thus, we can choose the parameters α_i so that the sampling zeros can be located in a stable region for a sufficiently small T. The stability condition of the limiting zeros for a sufficiently small T is shown by the following theorem.

Theorem 5 *Under the same assumption as Theorem 1, if all the zeros of S_C are stable and*

$$\begin{aligned}Re\Bigg\{\lambda\Bigg[&\frac{4(\alpha_3-\alpha_1)}{5\alpha_1+3\alpha_2+\alpha_3}C_{20}A_0B_0D_R \\ &+\frac{2(\alpha_2-7\alpha_1)}{9(5\alpha_1+9\alpha_2+\alpha_3)}C_{20}A_0^2B_0D_RT\Bigg]\Bigg\}< 0\end{aligned} \tag{34}$$

where $Re[\lambda(.)]$ denotes a real part of the eigenvalues of a matrix. Then all the zeros of S_D are stable for sufficiently small sampling periods.

Proof: When $N = 3$, we have [34]

$$\begin{aligned}c_N^1(\alpha) &= \frac{1}{3}(\alpha_1+\alpha_2+\alpha_3) \\ c_N^2(\alpha) &= \frac{5}{9}\alpha_1+\frac{1}{3}\alpha_2+\frac{1}{9}\alpha_3 \\ c_N^3(\alpha) &= \frac{19}{27}\alpha_1+\frac{7}{27}\alpha_2+\frac{1}{9}\alpha_3\end{aligned}$$

From (32), simple straightforward calculation can verify that all of the sampling zeros with a PC GSHF are stable, i.e., located strictly inside the unit circle if (34) holds.

As a result, the proof is complete. Q.E.D.

Remark 6 *As in the foregoing statement, the sampling zeros can be assigned inside the unit circle by choosing design parameters $\alpha_i(i = 1,\cdots,N, N \geq \max\{r_j\}, j = 1,\cdots,m)$ so that the sampling zero polynomial of (10) is identical to a desired stable one, though it seems difficult to derive explicit inequality relations for $N \geq 4$.*

Remark 7 *Similarly, we can also obtain approximate expressions of the zeros of sampled-data models as power series with respect to a sampling period T in the case of a PC GSHF when all the relative degrees of continuous-time systems S_C are one and two. Indeed, when some of the relative degrees of S_C are one and the rest are two, it is difficult to derive higher approximate expressions than in the following two cases which all the relative degrees are one or two.*

Remark 8 *When some of the relative degrees of S_C are one and the rest are two, and ZOH is used, the limiting zeros of the discrete-time system for a sufficiently small T are inside the unit circle only if the corresponding conditions are satisfied [9, 11]. Theorem 5 means that the limiting zeros of the discrete-time system can be assigned inside the unit circle by choosing design parameters $\alpha_j(j = 1,\cdots,N)$ such that (34) is hold, even the corresponding conditions with ZOH is not satisfied. Therefore, the PC GSHF with (34) will produce discrete-time models with all the zeros stable for a wider class of continuous-time plants than ZOH.*

Remark 9 *When PC GSHF is used as the signal reconstruction device, there exists some suitable parameters $\alpha_j(j = 1,\cdots,N)$ so that all of the zeros can be located in a stable region for a sufficiently small T. However, the use of PC GSHF can give the theoretical negative aspects [35, 36, 37, 38]. For example, it is well known that essential characteristics of the continuous-time system, such as intersample ripples etc, can be artificially exhibited in the discrete-time transfer function [39, 40]. Thus, it is still necessary to preserve the stability of limiting zeros by appropriately selecting design parameters of PC GSHF while satisfying other performance requirements, such as gain margin, intersample ripple, etc.*

4 Example

This section presents an interesting example to show the stability of sampling zeros with ZOH and PC GSHF. It has shown that the stability of zeros will be improved by using PC GSHF stead of ZOH. Both kinds of the zeros are calculated by use of MATLAB.

Example. Consider the following two-input-two-output, fourth-order continuous-time system S_C of a helicopter [41] with the relative degrees $r_1 = 1, r_2 = 2$ as an example of Theorem 1 and Theorem 5.

$$A = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & -1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, continuous-time system S_C has one zero z_1 [42] and the value of the zero is calculated as $z_1 = -0.018$. Meanwhile, the discretized system S_D with a PC GSHF has two zeros for sufficiently small sampling periods. Because the zero of S_C is stable, one zero of S_D approaches the point $z = 1$ from inside the unit circle as the sampling period tends to zero. The remaining zero of S_D reaches the point $z = -1$.

