The existence of positive solutions for a class of singular third-order three-point boundary value problem

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Abstract: The existence of positive solutions for a class of singular third-order three-point boundary value problem is considered by using Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type. In this class of problem, the nonlinear term is allowed to be singular. Main results show that this class of problem can have positive solutions provided that the conditions on the nonlinear term on some bounded sets are appropriate.

Key–Words: Third-order ordinary differential equation, Three-point boundary value problem, Singular nonlinearity, Fixed point theorem on cone, Existence of positive solutions

1 Introduction

Equations play an important role in practical application, and many actual problems can be converted into equation boundary value problems to solve. Recently, the study of positive solutions for equation (or system) boundary value problems has been widely concerned, see references [1-5]. However, due to boundary value problems of three-order differential equations produced in applied mathematics, applied physics, economics, and many other fields, though the development of boundary value problems of third-order differential equations is relatively slow, especially, when nonlinear term contains derivative or discontinuous and singular at any point. But third-order boundary value problems of singular differential equations have extensive application in applied mathematics and applied physics, for example, they have a wide range of backgrounds in the air convection, celestial evolution and fluid mechanics, etc. In recent years, these problems have arose the attention of experts and scholars in this field, people also made a lot of research work, for instance, the related theory results in documents [6-13], Yao [14] and Jiang [15].

For example, references [16-20] discussed a few third-order two-point boundary value problems, literature [21] studied third-order two-point boundary value problem as follows:

$$u'''(t) + \lambda h(t) f(t, u(t)) = 0, t \in (a, b), \tag{1}$$

$$u(a) = u''(a) = u'(b) = 0, \tag{2}$$

where $\lambda > 0$ is a parameter, $h \in C((a, b), R^+)$, $h(t)$ may be singular at $t = a, b$, and $f \in C([a, b] \times (0, +\infty), R^+)$ is a continuous function, $f(t, s)$ may be singular at $s = 0$.

In 2005, Sun [10] established the following third-order boundary value problem, and then, he got the existence of one and multiple positive solutions:

$$u'''(t) - \lambda h(t) f(t, u(t)) = 0, 0 < t < 1, \tag{3}$$

$$u(0) = u'(\eta) = u''(1) = 0, \tag{4}$$

where λ is a positive parameter and $\eta \in [\frac{1}{2}, 1)$ is a constant, $h(t)$ is a nonnegative continuous function defined on $(0, 1)$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

In 2008, Guo, Sun and Zhao [22] studied the third-order three-point boundary value problem for the following:

$$u'''(t) + h(t) f(u(t)) = 0, t \in (0, 1), \tag{5}$$

$$u'(1) = \alpha u'(\eta), u(0) = u'(0) = 0, \tag{6}$$

where $\eta \in (0, 1), \alpha \in (1, \frac{1}{\eta})$ are constants, $h \in C((a, b), R^+)$ is not zero in $[\frac{\eta}{\alpha}, \eta]$, also is a continuous function.

In 2009, Sun [23] discussed the following three-point nonhomogeneous boundary value problem of three-order differential equation:

$$u'''(t) + h(t) f(u(t)) = 0, t \in (0, 1), \tag{7}$$

$$u(0) = u'(0) = 0, u'(1) - \alpha u'(\eta) = \lambda, \quad (8)$$

where $\eta \in (0,1), \alpha \in [0, \frac{1}{\eta})$ are constants and $\lambda \in (0,+\infty)$ is a parameter, the nonlinear term is superlinear or sublinear.

Inspired by the above work, in this paper, we will study the existence, the nonexistence and the multiplicity of positive solutions for the following question:

$$u'''(t) = h(t)f(t,u(t)), t \in (0,1), \quad (9)$$

$$u(0) = \sigma u(\eta), u'(\eta) = 0, u''(1) = 0, \quad (10)$$

where $\sigma \in (0,1), \eta \in [\frac{1}{2}, 1)$ are constants, we allow nonlinear term $h(t)f(t,u(t))$ is singular at $u = 0, t = 0, t = 1$. Here, by a positive solution u^* of boundary value problem (9) and (10) which satisfies $u^* > 0, t \in (0,1)$.

This paper is organized as follows. In section 2, we first present relevant definitions, theorems and lemmas that will be used to prove our main results; then construct a suitable cone and transform the problem (9) and (10) into an integral equation, and we prove that the related integral operator is local completely continuous. In section 3, we establish the local existence of positive solutions by applying the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type. In section 4, we give an example to demonstrate the results which are obtained in section 3. Finally, section 5 concludes this paper.

2 Preliminary knowledge

Throughout this paper, we assume that

(H1) $h : (0,1) \to [0,+\infty)$ is continuous, and

$$0 < \int_0^1 g(s)h(s)ds < +\infty, \quad (11)$$

where $g(s) = \frac{1}{2(1-\sigma)}\min\{s^2, \eta^2\}$.

(H2) $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous.

