Abstract: Given an $k$-tuple of vectors, $S = (v_1, v_2, \ldots, v_k)$, the neighbourhood adjacency code of a vertex $v$ with respect to $S$, denoted by $nc_S(v)$ and defined by $(a_1, a_2, \ldots, a_k)$ where $a_i$ is 1 if $v$ and $v_i$ are adjacent and 0 otherwise. $S$ is called a neighbourhood resolving set or a neighbourhood resolving number of $G$ and is denoted by $nr(G)$ ($NR(G)$). In this article, we consider the $nr$-excellent graphs. For any graph $G$, $G$ is $nr$-excellent if every vertex of $G$ is contained in a minimum neighbourhood resolving set of $G$. We first prove that the union and join of two given $nr$-excellent graphs is $nr$-excellent under certain conditions. Also we prove that a non $nr$-excellent graph $G$ can be embedded in a $nr$-excellent graph $H$ such that $nr(H) = nr(G) + \text{number of } nr^{-\text{bad}} \text{ vertices of } G$. Some more results are also discussed.

Key Words: neighbourhood resolving set, neighbourhood resolving number, $nr$-excellent, $nr$-bad vertex

1 Introduction

Let $G = (V, E)$ be a simple graph. Given a $k$-tuple of vertices $(v_1, v_2, \ldots, v_k)$, assign to each vertex $v \in V(G)$, the $k$-tuple of its distances to these vertices, $f(v) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. Letting $S = (v_1, v_2, \ldots, v_k)$, the $k$-tuple $f(v)$ is called the $S$-location of $v$. A set $S$ is a locating set for $G$, if no two vertices have the same $S$-location and the location number $L(G)$ is the minimum cardinality of a locating set. This concept was introduced by P. J. Slater in [25] [27] [26]. It can be easily seen that $L(G) = 1$ if and only if $G$ is a path and any path $P_n$, ($n \geq 2$) has two $L(P_n)$-sets, each consisting of an end vertex. Also for any tree $T$ with at least three vertices, a subset $S$ is a locating set if and only if for each vertex $u$ there are vertices in $S$, contained in at least $\text{deg}(u) - 1$ of the $\text{deg}(u)$ components of $T - u$.

Slater [25] described the usefulness of locating sets when working with U.S. sonar and Coast Guard Loran stations. Independently Harary and Melter [14] found these concepts as well but used the term metric dimension, rather than location number, the terminology which has been adopted by subsequent authors. Recently, these concepts were rediscovered by M. A. Johnson [18] [19] of the pharmacia company while attempting to develop a capability of large datasets of chemical graphs.

If $V$ is a finite dimensional vector space over a field $F$ and $B = (v_1, v_2, \ldots, v_k)$ is an ordered basis, then every vector $v$ in $V$ can be associated with a unique $k$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k$. If $v$ and $w$ are distinct vectors, then their associated $k$-tuples of scalars are distinct. Thus any ordered basis gives rise to a coding for the elements of the vector space. The dimension of the vector space is the order of a basis. Since locating sets in a graph play the same role as bases in a vector space, the locating number is termed as metric dimension. Chartrand et al [4] called locating sets as resolving sets and retained the term metric dimension for locating number.

Many papers have appeared in this area since 2000. For example dominating resolving sets, independent resolving sets, acyclic resolving sets, connected resolving sets, resolving partitions and resolving decompositions were studied by many [4], [5], [6],[7], [8], [9], [11], [12], [13], [2], [3], [44].

In all these studies, the graph considered are connected and the codes of the vertices are $k$-tuples whose entries are positive integers ranging from 0 to the diameter of the graph. An alternate coding of the vertices using 0 and 1 (binary digits) can be thought of, which is possible in disconnected graphs also.

A new type of binary coding for vertices is defined through adjacency. A vertex $u$ in a graph
G with respect to a \(k\)-tuple of vertices (say) \(S = (v_1, v_2, \ldots, v_k)\) is assigned the code (which is written as \(nc_S(u) = (a_1, a_2, \ldots, a_k)\) where \(a_i = 1\) if \(u\) and \(v_i\) are adjacent and 0 otherwise. If \(nc_S(u) \neq nc_S(v)\) for any \(u \neq v, u, v \in V(G)\), then \(S\) is called a neighbourhood resolving set of \(G\). If we consider the vector spaces over the field \(Z_2\), then the code of any vector is a binary code. Thus neighbourhood resolving sets correspond to bases in a vector space over \(Z_2\). Obviously, neighbourhood resolving sets can be defined in disconnected graphs also. But not all graphs can admit neighbourhood resolving sets. For example, a graph \(G\) in which \(N(u) = N(v)\), for two non-adjacent vertices \(u, v \in V(G)\) will not have any neighbourhood resolving set. But these graphs can be embedded in graphs having neighbourhood resolving sets. Obviously a graph with more than one isolated vertex will not have neighbourhood resolving sets. The minimum cardinality of a neighbourhood resolving set in a graph \(G\) which admits neighbourhood resolving sets is called the neighbourhood resolving number of \(G\) and is denoted by \(nr(G)\). Any \(nr\)-set of \(G\) with \(nr(G) \geq 2\) cannot be independent. Graphs are often used to model different physical networks. \(nr\)-sets can be used to detect intruders on models of networks of facilities and it is used to detect the failures on networks of routers or processors.

Suk J. Seo and P. Slater [34] defined the same type of problem as an open neighborhood locating dominating set (OLD-set), is a minimum cardinality vertex set \(S\) with the property that for each vertex \(v\) its open neighborhood \(N(v)\) has a unique non-empty intersection with \(S\). But in Neighbourhood resolving sets \(N(v)\) may have the empty intersection with \(S\). Clearly every OLD-set of a graph \(G\) is a neighbourhood resolving set of \(G\), but the converse need not be true.