The decoupling matrix is given by

$$D = \begin{bmatrix} C_{10}B_0 \\ C_{20}A_0B_0 \end{bmatrix} = \begin{bmatrix} 0.36 & -8.6 \\ 0.35 & 0.009 \end{bmatrix}$$

Further

$$\begin{aligned} C_{20}A_0B_0D_R &= -3.1 < 0 \\ C_{20}A_0^2B_0D_R &= -1.6 < 0 \end{aligned}$$

From Theorem 1 and Theorem 5, a sampling zero is located strictly inside the unit circle for sufficiently small sampling periods if

$$-\frac{62(\alpha_3 - \alpha_1)}{5(5\alpha_1 + 3\alpha_2 + \alpha_3)} - \frac{16(\alpha_2 - 7\alpha_1)}{45(5\alpha_1 + 9\alpha_2 + \alpha_3)}T < 0$$

Hence, the intrinsic zero z_1 and the sampling zero z_2 are expressed approximately as, respectively

$$\begin{aligned} z_1 &\approx 1 - 0.018T \\ z_2 &\approx -1 + \frac{62(\alpha_3 - \alpha_1)}{5(5\alpha_1 + 3\alpha_2 + \alpha_3)} + \frac{16(\alpha_2 - 7\alpha_1)}{45(5\alpha_1 + 9\alpha_2 + \alpha_3)}T \end{aligned}$$

On the other hand, we choose the suitable parameters α_1, α_2 and α_3 of a PC GSHF such that the sampling zeros can be placed inside the unit circle. Furthermore, there exists a set of solutions $\alpha_1 = 0.1, \alpha_2 = 0.8, \alpha_3 = 0.3$, then it follows that the approximate values and the exact values of the intrinsic zero and the sampling zero of the sampled-data system are respectively shown in Table 1 and Table 2 by use of MATLAB, and Fig. 3 and Fig. 4 present respectively the intrinsic zero and sampling zero in the case of a PC GSHF.

Table 1: Intrinsic zero of the sampled-data system with PC GSHF

T	Approximate values (33)	Exact values
0.01	0.999820000	0.999820115
0.02	0.999640000	0.999640263
0.05	0.999100000	0.999100901
0.1	0.998200000	0.998202612
0.2	0.996400000	0.996408465

Table 2: Sampling zero of the sampled-data system with PC GSHF

T	Approximate values (32)	Exact values
0.01	−0.225047830	−0.224967909
0.02	−0.224939088	−0.224821314
0.05	−0.224859822	−0.224671422
0.1	−0.224645156	−0.224111129
0.2	−0.224125731	−0.222907696

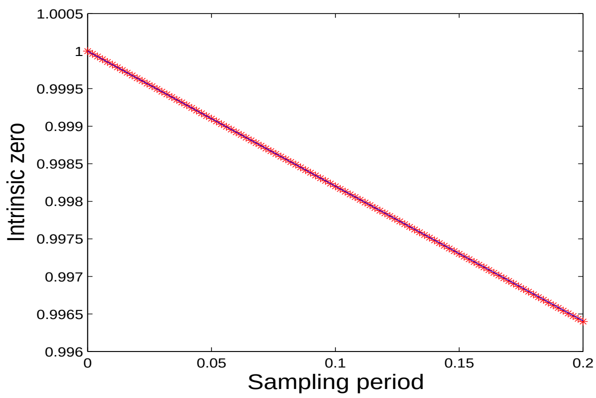

Figure 3: Intrinsic zero of the sampled-data model with PC GSHF

Remark 10 *When a continuous-time input $u(t)$ is generated by a ZOH or a PC GSHF, they have the same intrinsic zero from Remark 2 and Example.*

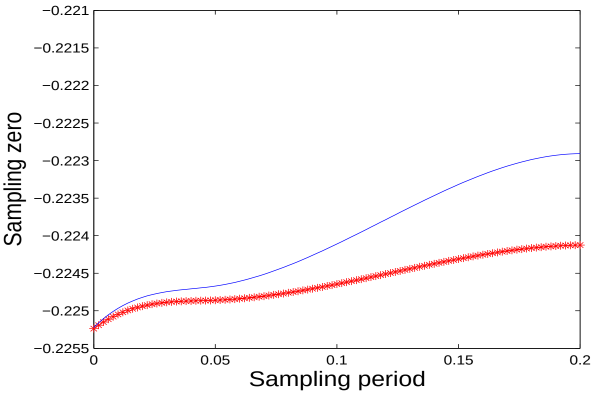

Figure 4: Sampling zero of the sampled-data model with PC GSHF

Remark 11 *It is obvious that PC GSHF reduces to ZOH for $N = 1$ or $\alpha_1 = \cdots = \alpha_N$. The limiting zeros for sufficiently small T in the case of a ZOH are stable with relative degree 1 while it is unstable with relative degree 2 in some case when the sum of the zeros is less than or equal to the sum of the poles. Thus, a ZOH provides no advantage over PC GSHF with stability of the limiting zeros [21, 23].*

5 Conclusions

This paper analyzes the asymptotic behavior of zeros of sampled multivariable systems when a continuous-time system with relative degrees being one and two is discretized using a PC GSHF, and gives approximate expression of the zeros as power series expansions with respect to a sampling period. And further the stability of the zeros is discussed when the sampling periods tend to zero. Moreover, the results obtained are useful for stability test of zeros of a sampled-data system for small sampling periods without direct calculation of the zeros. As a results of this work, it has been shown that if the zeros of S_C are stable and (34) holds, all of the zeros of the discrete-time multivariable system with the relative degrees being one and two stay inside the stability for sufficiently small sampling periods. The GSHF will produce discrete-time multivariable systems with all stable zeros for a wider class of continuous-time systems than ZOH. More importantly, there always exists a range of α_j for which the GSHF zeros are more stable than their counterparts when using a ZOH or FROH device, namely, as stable as possible or improved stability properties. Hence, the main theorem of this paper is an extension of the results of SISO systems to multivariable models.

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