(H3) There exist continuous functions $k : [0,1] \times (0,+\infty) \to [0,+\infty)$ and $p : (0,1) \times (0,+\infty) \to [0,+\infty)$ such that

$$\begin{aligned} &0 \leqslant f(t,u) - k(t,u) \leqslant p(t,u), \\ &(t,u) \in (0,1) \times (0,+\infty). \end{aligned} \quad (12)$$

(H4) $p : (0,1) \times (0,+\infty) \to [0,+\infty)$ is a nonincreasing function about u for any $0 < t < 1$.

(H5) For any $\nu > 0$, we have

$$\int_0^1 g(s)h(s)p(s,\tau\nu(s))ds < +\infty. \quad (13)$$

We allow that the nonlinear term $h(t)f(t,u)$ is singular at $t = 0, t = 1$ and $u = 0$.

This paper will establish some general criteria for the existence of single and multiple positive solutions of the problem (9) and (10) under the assumptions **(H1)-(H5)**.

Then we set up a few relevant definitions, theorems and lemmas that can be used in some main results' proof.

Definition 1 *Assume D and E are Banach spaces, the operator $T : D \to E$. If T put any bounded set S onto the compact set (or the relatively compact set) of the Banach space E in the Banach space D, then $T : D \to E$ is called a compact operator.*

Definition 2 *Assume D and E are Banach spaces, if the operator $T : D \to E$ is continuous and compact, then the operator T that hit D into E is completely continuous.*

Definition 3 *Assume E is a real Banach space, P is a nonempty closed set, if it satisfies the following two conditions:*
(i) if $x \in P$ and $\lambda \geqslant 0$, then $\lambda x \in P$;
(ii) if $x \in P$ and $-x \in P$, then $x = 0$.
Then P is called a cone in E.

Definition 4 *Assume S is Banach space, $T_n : S \to S(n = 1,2,3,\cdots)$ is a completely continuous operator, $T : S \to S$, if*

$$\lim_{n\to\infty} \max_{||u||<r} ||T_n u - Tu|| = 0, r > 0$$

then T is a completely continuous operator.

Theorem 5 ***The Arzela-Ascoli Theorem*** *Assume X is a compact metric space, $C(X)$ is a Banach space, if $\Phi \in C(X)$ is bounded and equicontinuous, i.e.:*

(i) "$\Phi \in C(X)$ is bounded" means that there exists a positive constant $M < \infty$ such that $|f(x)| \leqslant M$ for each $x \in X$ and each $f \in \Phi$.

(ii) "$\Phi \in C(X)$ is equicontinuous" means that: for every $\epsilon > 0$ there exists $\delta > 0$ (which depends only on ϵ) such that for $x, y \in X$:

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon, \forall f \in \Phi,$$

where d is the metric on X. Then Φ is totally bounded in $C(X)$.

Proof: In this proof's processing, we consider that the proof is divided into three steps:

Step 1. We show that the compact metric space X is separable, i.e., has a countable dense subset S.

Given a positive integer n and a point $x \in X$, let

$$B(x, \frac{1}{n}) = \{y \in X : d(x, y) < \frac{1}{n}\},$$

the open ball of radius $\frac{1}{n}$, centered at x. For a given n, the collection of all these balls as x runs through X is an open cover of x, so (because X is compact) there is a finite subcollection that also covers X. Let S_n denote the collection of centers of the balls in this finite subcollection. Thus S_n is a finite subset of X that is "$\frac{1}{n}$-dense" in the sense that every point of X lies within $\frac{1}{n}$ of a point S_n. Clearly the union S of all the sets S_n is countable, and dense in X.

Step 2. We find a subsequence of $\{f_n\}$ that converges point wise on S.

This is a standard diagonal argument. Let's list the (countably many) elements of S as $\{x_1, x_2, \cdots\}$. Then the numerical sequence $\{f_{1,n}(x_1)\}_{n=1}^{\infty}$ is bounded, so by Bolzano-Weierstrass, it has a convergent subsequence, which we'll write using double subscripts: $\{f_{1,n}(x_1)\}_{n=1}^{\infty}$. Now the numerical sequence $\{f_{1,n}(x_2)\}_{n=1}^{\infty}$ is bounded, so it has a convergent subsequence $\{f_{2,n}(x_2)\}_{n=1}^{\infty}$. Note that the sequence of functions $\{f_{2,n}\}_{n=1}^{\infty}$, since it is a subsequence of $\{f_{1,n}\}_{n=1}^{\infty}$, converges at both x_1 and x_2. Proceeding in this fashion, we obtain a countable collection of subsequence of our original sequence:

$$\begin{array}{cccc} f_{11} & f_{12} & f_{13} & \cdots \\ f_{21} & f_{22} & f_{23} & \cdots \\ f_{31} & f_{32} & f_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}$$

where the sequence in the n-th row converges at the points $x_1, \cdots, x_n$, and each row is a subsequence of the one above it.

Thus the diagonal sequence $\{f_{n,n}\}$ is a subsequence of the original sequence $\{f_n\}$ that converges at each point of S.

Step3. Completion of the proof.