M.G. Karpovsky, K. Chakrabarty, L.B. Levitin [20] introduced the concept of identifying sets using closed neighbourhoods to resolve vertices of \(G\). This concept was elaborately studied by A. Lobestein [21].

Let \(\mu\) be a parameter of a graph. A vertex \(v \in V(G)\) is said to be \(\mu\)-good if \(v\) belongs to a \(\mu\)-minimum (\(\mu\)-maximum) set of \(G\) according as \(\mu\) is a super hereditary (hereditary) parameter. \(v\) is said to be \(\mu\)-bad if it is not \(\mu\)-good. A graph \(G\) is said to be \(\mu\)-excellent if every vertex of \(G\) is \(\mu\)-good. Excellence with respect to domination and total domination were studied in [10], [15], [30], [31], [32], [33]. N. Sridhara and Yamuna [31], [32], [33], have defined various types of excellence. In this paper, definition, examples and properties of \(nr\)-excellent graphs is discussed.

2 Neighbourhood Resolving sets in Graphs

Definition 1 Let \(G\) be any graph. Let \(S \subset V(G)\). Consider the \(k\)-tuple \((u_1, u_2, \ldots, u_k)\) where \(S = \{u_1, u_2, \ldots, u_k\}, k \geq 1\). Let \(v \in V(G)\). Define a binary neighbourhood code of \(v\) with respect to the \(k\)-tuple \((u_1, u_2, \ldots, u_k)\), denoted by \(nc_S(v)\) as a \(k\)-tuple \((r_1, r_2, \ldots, r_k)\) where

\[
r_i = \begin{cases} 1, & \text{if } v \in N(u_i), 1 \leq i \leq k \\ 0, & \text{otherwise} \end{cases}
\]

\(S\) is called a neighbourhood resolving set or a neighbourhood \(r\)-set if \(nc_S(u) \neq nc_S(v)\) for any \(u, v \in V(G)\).

The least cardinality of a minimal neighbourhood resolving set of \(G\) is called the neighbourhood resolving number of \(G\) and is denoted by \(nr(G)\). The maximum cardinality of a minimal neighbourhood resolving set of \(G\) is called the upper neighbourhood resolving number of \(G\) and is denoted by \(NR(G)\).

Clearly \(nr(G) \leq NR(G)\). A neighbourhood resolving set \(S\) of \(G\) is called a minimum neighbourhood resolving set or \(nr\)-set if \(S\) is a neighbourhood resolving set with cardinality \(nr(G)\).

Example 2

Now \(S_1 = \{u_1, u_2, u_3\}\) is a neighbourhood resolving set of \(G\), since \(nc_S(u_1) = (0, 1, 1)\); \(nc_S(u_2) = (1, 0, 1)\); \(nc_S(u_3) = (0, 1, 0)\); \(nc_S(u_4) = (0, 0, 1)\) and \(nc_S(u_5) = (1, 1, 0)\). Also \(S_2 = \{u_1, u_3, u_4\}\). \(S_3 = \{u_1, u_2, u_4\}\). \(S_4 = \{u_1, u_3, u_5\}\) are neighbourhood resolving sets of \(G\). For this graph, \(nr(G) = NR(G) = 3\).

Observation 3 The above definition holds good even if \(G\) is disconnected.

In the following theorem characterisation of connected graphs which admit neighbourhood resolving sets is given.
Theorem 4 [38] Let $G$ be a connected graph of order $n \geq 3$. Then $G$ does not have any neighbourhood resolving set if and only if there exist two non adjacent vertices $u$ and $v$ in $V(G)$ such that $N(u) = N(v)$.

Definition 5 [40] A subset $S$ of $V(G)$ is called an $nr$-irredundant set of $G$ if for every $u \in S$, there exist $x, y \in V$ which are privately resolved by $u$.

Theorem 6 [40] Every minimal neighbourhood resolving set of $G$ is a maximal neighbourhood resolving irredundant set of $G$.

Definition 7 [40] The minimum cardinality of a maximal neighbourhood resolving irredundant set of $G$ is called the neighbourhood resolving irredundance number of $G$ and is denoted by $nr(G)$. The maximum cardinality is called the upper neighbourhood resolving irredundance number of $G$ and is denoted by $IR_{nr}(G)$.

Observation 8 [40] For any graph $G$,

$$ir_{nr}(G) \leq nr(G) \leq NR(G) \leq IR_{nr}(G).$$

Theorem 9 [41] For any graph $G$,

$$nr(G) \leq n - 1.$$ 

Theorem 10 [39] Let $G$ be a connected graph of order $n$ such that $nr(G) = k$. Then $\log_2 n \leq k$.

Observation 11 [39] There exists a graph $G$ in which $n = 2k$ and there exists a neighbourhood resolving set of cardinality $k$ such that $nr(G) = k$. Hence all the distinct binary $k$-vectors appear as codes for the $n$ vertices.

Theorem 12 [41] Let $G$ be a connected graph of order $n$ admitting neighbourhood resolving sets of $G$ and let $nr(G) = k$. Then $k = 1$ if and only if $G$ is either $K_2$ or $K_1$.

Theorem 13 [41] Let $G$ be a connected graph of order $n$ admitting neighbourhood resolving sets of $G$. Then $nr(G) = 2$ if and only if $G$ is either $K_3$ or $K_3 + a$ pendant edge or $K_3 \cup K_1$ or $K_2 \cup K_1$.