Let g_n be the diagonal subsequence produced in the previous step, convergent at each point of the dense set S. Let $\epsilon > 0$ be given, and choose $\delta > 0$ by equicontinuity of the original sequence, so that $d(x, y) < \delta$ implies

$$|g_n(x) - g_n(y)| < \frac{\epsilon}{3}$$

for each $x, y \in X$ and each positive integer n. Fix $M > \frac{1}{\delta}$ so that the finite subset $S_M \subset S$ that we produced in Step 1 is δ-dense in X. Since $\{g_n\}$ converges at each point of S_M, there exists $N > 0$ such that $n, m > N$,

$$|G_n(s) - g_m(s)| < \frac{\epsilon}{3}, \quad \forall s \in S_M \tag{14}$$

Fix $x \in X$. Then x lies within δ of some $s \in S_M$, so if $n, m > M$:

$$\begin{aligned} &|g_n(x) - g_m(x)| \le |g_n(x) - g_n(s)| \\ &+|g_n(s) - g_m(s)| + |g_m(s) - g_n(x)| \end{aligned}$$

The first and last terms on the right are $< \frac{\epsilon}{3}$ by our choices of δ (which was possible because of the equicontinuity of the original sequence), and the same estimate holds for the middle term by our choice of N in (2.4). In summary: given $\epsilon > 0$ we have produced N so that for each $x \in X$, as $m, n > N$,

$$|g_n(x) - g_m(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus on X the subsequence $\{g_n\}$ of $\{f_n\}$ is uniformly Cauchy, and therefore uniformly convergent. This complete the proof of the Arzela-Ascoli Theorem.

Lemma 6 *Let $u \in C_{[0,1]^+} = \{u \in C[0,1], u(t) \geqslant \sigma, t \in [0,1]\}$, then the boundary value problem (9) and (10) has only the solution:*

$$u(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds, \tag{15}$$

where

$$G(t,s) = \begin{cases} \dfrac{s^2}{2(1-\sigma)}, \ 0 \leqslant s \leqslant \min\{t,\eta\} \\ -\dfrac{1}{2}t^2 + ts + \dfrac{\sigma s^2}{2(1-\sigma)}, \\ (0 \leqslant t \leqslant s \leqslant \eta < 1) \\ \dfrac{1}{2}s^2 - ts + \eta t + \dfrac{\sigma \eta^2}{2(1-\sigma)}, \\ (\frac{1}{2} \leqslant \eta \leqslant s \leqslant t \leqslant 1) \\ -\dfrac{1}{2}t^2 + \eta t + \dfrac{\sigma\eta^2}{2(1-\sigma)}, \\ (s \geqslant \max\{t,\eta\}) \end{cases} \tag{16}$$

Proof: By assuming **(H1), (H2), (H3)**, we know that there is integral $u(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds$, in fact, if $u(t)$ is the solution of the boundary value problem (9) and (10), then let

$$\begin{aligned} u(t) &= \frac{1}{2}\int_0^1 (t-s)^2 h(s)f(s,u(s))ds \\ &+mt^2 + nt + q, \end{aligned} \tag{17}$$

Through the boundary value conditions (10), we can get:

$$m = -\frac{1}{2}\int_0^1 h(s)f(s,u(s))ds, \tag{18}$$

$$n = \eta\int_0^1 h(s)f(s,u(s))ds - \int_0^\eta (\eta-s)h(s)f(s,u(s))ds, \tag{19}$$

$$q = \frac{\sigma\eta^2}{2(1-\sigma)}\int_0^1 h(s)f(s,u(s))ds - \frac{\sigma}{2(1-\sigma)}\times \int_0^\eta (\eta^2-s^2)h(s)f(s,u(s))ds. \tag{20}$$

So the boundary value problem (9) and (10) has a unique solution:

$$\begin{aligned} u(t) &= \frac{1}{2}\int_0^1 (t-s)^2 h(s)f(s,u(s))ds \\ &\quad -\frac{1}{2}t^2\int_0^1 h(s)f(s,u(s))ds \\ &\quad +t\eta\int_0^1 h(s)f(s,u(s))ds \\ &\quad -t\int_0^\eta (\eta-s)h(s)f(s,u(s))ds \\ &\quad +\frac{\sigma\eta^2}{2(1-\sigma)}\int_0^1 h(s)f(s,u(s))ds \\ &\quad -\frac{\sigma}{2(1-\sigma)}\int_0^\eta (\eta^2-s^2)h(s)f(s,u(s))ds \\ &= \frac{1}{2}\int_0^t (t-s)^2 h(s)f(s,u(s))ds - \\ &\quad \int_0^\eta \left(\frac{\sigma(\eta^2-s^2)}{2(1-\sigma)} + t(\eta-s)\right)h(s)f(s,u(s))ds \\ &\quad -\int_0^1 \left(\frac{t^2}{2} - \eta t - \frac{\sigma\eta^2}{2(1-\sigma)}\right)h(s)f(s,u(s))ds \end{aligned}$$

$$= \begin{cases} \int_0^t \frac{s^2}{2(1-\sigma)}h(s)f(s,u(s))ds + \\ \int_t^\eta \left(-\frac{1}{2}t^2 + ts + \frac{\sigma s^2}{2(1-\sigma)}\right)h(s)f(s,u(s))ds + \\ \int_\eta^1 \left(-\frac{1}{2}t^2 + \eta t + \frac{\sigma\eta^2}{2(1-\sigma)}\right)h(s)f(s,u(s))ds \\ (t \leqslant \eta) \\ \int_0^\eta \frac{s^2}{2(1-\sigma)}h(s)f(s,u(s))ds + \\ \int_\eta^t \left(-\frac{1}{2}s^2 - ts + \eta t + \frac{\sigma\eta^2}{2(1-\sigma)}\right)h(s)f(s,u(s))ds \\ + \int_t^1 \left(-\frac{1}{2}t^2 + \eta t + \frac{\sigma\eta^2}{2(1-\sigma)}\right)h(s)f(s,u(s))ds \\ (t \geqslant \eta) \end{cases}$$

$$= \int_0^1 G(t,s)h(s)f(s,t(s))ds.$$

Lemma 6 is proved. □

Lemma 7 *For all the $(t,s) \in [0,1]\times[0,1]$, when $0<\sigma<1$, $\frac{1}{2} \leqslant \eta < 1$, we have*

$$0 \leqslant G(t,s) \leqslant g(s)$$

and

$$G(t,s) \geqslant \sigma g(s),$$

where $g(s) = \frac{1}{2(1-\sigma)}\min\{s^2,\eta^2\}$.

Proof: Clearly $G(t,s) \geqslant 0$, and when $s \leqslant \min\{t,\eta\}$, the conclusion is established.
When $t \leqslant s \leqslant \eta$, then

$$\begin{aligned} G(t,s) &= -\frac{1}{2}t^2 + ts + \frac{\sigma s^2}{2(1-\sigma)} \\ &\leqslant \frac{1}{2}s^2 + \frac{\sigma s^2}{2(1-\sigma)} \\ &= \frac{s^2}{2(1-\sigma)}, \end{aligned} \tag{21}$$

$$\begin{aligned} G(t,s) &= -\frac{1}{2}t^2 + ts + \frac{\sigma s^2}{2(1-\sigma)} \\ &\geqslant \frac{\sigma s^2}{2(1-\sigma)}. \end{aligned} \tag{22}$$

When $\eta \leqslant s \leqslant t$, we have

$$\begin{aligned} G(t,s) &= \frac{1}{2}s^2 - ts + t\eta + \frac{\sigma\eta^2}{2(1-\sigma)} \\ &= \frac{1}{2}s^2 - \frac{1}{2}\eta^2 - ts + t\eta + \frac{1}{2}\eta^2 + \frac{\sigma\eta^2}{2(1-\sigma)} \\ &= \frac{1}{2}(s-\eta)(s+\eta) - t(s-\eta) + \frac{\eta^2}{2(1-\sigma)} \\ &= (s-\eta)\left(\frac{1}{2}s + \frac{1}{2}\eta - t\right) + \frac{\eta^2}{2(1-\sigma)} \\ &\leqslant (s-\eta)\left(\frac{1}{2}t + \frac{1}{2}t - t\right) + \frac{\eta^2}{2(1-\sigma)} \\ &= \frac{\eta^2}{2(1-\sigma)}, \end{aligned} \tag{23}$$

$$
\begin{aligned}
G(t,s) &= \frac{1}{2}s^2 - ts + t\eta + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&= \frac{1}{2}s^2 - t(s-\eta) + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&\geqslant \frac{1}{2}s^2 - (s-\eta) + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&= \frac{1}{2}s^2 - s + \eta + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&\geqslant \frac{1}{2}s^2 - s + \frac{1}{2} + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&= \frac{1}{2}(s-1)^2 + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&\geqslant \frac{\sigma\eta^2}{2(1-\sigma)}.
\end{aligned} \tag{24}
$$

When $\mathrm{maxt}\{\eta, t\} \leqslant s$, then

$$
\begin{aligned}
G(t,s) &= -\frac{1}{2}t^2 + \eta t + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&\leqslant \frac{\eta^2}{2(1-\sigma)},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
G(t,s) &= -\frac{1}{2}t^2 + \eta t + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&\geqslant \frac{1}{2}t^2 + \frac{t}{2} + \frac{\sigma\eta^2}{2(1-\sigma)} \\
&\geqslant \frac{\sigma\eta^2}{2(1-\sigma)}.
\end{aligned} \tag{26}
$$

Lemma 7 is proved. □

By Lemma 7, We can see that $u(t)$ is the solution of the boundary value problem (1.9) and (1.10) if and only if $u(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds, 0 \leqslant t \leqslant 1$.

In the following paper, we will consider the problem (9) and (10) in Banach space. We denote Banach space $E = C[0,1]$, define the standard norm $||u|| = \max\limits_{0\leqslant t\leqslant 1} |u|, u \in E$, cone

$$P = \{u \in E/u(t) \geqslant 0, u(t) \geqslant \sigma||u||, t \in [0,1]\}$$

then P is a cone of nonnegative functions in $C[0,1]$.