Result 14 [40] For a complete graph $K_n$,

$$nr(K_n) = n - 1, n \geq 2.$$ 

Result 15 [40] For a path $P_n, n \geq 6$,

$$nr(P_n) = \left\lceil \frac{2n}{3} \right\rceil.$$ 

3 $nr$-excellent graphs

Definition 16 Let $G = (V, E)$ be a simple graph. Let $u \in V(G)$. Then $u$ is said to be $nr$-good if $u$ is contained in a minimum neighbourhood resolving set of $G$. A vertex $u$ is said to be $nr$-bad if there exists no minimum neighbourhood resolving set of $G$ containing $u$.

Definition 17 A graph $G$ is said to be $nr$-excellent if every vertex of $G$ is $nr$-good.

Example 18 Consider the graph $G$:

```
\begin{center}
\begin{tikzpicture}
\node (1) at (1,1) {1};
\node (2) at (2,1) {2};
\node (3) at (2,2) {3};
\node (4) at (3,1) {4};
\node (5) at (3,2) {5};
\node (6) at (4,1) {6};
\node (7) at (5,1) {7};
\draw (1) -- (2) -- (3) -- (4) -- (5) -- (6) -- (7);
\end{tikzpicture}
\end{center}
```

The $nr$-sets of $G$ containing the vertices $1, 2, 3, 4, 5, 6, 7$ are $\{1, 2, 3, 4\}, \{1, 5, 6, 7\}, \{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{1, 2, 3, 5\}, \{1, 4, 6, 7\}, \{1, 2, 3, 6\} \text{ and } \{3, 4, 5, 6\}$.

Therefore every vertex of $G$ belongs to some $nr$-set of $G$ and hence $nr$-good. Therefore $G$ is $nr$-excellent.

Theorem 19 The following results are obvious from the definition.

(i) $K_n$ is $nr$-excellent, for every $n$.
(ii) $P_n$ is $nr$-excellent if $n \equiv 0, 2, 3, 4 \pmod{6}$ and $P_n$ is non-$nr$-excellent if $n \equiv 1, 5 \pmod{6}$.
(iii) $C_n$ is $nr$-excellent, for every $n \not\equiv 4$.

Remark 20 Consider the graph $G$ obtained from two complete graphs $K_m$ and $K_n$, $m, n \geq 4$ having exactly one vertex in common. Then $nr(G) = m + n - 4$. $G$ is not $nr$-excellent, since there is no $nr$-set containing the common vertex.

Remark 21 Suppose $G$ has a unique $nr$-set. Then $G$ is not $nr$-excellent.

Theorem 22 Any vertex transitive graph is $nr$-excellent.

Proof: Let $G$ be a vertex transitive graph. Let $S$ be an $nr$-set of $G$. Let $u \in V(G)$. Suppose $u \notin S$. Select any vertex $v \in S$. As $G$ is vertex transitive, there exists an automorphism $\phi$ of $G$ which maps $v$ to $u$. Let $S^1 = \{\phi(w) \mid w \in S\}$. Since $S$ is a $nr$-set and $\phi$ is an automorphism, $S^1$ is also a $nr$-set. Since $v \in S$, $\phi(v) = u \in S^1$. Hence $G$ is $nr$-excellent.

Remark 23 Any vertex transitive graph is regular.
Remark 24 1. There exists a regular graph which is nr-excellent, but not vertex transitive.

![Graph G](image)

The nr-sets of G are \{1, 2, 5, 6, 9\} and 3, 4, 7, 8, 9. G is nr-excellent but not vertex transitive.

2. The path \(P_n\) where \(n \equiv 1, 5 \pmod{6}\) are not nr-excellent and not vertex transitive, but \(P_n\) where \(n \equiv 0, 2, 3, 4 \pmod{6}\) are nr-excellent but not vertex transitive.

3. There are regular graphs which are not vertex transitive without neighborhood resolving sets. The following graph G is an example.

![Graph G](image)

Since 1 and 4 are non-adjacent and \(N(1) = N(4)\), G has no neighborhood resolving sets.

Theorem 25 Let \(G_1\) and \(G_2\) be nr-excellent graphs. Then \(G_1 \cup G_2\) is nr-excellent if and only if at least one of \(G_1, G_2\) satisfies one of the following conditions

(i) If every nr-set of \(G_1, G_2\) containing a vertex \(x \in V(G_1)(V(G_2))\) admits a 0-code, then there exists an nr-set of \(G_2(G_1)\) not allowing 0-code for any vertex of \(V(G_2)(V(G_1))\).

(ii) Every nr-set \(S\) of \(G_i\), \(i = 1, 2\), allows 0-code.

Proof: Let \(G_1\) and \(G_2\) be nr-excellent graphs.

Suppose (i) holds. Then \(nr(G_1 \cup G_2) = nr(G_1) + nr(G_2)\). Let \(u \in V(G_1)\). Suppose there exists an nr-set \(S\) of \(G_1\) containing \(u\) and not admitting 0-code. Then for any nr-set \(T\) of \(G_2\), \(S \cup T\) is an nr-set of \(G_1 \cup G_2\) containing \(u\).

Suppose every nr-set of \(G_1\) containing \(u\) admits 0-code. Then by condition (i), there exists an nr-set \(T\) of \(G_2\) not allowing 0-code. Then \(S \cup T\) is an nr-set of \(G_1 \cup G_2\) containing \(u\). A similar proof holds for any vertex in \(V(G)\). Therefore \(G_1 \cup G_2\) is nr-excellent.