Define the integral operator $T : P \to E$ as follows

$$
\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)h(s)f(s,u(s))ds \\
&\quad 0 \leqslant t \leqslant 1, u \in P.
\end{aligned} \tag{27}
$$

By the Lemma 7, we know $(Tu)(t) \geqslant 0, u \in P, 0 \leqslant t \leqslant 1$, and

$$
\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s)h(s)f(s,u(s))ds \\
&\geqslant \sigma \int_0^1 g(s)h(s)f(s,u(s))ds \\
&\geqslant \sigma||Tu||
\end{aligned} \tag{28}
$$

Therefore, $T(P) \subset P$.

We also define $\Omega = \{u \in P/||u|| < r\}$, $\partial\Omega = \{u \in P/||u|| = r\}$, and let

$$
\begin{aligned}
A &= \left[\int_0^1 g(s)h(s)ds\right]^{-1}, \\
B &= \left[\int_\alpha^\beta g(s)h(s)ds\right]^{-1}, \\
&0 < \alpha < \beta < 1.
\end{aligned} \tag{29}
$$

Lemma 8 *Let $0 < a < b$. Then $T : (\overline{\Omega}(b)\backslash\Omega(a)) \to P$ is completely continuous.*

Proof: By direct calculation, we know $T(P) \subset P$. Let $u \in (\overline{\Omega}(b)\backslash\Omega(a))$, then

$$a\sigma \leqslant ||u||\sigma \leqslant u(t) \leqslant ||u|| \leqslant b,\ 0 \leqslant t \leqslant 1, \tag{30}$$

By the assumption **(H4)**,

$$p(t,u(t)) \leqslant p(t,a\sigma),\ 0 < t < 1, \tag{31}$$

By **(H5)**,

$$\int_0^1 g(s)h(s)p(s,a\sigma)ds < +\infty, \tag{32}$$

Let

$$
p_n(t,a\sigma) = \begin{cases}
0, \quad 0 \leqslant t < \dfrac{1}{2n}, \\
(2nt-1)p(t,a\sigma), \\
\qquad \dfrac{1}{2n} \leqslant t < \dfrac{1}{n} \\
p(t,a\sigma), \quad \dfrac{1}{n} \leqslant t \leqslant 1 - \dfrac{1}{n} \\
[2n(1-t)-1]p(t,a\sigma), \\
\qquad 1 - \dfrac{1}{n} < t \leqslant 1 - \dfrac{1}{2n} \\
0, \quad 1 - \dfrac{1}{2n} < t \leqslant 1,
\end{cases} \tag{33}
$$

Then

$$
\begin{aligned}
&\int_0^1 g(s)h(s)[p(s,a\sigma) - p_n(s,a\sigma)]ds \\
&\leqslant \int_0^{\frac{1}{n}} g(s)h(s)p(s,a\sigma)ds \\
&+ \int_{1-\frac{1}{n}}^1 g(s)h(s)p(s,a\sigma)ds \\
&\to 0, (n \to +\infty)
\end{aligned} \tag{34}
$$

Let $f_n(t,u)=\max\{f(t,u),k(t,u)+p_n(t,a\sigma)\}$, then $f_n:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. Define the operator T_n as follows

$$(T_nu)(t) = \int_0^1 G(t,s)h(s)f_n(s,u(s))ds, \quad 0\leqslant t\leqslant 1. \tag{35}$$

Modeling the proof of Theorem 2.3 in [5] we can prove $T_n : P \to C[0,1]$ is completely continuous by the continuity of $f_n(t,u(t))$ and the Theorem 5. Direct computations give that

$$\begin{aligned}
&\sup||Tu-T_nu||\\
&=\sup_{u\in\overline{\Omega}(b)\setminus\Omega(a)}\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)f(s,u(s))ds-\\
&\sup_{u\in\overline{\Omega}(b)\setminus\Omega(a)}\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)f_n(s,u(s))ds\\
&\leqslant\sup_{u\in\overline{\Omega}(b)\setminus\Omega(a)}\int_0^1 g(s)h(s)p(s,a\sigma)ds-\\
&\sup_{u\in\overline{\Omega}(b)\setminus\Omega(a)}\int_0^1 g(s)h(s)p_n(s,a\sigma)ds\\
&=\int_0^1 g(s)h(s)[p(s,a\sigma)-p_n(s,a\sigma)]ds\\
&\to 0,(n\to+\infty)
\end{aligned} \tag{36}$$

It shows that the completely continuous operator T_n converges to the operator T uniformly on the set $\overline{\Omega}(b)\setminus\Omega(a)$. Hence, $T:(\overline{\Omega}(b)\setminus\Omega(a))\to P$ is a completely continuous operator.

Our approach is based on the following Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type.

Theorem 9 *Let E be a Banach space, and $P\subset E$ be a cone in E. Assume Ω_1,Ω_2 are bounded open subset of E with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let $T:P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that, either*

(A): $||Tu||\leqslant||u||,\forall u\in P\cap\partial\Omega_1$, and $||Tu||\geqslant||u||,\forall u\in P\cap\partial\Omega_2$,
or
(B): $||Tu||\geqslant||u||,\forall u\in P\cap\partial\Omega_1$, and $||Tu||\leqslant||u||,\forall u\in P\cap\partial\Omega_2$.
Then T has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3 Main results and proof

We introduce the following height functions:

$$\begin{aligned}
&\varphi(r)=\max\{k(t,u):0\leqslant t\leqslant 1,r\sigma\leqslant u\leqslant r\},\\
&\psi(r)=\min\{f(t,u):\alpha\leqslant t\leqslant\beta,r\sigma\leqslant u\leqslant r\}.
\end{aligned} \tag{37}$$

In addition, we denote

$$w(r)=\int_0^1 g(s)h(s)p(s,r\sigma)ds.$$

Theorem 10 is an existence criterion for a positive solution of the singular problem (1.9) and (1.10). The theorem shows that the existence of a positive solution depends only on the properties of the nonlinear term $f(t,u(t))$ on the bounded set $\{(t,u):0<t<1,a\sigma\leqslant u\leqslant b\}$ and is independent of the states of $f(t,u(t))$ outside the set.

Theorem 10 *Assume that there exist two positive numbers $a<b$ such that one of the following conditions is satisfied:*
(a) $\varphi(a)\leqslant(a-w(a))A,\psi(b)\geqslant bB$.
(b) $\varphi(b)\leqslant(b-w(b))A,\psi(a)\geqslant aB$.
Then the problem (9) and (10) has at least one positive solution $u^*\in P$ such that

$$a\leqslant||u^*||\leqslant b.$$

Proof: Without loss of generality, we prove only the case (a).

If $u\in\partial\Omega(a)$, then $||u||=a$, and $a\sigma\leqslant u(t)\leqslant a, 0\leqslant t\leqslant 1$. From this, $k(t,u(t))\leqslant\varphi(a)\leqslant(a-w(a))A, p(t,u(t))\leqslant p(t,b\sigma), 0\leqslant t\leqslant 1$. It follows

$$\begin{aligned}
||Tu|| &=\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)h(s)f(s,u(s))ds\\
&\leqslant\int_0^1 g(s)h(s)f(s,u(s))ds\\
&\leqslant\int_0^1 g(s)h(s)[k(s,u(s))+p(s,u(s))]ds\\
&\leqslant\int_0^1 g(s)h(s)k(s,u(s))ds\\
&+\int_0^1 g(s)h(s)p(s,a\sigma)ds\\
&\leqslant(a-w(a))A\int_0^1 g(s)h(s)ds+w(a)\\
&=(a-w(a))AA^{-1}+w(a)\\
&=a=||u||.
\end{aligned} \tag{38}$$

If $u\in\partial\Omega(b)$, then $||u||=b$, and $b\sigma\leqslant u(t)\leqslant b, 0\leqslant t\leqslant 1$. Thus $f(t,u(t))\geqslant\psi(b)\geqslant bB,\alpha\leqslant t\leqslant\beta$. It follows

$$\begin{aligned}
||Tu|| &=\max_{\alpha\leqslant t\leqslant\beta}\int_\alpha^\beta G(t,s)h(s)f(s,u(s))ds\\
&\geqslant\int_\alpha^\beta\sigma g(s)h(s)f(s,u(s))ds\\
&\geqslant bB\int_\alpha^\beta g(s)h(s)ds\\
&\geqslant bBB^{-1}=b=||u||.
\end{aligned} \tag{39}$$

So, by Theorem 9 and Lemma 8, the operator T has one fixed point $u^* \in P \cap (\overline{\Omega}(b) \backslash \Omega(a))$. Thus (9) and (10) has at least one positive solution. □

Theorem 11 *Assume that there exist three positive numbers $a < b < c$ such that one of the following conditions is satisfied:*

(c) $\psi(a) \geqslant aB, \varphi(b) < (b - w(b))A, \psi(c) \geqslant cB$.

(d) $\varphi(a) \leqslant (a - w(a))A, \psi(b) > bB$, $\varphi(c) \leqslant (c - w(c))A$.

Then the boundary value problem (9) and (10) has at least two different positive solutions $u_1^*, u_2^* \in P$ such that

$$a \leqslant ||u_1^*|| < b < ||u_2^*|| \leqslant c.$$

Proof: Without loss of generality, we prove only the case (d).

Applying the assumptions $\varphi(a) \leqslant (a - w(a))A, \psi(b) > bB$ and copying the proof of Theorem 10, we see that the boundary value problem (1.9) and (1.10) has at least one positive solution $u_1^* \in P$ such that $a \leqslant ||u_1^*|| < b$. Similarly, the problem (9) and (10) has at least one positive solution $u_2^* \in P$ such that $b < ||u_2^*|| \leqslant c$ by the assumption $\psi(b) > bB, \varphi(c) \leqslant (c - w(c))A$. Thus, the conclusion is established. □

Theorem 12 *Assume that there exist four positive numbers $a < b < c < d$ such that one of the following conditions is satisfied:*

(e) $\psi(a) \geqslant aB, \varphi(b) < (b - w(b))A$, $\psi(c) > cB, \varphi(d) \leqslant (d - w(d))A$.

(f) $\varphi(a) \leqslant (a - w(a))A, \psi(b) > bB$, $\varphi(c) < (c - w(c))A, \psi(d) \geqslant dB$.