Suppose (ii) holds. Then for any nr-set \(S\) of \(G_1\) and an nr-set \(T\) of \(G_2\), \(S \cup T \cup \{x\}\) where \(x \in V(G_1 \cup G_2)\) and \(x\) is adjacent to at least one of the two vertices which receive 0-code with respect to \(S\) or with respect to \(T\), is an nr-set of \(G_1 \cup G_2\). Since \(G_1\) and \(G_2\) are nr-excellent, we get that \(G_1 \cup G_2\) is nr-excellent.

Conversely, Suppose \(G_1 \cup G_2\) is nr-excellent.

Case (i) \(nr(G_1 \cup G_2) = nr(G_1) + nr(G_2)\). Therefore at least one of \(G_1, G_2\) has an nr-set for which there exists a vertex that receives 0-code with respect to \(S\). Without loss of generality, let \(S \subseteq V(G_1)\). Suppose there exists a vertex \(u \in V(G_1)\) such that every nr-set of \(G_1\) containing \(u\) admits 0-code. Since \(G_1 \cup G_2\) is nr-excellent, there exists an nr-set of \(G_1 \cup G_2\) containing \(u\).

Let \(S_1 = S \cup V(G_1)\) and \(S_2 = S \cup V(G_2)\). By our assumption, \(S_1\) admits 0-code. Since \(S_1\) is an nr-set of \(G_1 \cup G_2\), \(S_2\) is an nr-set of \(G_2\) which does not allow 0-code in \(G_2\). Hence condition (i) hold.

Case (ii) \(nr(G_1 \cup G_2) = nr(G_1) + nr(G_2) + 1\). Therefore every nr-set of \(G_1\) as well as that of \(G_2\), allows 0-code. Hence (ii) holds.

Definition 26 A graph G is said to be of type-I if there exists an nr-set S in G such that no vertex in \(V - S\) is adjacent to every vertex of \(S\). (That is every vertex in \(V - S\) is not adjacent to at least one vertex of \(S\)).

Definition 27 A graph G is said to be of type-II if there exists an nr-set S in G such that there exists a vertex in \(V - S\) which is adjacent to every vertex of \(S\).

Definition 28 (i) A graph G is of type-I nr-full if for every nr-set S in G, every vertex outside S is not adjacent to at least one vertex in S.

(ii) A graph G is type-II nr-full if for every nr-set S in G, there exists a vertex outside S adjacent to every vertex of S.

Example 29 (i) \(K_n\), \(n \geq 3\) is of type-II nr-full, but type-I non-nr-full.

(ii) Consider the following graph G.

![Graph G](image)

The nr-sets of G are \{1, 3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 4, 5\}, \{3, 4, 5\} and \{1, 3, 5\}. For an nr-set \(S = \{1, 3, 4\}\) there exists a vertex 2 outside S such that 2 is adjacent to 1, 3, 4. Similarly the property holds for other nr-sets also.

Therefore, G is type-II nr-full, but not type-I nr-full.
The $nr$-sets of $G$ are $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 2, 5\}$.

The type-I $nr$-sets are $\{1, 3, 4\}$, $\{1, 3, 5\}$ and $\{1, 2, 5\}$, since every vertex outside the $nr$-set is not adjacent to at least one vertex of the set.

The only type-II $nr$-set is $\{1, 2, 4\}$, since there exists a vertex $5$ outside the set $\{1, 2, 4\}$ such that $5$ is adjacent to $1, 2$ and $4$.

Therefore $G$ is type-II non-$nr$-full and type-I non-$nr$-full.

(iii) $G :$

The vertex $5$ outside the set $\{1, 2, 3\}$.

(v) $P_n$ and $C_n$ are type-I $nr$-full, but type-II non-$nr$-full.

(vi) Consider the graph $G$.

$G :$

The $nr$-sets of $G$ are $S_1 \cup S_2$ where $S_1$ and $S_2$ are 3-element subsets of $\{1, 2, 3\}$ and $\{5, 6, 7, 8\}$ respectively.

$G$ is type-I $nr$-full and type-II non-$nr$-full.

Remark 30 If $G$ is type-I $nr$-full then it is type-II non-$nr$-full and vice versa.

Definition 31 Let $G$ be a graph. A vertex $u \in V(G)$ is said to be type-I (type-II) $nr$-good if $u$ belongs to some type-I(type-II) $nr$-set. Otherwise $u$ is type-I(type-II) $nr$-bad.

A graph $G$ is type-I (type-II) $nr$-excellent if every vertex of $G$ is type-I (type-II) $nr$-good.

Example 32 (i) $K_n$ is type-II $nr$-excellent, but not type-I $nr$-excellent.

(ii) Consider the following graph $G$.

$G :$

The type-I $nr$-sets of $G$ are $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 4, 5, 6\}$, $\{1, 2, 3, 6\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 5, 6\}$, $\{1, 4, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$, $\{2, 3, 5, 6\}$, $\{2, 3, 6, 7\}$, $\{3, 4, 5, 6\}$, $\{3, 4, 5, 7\}$, $\{3, 4, 6, 7\}$, $\{3, 5, 6, 7\}$, $\{2, 3, 6, 7\}$ and $\{2, 5, 6, 7\}$. The only type-II $nr$-set is $\{2, 3, 5, 6\}$. Note that $G$ is both type-II and type-I non-$nr$-full. Therefore $G$ is type-I $nr$-excellent but not type-II $nr$-excellent.

(iii) Consider the following graph $G$.

$G :$

The only type-I $nr$-set of $G$ is $\{1, 2, 6\}$ and the only type-II $nr$-set of $G$ is $\{1, 2, 4\}$. Therefore $G$ is neither type-I $nr$-excellent nor type-II $nr$-excellent.