Then the boundary value problem (9) and (10) has at least three different positive solutions $u_1^*, u_2^*, u_3^* \in P$ such that

$$a \leqslant ||u_1^*|| < b < ||u_2^*|| < c < ||u_3^*|| \leqslant d.$$

Theorem 13 *Assume that there exist $n + 1$ positive numbers $a_1 < a_2 < \cdots < a_{n+1}$ such that one of the following conditions is satisfied:*

(g) $\psi(a_{2k}) > a_{2k}B, k = 1, 2, \cdots, [\frac{n+1}{2}]$, $\varphi(a_{2k-1}) < (a_{2k-1} - w(a_{2k-1}))A$, $k = 1, 2, \cdots, [\frac{n+2}{2}]$.

(h) $\psi(a_{2k-1}) > a_{2k-1}B, k = 1, 2, \cdots, [\frac{n+2}{2}]$, $\varphi(a_{2k}) < (a_{2k} - w(a_{2k}))A$, $k = 1, 2, \cdots, [\frac{n+1}{2}]$.

Then the boundary value problem (9) and (10) has at least n positive solutions $u_k^* \in P, k = 1, 2, \cdots, n$ such that

$$a_k < ||u_k^*|| < a_{k+1}.$$

If $\lim\limits_{u \to 0^+} \inf \min\limits_{\alpha \leqslant t \leqslant \beta} f(t, u) > 0$ (particulary, $\lim\limits_{u \to 0^+} \inf \min\limits_{\alpha \leqslant t \leqslant \beta} f(t, u) = +\infty$), we have the following existence theorems.

Corollary 14 *Assume $\lim\limits_{u \to 0^+} \inf \min\limits_{\alpha \leqslant t \leqslant \beta} f(t, u) > 0$ and there exists a positive number $b > 0$ such that*

$$\varphi(b) \leqslant (b - w(b))A.$$

Then the boundary value problem (1.9) and (1.10) has at least one positive solution $u^* \in P$ such that

$$0 < ||u^*|| \leqslant b.$$

Proof: Since $\lim\limits_{u \to 0^+} \inf \min\limits_{\alpha \leqslant t \leqslant \beta} f(t, u) > 0$, then there exists $a > 0$ such that $0 < a < b$, and $f(t, u) \geqslant aB, (t, u) \in [\alpha, \beta] \times [0, a]$. It follows $\psi(a) \geqslant aB$.

Thus the boundary value problem (9) and ((10) has at least one positive solution $u^* \in P$ such that $0 < a \leqslant ||u^*|| \leqslant b$ by Theorem 10. □

Corollary 15 *Assume $\lim\limits_{u \to 0^+} \inf \min\limits_{\alpha \leqslant t \leqslant \beta} f(t, u) > 0$ and there exist two positive numbers $0 < a < b$ such that*

$$\varphi(a) \leqslant (a - w(a))A, \psi(b) \geqslant bB.$$

Then the boundary value problem (1.9) and (1.10) has at least two positive solutions $u_1^*, u_2^* \in P$ such that

$$0 < ||u^*|| \leqslant a \leqslant ||u_2^*|| \leqslant b.$$

Proof: The proof is completed by Theorem 10 and Corollary 14.

Corollary 16 *Assume $\lim\limits_{u \to 0^+} \inf \min\limits_{\alpha \leqslant t \leqslant \beta} f(t, u) > 0$ and there exist n positive numbers $a_1 < a_2 < \cdots < a_n$ such that*

$$\begin{aligned} \varphi(a_{2k-1}) \quad &< (a_{2k-1} - w(a_{2k-1}))A, \\ &k = 1, 2, \cdots, \frac{n+1}{2}, \end{aligned}$$

$$\psi(a_{2k}) > a_{2k}B, k = 1, 2, \cdots, \frac{n}{2}.$$

Then the boundary value problem (9) and (10) has at least n positive solutions $u_k^* \in P, k = 1, 2, \cdots, n$ such that

$$0 < ||u_1^*|| < a_1 < ||u_2^*|| < a_2 < \cdots < ||u_n^*|| < a_n.$$

The following corollary is convenient for the existence of a single positive solution.

Corollary 17 *Assume $\psi(a) \geqslant aB$, and $\lim\limits_{r \to +\infty} \varphi(r) - (r - w(r))A \leqslant 0, a \in (0, +\infty)$. Then the boundary value problem (9) and (10) has at least one positive solution $u^* \in P$.*

Proof: If $u \in \partial\Omega(a)$, then $||u|| = a$ and $a\sigma \leqslant u \leqslant a$, it follows

$$f(t,u) \geqslant \psi(a) \geqslant aB, \alpha \leqslant t \leqslant \beta.$$

Thus we have

$$\begin{aligned} ||Tu|| &\geqslant \int_\alpha^\beta \sigma g(s)h(s)f(s,u(s))ds \\ &\geqslant aB\int_\alpha^\beta \sigma g(s)h(s)ds \\ &= aBB^{-1} = a = ||u||, \\ &\forall u \in P\cap\partial\Omega(a). \end{aligned} \tag{40}$$

There exists $R > 0$ such that $\varphi(r) \leqslant (r - w(r))A$, and $||u|| = R$ for any $u \in P$ by $\lim\limits_{r\to+\infty} \varphi(r) - (r - w(r))A \leqslant 0$. Therefore, we can get

$$\begin{aligned} ||Tu|| &\leqslant \int_0^1 g(s)h(s)f(s,u(s))ds \\ &\leqslant \int_0^1 g(s)h(s)[k(s,u(s)) + p(s,u(s))]ds \\ &\leqslant \int_0^1 g(s)h(s)k(s,u(s))ds \\ &+ \int_0^1 g(s)h(s)p(s,R\sigma)ds \\ &\leqslant (R - w(R))A\int_0^1 g(s)h(s)ds + w(R) \\ &= R = ||u||. \end{aligned} \tag{41}$$

Therefore the operator T has at least one fixed point $u^* \in P\cap(\overline{\Omega}(R)\backslash\Omega(a))$ such that $||Tu^*|| = ||u^*||$, that is to say u^* is a positive solution of the boundary value problem (9) and (10). □

Corollary 18 *Assume for any $r \in (0,+\infty)$, we have $\psi(r) - \frac{rB}{\sigma^2} > 0$. Then the boundary value problem (9) and (10) has no positive solution.*

Proof: Here we apply reduction to absurdity. Assume the boundary value problem (9) and (10) has at least one positive solution $\widetilde{u}(t)$, clearly, $\widetilde{u}(t) \in P$.

Due to $\psi(r) - \frac{rB}{\sigma^2} > 0$, then there exists a constant $r \in [0,+\infty)$ such that

$$f(t,r) > \frac{rB}{\sigma^2}, \quad t \in (\alpha,\beta).$$

Thus

$$\begin{aligned} ||\widetilde{u}(t)|| &= \int_0^1 G(t,s)h(s)f(s,\widetilde{u}(s))ds \\ &\geqslant \int_0^1 \sigma g(s)h(s)f(s,\widetilde{u}(s))ds \\ &> \int_\alpha^\beta \sigma g(s)h(s)f(s,\widetilde{u}(s))ds \\ &\geqslant \frac{B}{\sigma^2}\int_\alpha^\beta \sigma g(s)h(s)\widetilde{u}(s)ds \\ &\geqslant \frac{B}{\sigma}\int_\alpha^\beta g(s)h(s)ds\cdot\sigma||\widetilde{u}|| \\ &= BB^{-1}||\widetilde{u}|| = ||\widetilde{u}||. \end{aligned} \tag{42}$$

That is a contradiction. Therefore the boundary value problem (9) and (10) has no positive solution. □

4 Example

We consider the boundary value problem as follows

$$\begin{aligned} u'''(t) &= \frac{1}{t(1-t)^{\frac14}}\cdot[\frac{ut}{100} + \frac{e^u}{u^{\frac18}t^{\frac14}(1-t)^{\frac14}}], \\ &0 < t < 1, \end{aligned} \tag{43}$$

$$u(0) = \frac14 u(\frac34),\ u'(\frac34) = 0,\ u''(1) = 0. \tag{44}$$

where

$\sigma = \frac14, \quad \eta = \frac34, \qquad a(t) = \frac{1}{t(1-t)^{\frac14}},$
$f(t,u) = \frac{ut}{100} + \frac{e^u}{u^{\frac18}t^{\frac14}(1-t)^{\frac14}}, \qquad \varphi(r) = \frac{r}{100},$
$w(r) = \int_0^1 g(s)a(s)p(s,r\sigma)ds.$

When $s < \dfrac34$, then

$$g(s) = \frac{s}{2(1-\sigma)} = \frac{2s}{3},$$

$$\begin{aligned} A &= [\int_0^1 g(s)a(s)ds]^{-1} \\ &= [\int_0^1 \frac{1}{s(1-s)^{\frac14}}\cdot\frac{2s}{3}ds]^{-1} \\ &= 1.875, \end{aligned}$$

$$\begin{aligned} B &= [\int_\alpha^\beta g(s)a(s)ds]^{-1} \\ &= [\int_{\frac14}^{\frac34} \frac{1}{s(1-s)^{\frac14}}\cdot\frac{2s}{3}ds]^{-1} \\ &= 0.7538, \end{aligned}$$

$$\begin{aligned} k(t,u) &= \frac{ut}{100} \\ p(t,u) &= \frac{1}{u^{\frac18}t^{\frac14}(1-t)^{\frac14}} \\ \varphi(r) &= \frac{r}{100} \\ \psi(r) &\geqslant \frac{\sqrt2}{r^{\frac18}}. \end{aligned}$$

Let $a = 0.2, b = 4$, then

$$\varphi(b) \leqslant (b - w(b))A, \psi(a) \geqslant aB.$$

Therefore, the boundary value problem (43)-(44) has at least one positive solution by Theorem 10.

5 Conclusion

In this paper we considered the existence and multiplicity of positive solutions for a class of singular third-order three-point boundary problem (9) and (10) by using Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type, we got the conditions that the boundary value problem (9) and (10) had $one, two, \cdots, n$ positive solutions, meanwhile, we gave the conditions that the problem (9) and (10) had no positive solution. Finally, we validated the results that we have obtained by an example.

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