Note that $G$ is not $nr$-full.

Theorem 33 Let $G$ and $H$ be $nr$-excellent graphs. Then $G + H$ is $nr$-excellent if and only if

(i) $G$ and $H$ are type-I $nr$-full. (or)

(ii) $G$ and $H$ are type-II $nr$-full. (or)

(iii) $G$ is type-I $nr$-full and $H$ is not $nr$-full. (or)

(iv) $H$ is type-I $nr$-full and $G$ is not $nr$-full. (or)

(v) $G$ is type-II $nr$-full, $H$ is not $nr$-full and for every vertex $u \in V(H)$, there exists a type-I $nr$-set in $H$ containing $u$. (or)

(vi) $G$ is not $nr$-full, $H$ is type-II $nr$-full and for every vertex $u \in V(H)$, there exists a type-I $nr$-set in $H$ containing $u$. (or)

(vii) $G$ and $H$ are not $nr$-full and for every vertex $u \in V(G)$, there exists a type-I $nr$-set in $G$ containing $u$ and for every vertex $v \in V(H)$, there exists a type-I $nr$-set in $H$ containing $v$.
Proof: Suppose $G$ and $H$ are $nr$-full.

Case (i): $G$ and $H$ are type-I $nr$-full.

Then for any $nr$-set $S_1$ of $G$ and any $nr$-set $S_2$ of $H$, $S_1 \cup S_2$ is an $nr$-set of $G + H$. Therefore $G + H$ is $nr$-excellent.

Case (ii): $G$ and $H$ are type-II $nr$-full.

Then for any $nr$-set $S_1$ of $G$ and any $nr$-set $S_2$ of $H$, $S_1 \cup \{u\} \cup S_2$ and $S_1 \cup S_2 \cup \{v\}$ is an $nr$-set of $G + H$, where $u \in \{V(G) - S_1\}$ is adjacent with every vertex of $S_1$ and $v \in \{V(H) - S_2\}$ is adjacent with every vertex of $S_2$.

Therefore $nr(G + H) = nr(G) + nr(H) + 1$. Clearly $G + H$ is $nr$-excellent.

Case (iii): $G$ is not $nr$-full and $H$ is $nr$-full.

Therefore there exist $nr$-sets $S_1$ and $S_2$ in $G$ such that there exists a vertex $u \in V(G) - S_1$ which is adjacent to every vertex of $S_1$ and for any vertex $v \in V(G) - S_2$, $v$ is non-adjacent with some vertex of $S_2$.

Let $S_3$ be any $nr$-set of $H$.

Subcase (i): $H$ is type-I $nr$-full.

Then for every vertex $w \in V(H) - S_3$, $w$ is not adjacent to some vertex of $S_3$. In this case for any $nr$-set $S$ of $G$ and any $nr$-set $S$ of $H$, $S \cup S$ is an $nr$-set of $G + H$. Therefore $G + H$ is $nr$-excellent.

Subcase (ii): $H$ is type-II $nr$-full.

Let $S_2 \cup S_2$ be any $nr$-set of $H$. Then $S_2 \cup S_2$ is an $nr$-set of $G + H$ and $S_1 \cup S_2$ is an $nr$-set of $G + H$.

If for every vertex $u \in V(G)$, there exists a type-I $nr$-set $T_u$ in $H$ containing $u$, then $G + H$ is $nr$-excellent.

If there exists a vertex $u \in V(G)$ such that if it is contained only in type-II $nr$-sets, then $G + H$ is not $nr$-excellent.

Case (iv): $G$ is $nr$-full and $H$ is not $nr$-full.

If $G$ is type-I then arguing as in Subcase (i) of Case (iii), we get that $G + H$ is $nr$-excellent. If $G$ is type-II, then arguing as in Subcase (ii) of Case (iii), if for every vertex $u \in V(G)$, there exists a type-I $nr$-set $T_u$ in $H$ containing $u$, then $G + H$ is $nr$-excellent.

Case (v): $G$ and $H$ are not $nr$-full.

Then the union of any two type-II $nr$-sets in $G$ and $H$ will not be an $nr$-set in $G + H$. If there exists a vertex $u \in V(G)$ such that it is contained only in type-II $nr$-sets of $G$ (or) there exists a vertex $v \in V(H)$ such that it is contained only in type-II $nr$-sets of $H$. Then $G + H$ is not $nr$-excellent. Otherwise $G + H$ is $nr$-excellent. Converse is obviously true. \(\Box\)

The above theorem can be restated as follows:

Theorem 34 Let $G$ and $H$ be $nr$-excellent graphs. Then $G + H$ is $nr$-excellent if and only if

(i) $G$ and $H$ are type-II $nr$-full. (or)
(ii) $G$ or $H$ is type-I $nr$-full. (or)
(iii) $G$ is type-II $nr$-full and $H$ is type-I $nr$-excellent or vice versa. (or)

(iv) $G$ and $H$ are non-$nr$-full and both are type-I $nr$-excellent.

Remark 35 Let $G$ be a non-$nr$-excellent graph. Then the number of $nr$-bad vertices of $G$ is $nr(G)$.

When $nr(G) = 1$, $G = K_1$ or $K_2$ and hence $G$ is $nr$-excellent. Therefore $nr(G) \geq 2$ and number of $nr$-bad vertices of $G \leq n - 2$.

When $nr(G) = 2$, $G$ is $K_3$ or $K_3 + 2$ pendant edge or $K_3 \cup K_1$ or $K_2 \cup K_1$.

When $G = K_3$ or $K_3 + 2$ pendant edge or $K_3 \cup K_1$ or $K_2 \cup K_1$, then $G$ is not $nr$-excellent.

The number of $nr$-bad vertices of $G$ is $2$ or $1$.

Therefore the number of $nr$-bad vertices when $G$ is $K_3 + 2$ pendant edge $= 2 = n - 2$, since $n = 4$.

Theorem 36 Let $G$ be a non-$nr$-excellent graph. Then $G$ can be embedded in a $nr$-excellent graph (say) $H$ such that $nr(H) = nr(G) + number of nr-bad vertices of G$.

Proof: Let $G$ be a non-$nr$-excellent graph. Let $B = \{u_1, u_2, \ldots, u_k\}$ be the set of all $nr$-bad vertices of $G$. Add a vertex $v_1$ to $V(G)$ and join $v_1$ with every vertex of $N(u_1)$.

Let the resulting graph be $G_1$. Let $S$ be an $nr$-set of $G$. Then $N_S(u_1) = N_S(v_1)$. Therefore $S$ is not a neighbourhood resolving set of $G$.

Let $S_1 = S \cup \{u_1\}$ and $S_2 = S \cup \{v_1\}$. Let $x, y \in V(G)$. Then $nc_{S_1}(x) = (a_1, a_2, \ldots, a_r)$ and $nc_{S_1}(y) = (b_1, b_2, \ldots, b_r, c_1)$.

Since $S$ is an $nr$-set of $G$, there exist $i, j, k$ such that $1 \leq i \leq r$ and $a_i \neq b_j$, $1 \leq j \leq r$ and $a_j \neq c_i$ and $1 \leq k \leq r$ and $b_k \neq c_k$. Therefore $S_1$ is a neighbourhood resolving set of $G_1$. Similarly $S_2$ is a neighbourhood resolving set of $G_1$. Therefore $nr(G_1) \leq nr(G) + 1$.

Let $D$ be an $nr$-set of $G_1$. If $D \subseteq V(G)$, then $nc_D(u_1) = nc_D(v_1)$, a contradiction, since $D$ is an $nr$-set of $G$. Therefore $D$ must contain either $u_1$ or $v_1$.

Case (i): $u_1 \in D$ and $v_1 \notin D$. Then $D \subseteq V(G)$. Clearly $D$ is a neighbourhood resolving set of $G$.

Since $u_1$ is a $nr$-bad vertex of $G$, $D$ is not an $nr$-set of $G$. Therefore $nr(G) < |D| = nr(G_1)$ and hence $nr(G) + 1 \leq nr(G_1)$. Therefore $nr(G_1) = nr(G) + 1$.

Case (ii): $v_1 \in D$ and $u_1 \notin D$.

Let $x, y \in V(G)$. If $x, y \in N[u_1]$, then $x$ and $y$ have the same code with respect to $v_1$ and hence $v_1$ does not resolve $x$ and $y$. Let $x = u_1$ and $y \notin N[u_1]$. Then $v_1$ resolves $u_1$ and $y$. Suppose $v_1$ privately resolves $u_1$ and $y$ where $y \notin N[u_1]$. Then $nc_{D \setminus \{u_1\}}(u_1) = nc_{D \setminus \{v_1\}}(y) =$
ncD−{v1}(v1). Since u1 is not adjacent to y, v1 is not adjacent to y. Therefore ncD(y) = ncD(v1), a contradiction, since D is an nr-set of G. Therefore 
v1 resolves only u1 and v1 in G. Thus D − {v1} is a neighbourhood resolving set of G and nr(G) ≤ |D| − 1 = nr(G1) − 1 which implies that nr(G1) ≥ nr(G) + 1. Therefore nr(G1) = nr(G) + 1.

Case (iii) : v1, v1 ∈ D.

As in Case(ii), v1 resolves only u1 and v1 in G. Since u1 ∈ D, u1 resolves u1 and v1 in G1. Therefore D − {u1} is a neighbourhood resolving set of G1, a contradiction. Thus S11 = S ∪ {u1} and S12 = S ∪ {v1} are nr-sets of G1, which means u1 and v1 are nr-good in G1.

Since any nr-good vertices of G belongs to an nr-set of G, these vertices are also nr-good in G1. More over G is an induced subgraph of G1 and nr(G1) = nr(G) + 1. Let B1 be the set of all nr-
bad vertices of G1. Then B1 = \{u2, u3, \ldots, uk\}. Proceeding as before, construct a graph G2 such that there is a new vertex v2 \notin V(G) such that ncG2[v2] =
\[ N_{G1}[u2] = N_G[u2] \] and S_{21}^{(1)} = S_{11} \cup \{u2\}; S_{21}^{(2)} = S_{12} \cup \{v1\} are nr-sets of G2. Therefore u2 and v2 are nr-good in G2 and all nr-good vertices in G1 are also nr-good in G2 and G1 and hence G is an induced subgraph of G2. Also nr(G2) = nr(G) + 2.

Proceeding in this way, the k-th stage yields a graph G_k such that G_k is nr-excellent, G is an induced subgraph of G_k and nr(G_k) = nr(G) + k. □

**Corollary 37** Let G be a non nr-excellent graph and let H be a nr-excellent graph containing G as an induced subgraph. Then
\[ nr(G) < nr(H) \leq nr(G) + n - 2. \]

**Theorem 38** Let G be a connected non-nr-excellent graph. Let \{v1, u2, \ldots, uk\} be the set of all nr-bad vertices of G. Add vertices v1, v2, v3, v4 with V(G). Join v1 with vj, 1 \leq i, j \leq 4, i \neq j. Join u1 with v1, 1 \leq i \leq k. Let H be the resulting graph. Suppose there exists no nr-set T of H such that v1 privately resolves nr-good vertices and nr-bad vertices of G. Then H is nr-excellent, G is an induced subgraph of H and nr(H) = nr(G) + 3.

**Proof:** Let G be a connected non-nr-excellent graph. Let \{v1, u2, \ldots, uk\} be the set of all nr-bad vertices of G. Add vertices v1, v2, v3, v4 with V(G). Join v1 with vj, 1 \leq i, j \leq 4, i \neq j. Join u1 with v1, 1 \leq i \leq k. Let H be the resulting graph.

Suppose there exists no nr-set of T of H such that v1 privately resolves nr-good vertices and nr-bad vertices of G. Let S be an nr-set of G. Let A = \{v1, v2, v3, v4\}. Let S1 = S \cup T where T is a three element subset of A containing v1. Let S2 = S \cup \{u1, v1, v2\} where v1, v2 \in \{v2, v3, v4\}, 1 \leq i \leq k and S3 = S \cup \{v2, v3, v4\}. Let x, y \in G where x, y \neq u1, 1 \leq i \leq k.

Now ncS1(x) = (a1, a2, \ldots, ar, 0, 0, 0); ncS1(y) = (b1, b2, \ldots, br, 0, 0, 0). ncS1(u1) = (c1, c2, \ldots, cr, 1, 0, 0). ncS1(v1) = (d1, d2, \ldots, dr, 1, 0, 0). ncS1(v2) = (0, 0, 0, 1, l1, l2). ncS1(v3) = (0, 0, 0, 0, 0, 1). ncS1(v4) = (0, 0, 0, 0, 0, 1, 1). Therefore S2 is a neighbourhood resolving set of H.

Now ncS2(x) = (a1, a2, \ldots, ar, 0, 0, 0); ncS2(y) = (b1, b2, \ldots, br, 0, 0, 0). ncS2(u1) = (c1, c2, \ldots, cr, 0, 0, 0); ncS2(v1) = (d1, d2, \ldots, dr, 1, 0, 0). ncS2(v2) = (0, 0, 0, 0, 0, 1, 1). ncS2(v3) = (0, 0, 0, 1, 1, 0); ncS2(v4) = (0, 0, 0, 0, 1, 1, 0). Therefore S3 is a neighbourhood resolving set of H. Therefore nr(H) \leq nr(G) + 1. Let D be an nr-set of H.

**Case (A) :** D contains at most two elements from \{u1, u2, \ldots, uk, v1, v2, v3, v4\}.

**Subcase (i) :** D contains u1, u1, 1 \leq i, j \leq k, i \neq j. Then ncD(v2) = ncD(v3) = ncD(v4) = (0, 0, 0), a contradiction.

**Subcase (ii) :** D contains u1, vj, 1 \leq i, j \leq 4, 2 \leq j \leq 4. Then ncD(vj) = ncD(vj) = (0, 0, 0, 0, 0, 1) where r, s \neq j, r \neq s \leq 4, a contradiction.

**Subcase (iii) :** D contains vj, vj, 1 \leq i, j \leq 4, 1 \neq j. Then D contains vj, vj, 1 \leq i, j \leq 4, r \neq s, r, s \leq 4, a contradiction.

**Subcase (iv) :** D contains u1, vj, 1 \leq i \leq k. Then ncD(v2) = ncD(v3) = (0, 0, 0, 0, 1, 1), a contradiction.

Therefore D contains at least three vertices from \{u1, u2, \ldots, uk, v1, v2, v3, v4\}.

**Case (B) :** D contains more than three elements from \{u1, u2, \ldots, uk, v1, v2, v3, v4\}.

**Subcase (i) :** D \subset \{u1, u2, \ldots, uk\}. Then v2, v3 and v4 have 0-code with respect to D, a contradiction.

**Subcase (ii) :** D contains v2, v3, v4 and the remaining vertices from \{u1, u2, \ldots, uk\}. Then D –
\{v_2, v_3, v_4\} is a neighbourhood resolving set of G. Since D contains nr-bad vertices of G, nr(G) < |D| - 3 = nr(H) - 3. That is nr(H) ≥ nr(G) + 4, a contradiction.

Subcase (iii) : D contains two of the vertices from \{v_1, v_2, v_3, v_4\} and at least two vertices from \{u_1, u_2, \ldots, u_k\}.

Subcase (iiiia) : D contains v_2, v_3 and at least two vertices from \{u_1, u_2, \ldots, u_k\}. Then D^1 = D - \{v_2, v_3\} is a neighbourhood resolving set of G.

Subcase (iiiib) : D contains v_1, v_2, v_3 and at least two vertices from \{u_1, u_2, \ldots, u_k\}. Then D^1 = D - \{v_1, v_2\} is a neighbourhood resolving set of G.

Subcase (iiiic) : D contains v_1, v_2, v_3 and at least one vertex from \{u_1, u_2, \ldots, u_k\}. Then D^1 = D - \{v_2, v_3\} is a neighbourhood resolving set of G.

As in Subcase (iiia), nr(H) = nr(G) + 3. If D contains v_1, v_3, u_1; v_1, v_2, v_4, similar result is arrived at.

Case (C) : D contains exactly three vertices from \{u_1, u_2, \ldots, u_k, v_1, v_2, v_3, v_4\}.

Subcase (i) : D contains u_{i_1}, u_{i_2}, u_{i_3}, 1 ≤ i_1, i_2, i_3 ≤ k. Then ncD(v_3) = ncD(v_4) = (0, 0, \ldots, 0), a contradiction.

Subcase (ii) : D contains u_1, u_2, v_1. Then ncD(v_2) = ncD(v_3) = ncD(v_4) = (0, 0, \ldots, 0), a contradiction. Similarly if D contains \{u_1, u_2, v_2\} or \{u_1, u_3, v_3\} or \{u_2, v_2, v_3\}, it gives rise to a contradiction.

Subcase (iii) : D contains u_1, v_1, v_2. Then ncD(v_3) = ncD(v_4) = (0, 0, \ldots, 0), a contradiction. If D contains \{u_1, v_1, v_3\} or \{u_1, v_1, v_4\}, then these cases also lead to a contradiction.

Subcase (iv) : D contains u_1, v_2, v_3. Then D^1 = D - \{v_2, v_3\} is a neighbourhood resolving set of G. Since D^1 contains nr-bad vertex u_1 of G, then nr(G) + 1 ≤ |D^1| = nr(H) - 2. Hence nr(H) ≥ nr(G) + 3. Therefore nr(H) = nr(G) + 3. The same result is true if D contains \{u_1, v_3, v_1\} or \{u_1, v_2, v_4\}.

Subcase (v) : D contains v_2, v_3, v_4. Then D^1 = D - \{v_2, v_3, v_4\} is a neighbourhood resolving set of G. Therefore nr(G) ≤ |D^1| = |D| - 3 = nr(H) - 3. That is nr(H) ≥ nr(G) + 3. Therefore nr(H) = nr(G) + 3. Hence in all cases, nr(H) = nr(G) + 3.

Since |S_1| = |S_2| = |S_3| = nr(G) + 3, S_1, S_2 and S_3 are nr-sets of H containing u_i, 1 ≤ i ≤ k, v_1, v_2, v_3, v_4. Therefore, u_i, 1 ≤ i ≤ k, v_1, v_2, v_3, v_4 are nr-good in H, also all nr-good vertices in G are also nr-good in H and G is an induced subgraph of H. Therefore H is nr-excellent graph containing G as an induced subgraph and also nr(H) = nr(G) + 3.

\[\square\]

Theorem 39 Let G be a graph with |G| = 2^{nr(G)}. Then Δ(G) ≥ 2^{nr(G)} - 1.

Proof: Let S be an nr-set of G. Let u_1 be the first vertex in the ordered set S. Since 2^{nr(G)} distinct codes are associated with the vertices of G, any vertex of G with the first element of its code with respect to S is 1, is adjacent to u_1. Since there are 2^{nr(G)} - 1 vertices with first element of its code 1, u has degree exactly 2^{nr(G)} - 1. Hence Δ(G) ≥ 2^{nr(G)} - 1.

\[\square\]

Remark 40 Every vertex of S has degree 2^{nr(G)} - 1.

Remark 41 There exists a graph G with |G| = 2^{nr(G)} and Δ(G) = 2^{nr(G)} - 1.

Consider the graph G :

\[G : \begin{array}{cccccc}
5 & 1 & 2 & 6 \\
4 & 3 & 7 & 8 \\
\end{array}\]

S = \{1, 2, 3\} is an nr-set of G. ncS(1) = (0, 1, 1) ; ncS(2) = (1, 0, 1) ; ncS(3) = (1, 1, 0) ; ncS(4) = (1, 1, 1) ; ncS(5) = (1, 0, 0) ; ncS(6) = (0, 1, 0) ; ncS(7) = (0, 0, 1) ; ncS(8) = (0, 0, 0).

Therefore G is a graph with |G| = 2^{nr(G)} = 2^4 = 8 and Δ(G) = 4 = 2^{nr(G)} - 1 = 2^2.

Remark 42 There exists a graph G with |G| = 2^{nr(G)} and Δ(G) > 2^{nr(G)} - 1.

Consider the graph G :

\[G = \begin{array}{cccc}
3 & 4 \\
1 & 2 \\
\end{array}\]
S = \{1, 2\} is an \(nr\)-set of \(G\). \(nr_S(1) = (0, 1) ; \ nr_S(2) = (1, 0) ; n S(3) = (1, 1) ; nr_S(4) = (0, 0) \). Therefore \(G\) is a graph with \(|G| = 2^{nr(G)} \), and \(\Delta(G) = 3 > 2^{nr(G)-1} = 2\).

Remark 43 Let \(G\) be a \(nr\)-excellent graph with \(|G| = 2^{nr(G)}\). Then \(G\) is regular with degree of regularity \(2^{nr(G)}-1\).

Proof: Suppose \(G\) is a \(nr\)-excellent graph with \(|G| = 2^{nr(G)}\). Then every vertex belongs to an \(nr\)-set of \(G\). Therefore degree of every vertex is \(2^{nr(G)}-1\). □

4 Conclusion

Using the concept of excellence in graphs and neighbourhood resolving set of a graph \(nr(G)\), we have defined \(nr\)-excellent graphs. We have characterized graphs \(G\) and \(H\) for which \(G \cup H\) and \(G + H\) are \(nr\)-excellent, when \(G\) and \(H\) are \(nr\)-excellent. We have also proved that a non \(nr\)-excellent graph \(G\) can be embedded in a \(nr\)-excellent graph \(H\) such that \(nr(H) = nr(G) + \) number of \(nr\)-bad vertices of \(G\). Also a new graph \(H\) can be constructed from a connected non \(nr\)-excellent graph \(G\) such that \(H\) is \(nr\)-excellent, \(G\) is an induced subgraph of \(H\) and \(nr(H) = nr(G) + 3\). Some more results are also discussed.

References:


