Some iterative algorithms for $k$-strictly pseudo-contractive mappings in a $CAT(0)$ space

AYNUR ŞAHİN
Sakarya University
Department of Mathematics
Sakarya, 54187
TURKEY
ayuce@sakarya.edu.tr

METİN BAŞARIR
Sakarya University
Department of Mathematics
Sakarya, 54187
TURKEY
basarir@sakarya.edu.tr

Abstract: In this paper, we prove the $\Delta$-convergence theorems of the cyclic algorithm and the new multi-step iteration for $k$-strictly pseudo-contractive mappings and give also the strong convergence theorem of the modified Halpern’s iteration for these mappings in a $CAT(0)$ space. Our results extend and improve the corresponding recent results announced by many authors in the literature.

Key–Words: $CAT(0)$ space, fixed point, strong convergence, $\Delta$-convergence, $k$-strictly pseudo-contractive mapping, iterative algorithm.

1 Introduction

Let $C$ be a nonempty subset of a real Hilbert space $X$. Recall that a mapping $T : C \rightarrow C$ is said to be $k$-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$.

A point $x \in C$ is called a fixed point of $T$ if $x = Tx$. We will denote the set of fixed points of $T$ by $F(T)$. Note that the class of $k$-strictly pseudo-contractive includes the class of nonexpansive mappings which are mappings $T$ on $C$ such that

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$ 

That is, $T$ is nonexpansive if and only if $T$ is 0-strictly pseudo-contractive. The mapping $T$ is also said to be pseudo-contractive if $k = 1$ and $T$ is said to be strongly pseudo-contractive if there exists a constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of $k$-strictly pseudo-contractive mappings is the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent from the class of $k$-strictly pseudo-contractive mappings (see, e.g., [1]-[3]). Recently, many authors have been devoted the studies on the problems of finding fixed points for $k$-strictly pseudo-contractive mappings (see, e.g., [4]-[10]).

We define the concept of $k$-strictly pseudo-contractive mapping in a $CAT(0)$ space as follows.

Let $C$ be a nonempty subset of a $CAT(0)$ space $X$. A mapping $T : C \rightarrow C$ is said to be $k$-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty)^2 \leq d(x, y)^2 + k \left( d(x, Tx) + d(y, Ty) \right)^2$$

for all $x, y \in C$.


For an arbitrary fixed order $k \geq 2$,

$$\begin{cases}
  x_0 \in C, \\
  x_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n, \\
  y_n^1 = (1 - \beta_n^1)y_n^1 + \beta_n^1 Ty_n^2, \\
  y_n^2 = (1 - \beta_n^2)y_n^2 + \beta_n^2 Ty_n^3, \\
  \vdots \\
  y_n^{k-2} = (1 - \beta_n^{k-2})y_n^{k-1} + \beta_n^{k-2}Ty_n^{k-1}, \\
  y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1}Tx_n, \quad \forall n \geq 0,
\end{cases}$$

or, in short,

$$\begin{cases}
  x_0 \in C, \\
  x_{n+1} = (1 - \alpha_n)y_n^1 + \alpha_n Ty_n^1, \\
  y_n^i = (1 - \beta_n^i)y_n^i + \beta_n^i Ty_n^{i+1}, \quad i = 1, 2, ..., k - 2, \\
  y_n^{k-1} = (1 - \beta_n^{k-1})x_n + \beta_n^{k-1}Tx_n, \quad \forall n \geq 0.
\end{cases}$$

By taking $k = 3$ and $k = 2$ in (2), we obtain the SP-iteration of Phuengrattana and Suantai [13] and the two-step iteration of Thianwan [14], respectively.
Acedo and Xu [15] introduced a cyclic algorithm in a Hilbert space. We modify this algorithm in a CAT(0) space.

Let \( x_0 \in C \) and \( \{a_n\} \) be a sequence in \([a, b]\) for some \( a, b \in (0, 1) \). The cyclic algorithm generates a sequence \( \{x_n\} \) in the following way:

\[
\begin{align*}
x_1 &= a_0x_0 \oplus (1 - a_0)T_0x_0, \\
x_2 &= a_1x_1 \oplus (1 - a_1)T_1x_1, \\
\vdots \\
x_N &= a_{N-1}x_{N-1} \oplus (1 - a_{N-1})T_{N-1}x_{N-1}, \\
x_{N+1} &= a_Nx_N \oplus (1 - a_N)T_0x_N, \\
\vdots
\end{align*}
\]

or, shortly,

\[
x_{n+1} = a_nx_n \oplus (1 - a_n)T_{[n]}x_n, \forall n \geq 0,
\]

where \( T_{[n]} = T_i, \) with \( i = n(modN) \), \( 0 \leq i \leq N - 1 \).

By taking \( T_{[n]} = T \) for all \( n \) in (3), we obtain the Mann iteration in [16].

In this paper, motivated by the above results, we prove the demiclosedness principle for \( k \)-strictly pseudo-contractive mappings in a CAT(0) space. Also we present the \( \Delta \)-convergence theorems of the cyclic algorithm and the new multi-step iteration and the strong convergence theorem of the modified Halpern’s iteration which is introduced for Hilbert space by Hu [17] for these mappings in a CAT(0) space.

### 2 Preliminaries on CAT(0) space

A metric space \( X \) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in \( X \) is at least as ‘thin’ as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [18]), Euclidean buildings (see [19]), R-trees (see [20]), the complex Hilbert ball with a hyperbolic metric (see [21]) and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [18].

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [22], [23]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in a CAT(0) space has been rapidly developed and many papers have appeared (see, e.g., [24]-[32]). It is worth mentioning that fixed point theorems in a CAT(0) space (specially in R-trees) can be applied to graph theory, biology and computer science (see, e.g., [20], [33]-[36]).

Let \( (X, d) \) be a metric space. A geodesic path joining \( x \in X \) to \( y \in X \) (or more briefly, a geodesic from \( x \) to \( y \)) is a map \( c \) from a closed interval \([0, l] \subset R \) to \( X \) such that \( c(0) = x \), \( c(l) = y \) and \( d(c(t), c(t')) = |t - t'| \) for all \( t, t' \in [0, l] \). In particular, \( c \) is an isometry and \( d(x, y) = l \). The image of \( c \) is called a geodesic (or metric) segment joining \( x \) and \( y \). When it is unique, this geodesic is denoted by \([x, y]\). The space \((X, d)\) is said to be a geodesic space if every two points of \( X \) are joined by a geodesic and \( X \) is said to be a uniquely geodesic if there is exactly one geodesic joining \( x \) to \( y \) for each \( x, y \in X \).

A geodesic triangle \( \Delta(x_1, x_2, x_3) \) in a geodesic metric space \((X, d)\) consist of three points in \( X \) (the vertices of \( \Delta \)) and a geodesic segment between each pair of vertices (the edges of \( \Delta \)). A comparison triangle for geodesic triangle \( \Delta(x_1, x_2, x_3) \) in \((X, d)\) is a triangle \( \Delta'(\pi_1, \pi_2, \pi_3) \) in the Euclidean plane \( \mathbb{R}^2 \) such that

\[
d_{\mathbb{R}^2}(\pi_i, \pi_j) = d(x_i, x_j)
\]

for \( i, j \in \{1, 2, 3\} \). Such a triangle always exists (see [18]).

A geodesic metric space is said to be a CAT(0) space [18] if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let \( \Delta \) be a geodesic triangle in \( X \) and \( \Delta' \) be a comparison triangle for \( \Delta \). Then, \( \Delta \) is said to satisfy the CAT(0) inequality if for all \( x, y \in \Delta \) and all comparison points \( \pi, \mu \in \Delta' \),

\[
d(x, y) \leq d_{\mathbb{R}^2}(\pi, \mu).
\]

If \( x, y_1, y_2 \) are points in a CAT(0) space and if \( y_0 \) is the midpoint of the segment \([y_1, y_2]\), then the CAT(0) inequality implies that

\[
d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.
\]

This is the (CN) inequality of Bruhat and Tits [37]. In fact (see [18, p.163]), a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality. It is worth mentioning that the results in a CAT(0) space can be applied to any CAT(0) space with \( k \leq 0 \) since any CAT(0) space is a CAT(0') space for every \( k' \geq k \) (see [18, p.165]).

Let \( x, y \in X \) and by Lemma 2.1(iv) of [27] for each \( t \in [0, 1] \), there exists a unique point \( z \in [x, y] \) such that

\[
d(x, z) = td(x, y), \quad d(y, z) = (1 - t)d(x, y).
\]
From now on, we will use the notation \((1 - t)x \oplus ty\) for the unique point \(z\) satisfying (4). We now collect some elementary facts about \(CAT(0)\) spaces which will be used in sequel the proofs of our main results.

**Lemma 1** Let \(X\) be a \(CAT(0)\) space. Then

(i) (see [27, Lemma 2.4]) for each \(x, y, z \in X\) and \(t \in [0, 1]\), one has

\[
d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z),
\]

(ii) (see [27, Lemma 2.5]) for each \(x, y, z \in X\) and \(t \in [0, 1]\), one has

\[
d((1 - t)x \oplus ty, z)^2 \\
\leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2.
\]

**3 Demiclosedness principle for \(k\)-strictly pseudo-contractive mappings**

In 1976 Lim [38] introduced a concept of convergence in a general metric space setting which is called \(\Delta\)-convergence. Later, Kirk and Panyanak [39] used the concept of \(\Delta\)-convergence introduced by Lim [38] to prove on the \(CAT(0)\) space analogs of some Banach space results which involve weak convergence. Also, Dhompongsa and Panyanak [27] obtained the \(\Delta\)-convergence theorems for the Picard, Mann and Ishikawa iterations in a \(CAT(0)\) space for nonexpansive mappings under some appropriate conditions.

We now give the definition and collect some basic properties of the \(\Delta\)-convergence.

Let \(X\) be a \(CAT(0)\) space and \(\{x_n\}\) be a bounded sequence in \(X\). For \(x \in X\), we set

\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).
\]

The asymptotic radius \(r(\{x_n\})\) of \(\{x_n\}\) is given by

\[
r(\{x_n\}) = \inf \{ r(x, \{x_n\}) : x \in X \}.
\]

The asymptotic center \(A(\{x_n\})\) of \(\{x_n\}\) is the set

\[
A(\{x_n\}) = \{ x \in X : r(x, \{x_n\}) = r(\{x_n\}) \}.
\]

It is known that in a complete \(CAT(0)\) space, \(A(\{x_n\})\) consists of exactly one point (see [40, Proposition 7]).

**Definition 2** ([38], [39]) A sequence \(\{x_n\}\) in a \(CAT(0)\) space \(X\) is said to be \(\Delta\)-convergent to \(x \in X\) if \(x\) is the unique asymptotic center of \(\{u_n\}\) for every subsequence \(\{u_n\}\) of \(\{x_n\}\). In this case, we write \(\Delta\)-lim_{n \to \infty} x_n = x\) and \(x\) is called the \(\Delta\)-limit of \(\{x_n\}\).

**Lemma 3** (i) Every bounded sequence in a complete \(CAT(0)\) space always has a \(\Delta\)-convergent subsequence. (see [39, p.3690])

(ii) Let \(C\) be a nonempty closed convex subset of a complete \(CAT(0)\) space and let \(\{x_n\}\) be a bounded sequence in \(C\). Then the asymptotic center of \(\{x_n\}\) is in \(C\). (see [41, Proposition 2.1])

**Lemma 4** ([27, Lemma 2.8]) If \(\{x_n\}\) is a bounded sequence in a complete \(CAT(0)\) space with \(A(\{x_n\}) = \{x\}\), \(\{u_n\}\) is a subsequence of \(\{x_n\}\) with \(A(\{u_n\}) = \{u\}\) and the sequence \(\{d(x_n, u)\}\) is convergent then \(x = u\).

Let \(C\) be a closed convex subset of a \(CAT(0)\) space \(X\) and \(\{x_n\}\) be a bounded sequence in \(C\). We denote the notation

\[
\{x_n\} \to w \Leftrightarrow \Phi(w) = \inf_{x \in C} \Phi(x) \tag{5}
\]

where \(\Phi(x) = \limsup_{n \to \infty} d(x, x_n)\).

Nanjaras and Panyanak [42] gave a connection between the "\(\to\)" convergence and \(\Delta\)-convergence.

**Proposition 5** ([42, Proposition 3.12]) Let \(C\) be a closed convex subset of a \(CAT(0)\) space \(X\) and \(\{x_n\}\) be a bounded sequence in \(C\). Then \(\Delta\)-lim_{n \to \infty} x_n = p\) implies that \(\{x_n\} \to p\).

The purpose of this section is to prove demiclosedness principle for \(k\)-strictly pseudo-contractive mappings in a \(CAT(0)\) space by using the convergence defined in (5).

**Theorem 6** Let \(C\) be a nonempty closed convex subset of a complete \(CAT(0)\) space \(X\) and \(T : C \to C\) be a \(k\)-strictly pseudo-contractive mapping such that \(k \in \left[0, \frac{1}{2}\right]\) and \(F(T) \neq \emptyset\). Let \(\{x_n\}\) be a bounded sequence in \(C\) such that \(\Delta\)-lim_{n \to \infty} x_n = w\) and \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\). Then \(Tw = w\).

**Proof:** By the hypothesis, \(\Delta\)-lim_{n \to \infty} x_n = w\). From Proposition 5, we get \(\{x_n\} \to w\). Then we obtain \(A(\{x_n\}) = \{w\}\) by Lemma 3 (ii) (see [42]). Since \(\lim_{n \to \infty} d(x_n, Tx_n) = 0\), then we get

\[
\Phi(x) = \limsup_{n \to \infty} d(x_n, x) = \limsup_{n \to \infty} d(Tx_n, x) \tag{6}
\]

for all \(x \in C\). In (6) by taking \(x = Tw\), we have

\[
\Phi(Tw)^2 = \limsup_{n \to \infty} d(Tx_n, Tw)^2 \\
\leq \limsup_{n \to \infty} \{d(x_n, w)^2 \\
+ k(d(x_n, Tx_n) + d(w, Tw))^2\} \\
\leq \limsup_{n \to \infty} d(x_n, w)^2 \\
+ k \limsup_{n \to \infty} \{d(x_n, Tx_n) + d(w, Tw)^2\} \\
= \Phi(w)^2 + kd(w, Tw)^2 \tag{7}
\]
Let \( T \) defined by (2). Then the sequence \( x_{n+1} = w, T x_{n}, T^2 x_{n}, \ldots, k = 1, 2 \) be a nonempty closed convex subset of a complete CAT(0) space \( X \), \( T : C \to C \) be a k-strictly pseudo-contractive mapping such that \( k \in [0, \frac{1}{2}) \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \), \( i = 1, 2, \ldots, k - 2 \) be sequences in \([a, b]\) for some \( a, b \in (0, 1) \) and \( k < 1 - b \). Let \( \{x_n\} \) be a sequence defined by (2). Then the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \).

**Theorem 7** Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space \( X \) and \( T : C \to C \) be a k-strictly pseudo-contractive mapping such that \( k \in [0, \frac{1}{2}) \) and \( F(T) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \), \( i = 1, 2, \ldots, k - 2 \) be sequences in \([a, b]\) for some \( a, b \in (0, 1) \) and \( k < 1 - b \). Let \( \{x_n\} \) be a sequence defined by (2). Then the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \).

**Proof:** Let \( p \in F(T) \). From (1), (2) and Lemma 1, we have

\[
d(x_{n+1} - p)^2 = d((1 - \alpha_n)y_n^1 + \alpha_n Ty_n^1, p)^2
\leq (1 - \alpha_n)d(y_n^1, p)^2 + \alpha_n d(Ty_n^1, p)^2
- \alpha_n(1 - \alpha_n)d(y_n^1, Ty_n^1)^2
\leq (1 - \alpha_n)d(y_n^1, p)^2 + \alpha_n \{d(y_n^1, p)^2 + kd(y_n^1, Ty_n^1)^2\}
- \alpha_n(1 - \alpha_n)d(y_n^1, Ty_n^1)^2
= d(y_n^1, p)^2 - \alpha_n((1 - \alpha_n) - k)d(y_n^1, Ty_n^1)^2
\leq d(y_n^1, p)^2.
\]

Also, we obtain

\[
d(y_n^1, p)^2 = d((1 - \beta_n^1)y_n^2 + \beta_n^1Ty_n^2, p)^2
\leq (1 - \beta_n^1)d(y_n^2, p)^2 + \beta_n^1d(Ty_n^2, p)^2
\leq (1 - \beta_n^1)d(y_n^2, p)^2 + \beta_n^1 \{d(y_n^2, p)^2 + kd(y_n^2, Ty_n^2)^2\}
- \beta_n^1((1 - \beta_n^1) - k)d(y_n^2, Ty_n^2)^2
\leq d(y_n^2, p)^2.
\]

Continuing the above process we have

\[
d(x_{n+1}, p) \leq d(y_n^1, p) \leq \ldots \leq d(y_n^k, p) \leq d(x_n, p).
\]

This inequality guarantees that the sequence \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in F(T) \). Let \( \lim_{n \to \infty} d(x_n, p) = r \). By using (9), we get

\[
\lim_{n \to \infty} d(y_{n+1}, p) = r.
\]

By Lemma 1, we also have

\[
d(y_{n+1}, p)^2 = d((1 - \beta_n^{k-1})x_n + \beta_n^{k-1}Tx_n, p)^2
\leq (1 - \beta_n^{k-1})d(x_n, p)^2 + \beta_n^{k-1}d(Tx_n, p)^2
- \beta_n^{k-1}((1 - \beta_n^{k-1}) - k)d(x_n, Tx_n)^2
\leq (1 - \beta_n^{k-1})d(x_n, p)^2 + \beta_n^{k-1} \{d(x_n, p)^2 + kd(x_n, Tx_n)^2\}
- \beta_n^{k-1}((1 - \beta_n^{k-1}) - k)d(x_n, Tx_n)^2
= d(x_n, p)^2 - \beta_n^{k-1}((1 - \beta_n^{k-1}) - k)d(x_n, Tx_n)^2,
\]

which implies that

\[
\lim_{n \to \infty} d(x_{n+1}, p) = r.
\]

Thus \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \). To show that the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \), we prove that

\[
W_\Delta(x_n) = \bigcup_{u_n \subseteq \{x_n\}} A(A(u_n)) \subseteq F(T)
\]

and \( W_\Delta(x_n) \) consists of exactly one point. Let \( u \in W_\Delta(x_n) \). Then, there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A(u_n) = \{u\} \). By Lemma 3, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta \)-lim \( v_n \to v = v \in K \). By Theorem 6, we have \( v \in F(T) \) and by Lemma 4, we have \( u = v \in F(T) \). This shows that \( W_\Delta(x_n) \subseteq F(T) \). Now, we prove that \( W_\Delta(x_n) \) consists of exactly one point. Let \( \{u_n\} \)
be a subsequence of \( \{ x_n \} \) with \( A(\{ u_n \}) = \{ u \} \) and let \( A(\{ x_n \}) = \{ x \} \). We have already seen that \( u = v \) and \( v \in F(T) \). Finally, since \( \{ d(x_n, v) \} \) is convergent, we have \( x = v \in F(T) \) byLemma 4. This shows \( W_\Delta(x_n) = \{ x \} \). This completes the proof. \( \square \)

Also, we prove the \( \Delta \)-convergence of the cyclic algorithm for \( k \)-strictly pseudo-contractive mappings in a \( CAT(0) \) space.

**Theorem 8** Let \( C \) be a nonempty closed convex subset of a complete \( CAT(0) \) space \( X \) and \( N \geq 1 \) be an integer. Let, for each \( 0 \leq i \leq N-1 \), \( T_i : C \to C \) be \( k_i \)-strictly pseudo-contractive mappings for some \( 0 \leq k_i < \frac{1}{2} \). Let \( k = \max \{ k_i ; 0 \leq i \leq N-1 \} \), \( \{ \alpha_n \} \) be a sequence in \([a, b]\) for some \( a, b \in (0, 1) \) and \( k < a \). Let \( F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset \). For \( x_0 \in C \), let \( \{ x_n \} \) be a sequence defined by (3). Then the sequence \( \{ x_n \} \) is \( \Delta \)-convergent to a common fixed point of the family \( \{ T_i \}_{i=0}^{N-1} \).

**Proof:** Let \( p \in F \). Using (1), (3) and Lemma 1, we have

\[
d(x_{n+1}, p)^2 = d(\alpha_n x_n + (1-\alpha_n) T[n] x_n, p)^2 \leq \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(T[n] x_n, p)^2 - \alpha_n (1-\alpha_n) d(x_n, T[n] x_n)^2 \\
\leq \alpha_n d(x_n, p)^2 + (1-\alpha_n) \left( d(x_n, p)^2 + kd(x_n, T[n] x_n)^2 \right) - \alpha_n (1-\alpha_n) d(x_n, T[n] x_n)^2 \\
= d(x_n, p)^2 - (1-\alpha_n)(\alpha_n - k) d(x_n, T[n] x_n)^2 \leq d(x_n, p)^2.
\]

This inequality guarantees that the sequence \( \{ x_n \} \) is bounded and \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in F \).

By (10), we also have

\[
d(x_n, T[n] x_n)^2 \leq \frac{1}{(1-\alpha_n)(\alpha_n - k)} d(x_n, p)^2 - (1-\alpha_n) d(x_{n+1}, p)^2 \leq \frac{1}{(1-b)(a - k)} d(x_n, p)^2 - d(x_{n+1}, p)^2.
\]

Since \( \lim_{n \to \infty} d(x_n, p) \) exists, we obtain \( \lim_{n \to \infty} d(x_n, T[n] x_n) = 0 \). The rest of the proof closely follows the proof of Theorem 7 and is therefore omitted. \( \square \)

### 4 The strong convergence theorem for the modified Halpern’s iteration

In [17], Hu introduced a modified Halpern’s iteration. We modify this iteration in a \( CAT(0) \) space as follows.

For an arbitrary initial value \( x_0 \in C \) and a fixed anchor \( u \in C \), the sequence \( \{ x_n \} \) is defined by

\[
x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \\
y_n = \frac{1}{1-\alpha_n} x_n + \frac{\gamma_n}{1-\alpha_n} T x_n, \quad \forall n \geq 0,
\]

where \( \{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \} \) are three real sequences in \((0, 1)\) satisfying \( \alpha_n + \beta_n + \gamma_n = 1 \).

Clearly, the iterative sequence (11) is a natural generalization of the well known iterations.

(i) If we take \( \beta_n = 0 \) for all \( n \) in (11), then the sequence (11) reduces to the Halpern’s iteration in [43].

(ii) If we take \( \alpha_n = 0 \) for all \( n \) in (11), then the sequence (11) reduces to the Mann iteration in [16].

In this section, we prove the strong convergence of the modified Halpern’s iteration in a \( CAT(0) \) space.

Recall that a continuous linear functional \( \mu \) on \( \ell_\infty \), the Banach space of bounded real sequences, is called a Banach limit if \( \| \mu \| = 1 \) and \( \mu(\{ a_n \}) = \mu(\{ a_n \}) \) for all \( \{ a_n \}_{n=1}^\infty \subset \ell_\infty \).

**Lemma 9** (see [44, Proposition 2]) Let \( \{ a_n \} \in \ell_\infty \) be such that \( \mu(a_n) \leq 0 \) for all Banach limits \( \mu \) and \( \limsup_{n \to \infty} (a_{n+1} - a_n) \leq 0 \). Then, \( \limsup_{n \to \infty} a_n \leq 0 \).

**Lemma 10** Let \( C \) be a nonempty closed convex subset of a complete \( CAT(0) \) space \( X \), \( T : C \to C \) be a \( k \)-strictly pseudo-contractive mapping with \( k \in [0, 1) \) and \( S : C \to C \) be a mapping defined by \( Sz = k z \oplus (1-k) T z \), for \( z \in C \). Let \( u \in C \) be fixed. For each \( t \in [0, 1] \), the mapping \( S_t : C \to C \) defined by

\[
S_t z = tu \oplus (1-t) S z = tu \oplus (1-t) (k z \oplus (1-k) T z)
\]

for \( z \in C \), has a unique fixed point \( z_t \in C \), that is,

\[
z_t = S_t(z_t) = tu \oplus (1-t) S(z_t).
\]

**Proof:** As it has been proven in [45], if \( T \) is a \( k \)-strictly pseudo-contractive mapping with \( k \in [0, 1) \), \( S \) is a nonexpansive mapping such that \( F(S) = F(T) \). Then, from Lemma 2.1 in [29], the mapping \( S_t \) has a unique fixed point \( z_t \in C \). \( \square \)
Lemma 11 Let $X, C, T$ and $S$ be as in Lemma 10. Then, $F(T) \neq \emptyset$ if and only if \{z$_t$\} given by (12) remains bounded as $t \to 0$. In this case, the following statements hold:

1. \{z$_t$\} converges to the unique fixed point $z$ of $T$ which is nearest to $u$;
2. $d(u, z)^2 \leq \mu d(u, x_n)^2$ for all Banach limits $\mu$ and all bounded sequences \{x$_n$\} with $\lim_{n \to \infty} d(x_n, T x_n) = 0$.

Proof: If $F(T) \neq \emptyset$, then we have $F(S) = F(T) \neq \emptyset$. Also, if $\lim_{n \to \infty} d(x_n, T x_n) = 0$, we obtain that
\[
d(x_n, S x_n) = d(x_n, k x_n \oplus (1 - k) T x_n) \leq (1 - k) d(x_n, T x_n) \to 0 \text{ as } n \to \infty.
\]
Thus, from Lemma 2.2 in [29], the rest of the proof of this lemma can be seen. \hfill $\Box$

The following lemma can be found in [46].

Lemma 12 (see [46, Lemma 2.1]) Let \{a$_n$\} be a sequence of non-negative real numbers satisfying the condition
\[
a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \sigma_n, \quad \forall n \geq 0,
\]
where \{\gamma$_n$\} and \{\sigma$_n$\} are sequences of real numbers such that
1. \{\gamma$_n$\} \subset [0, 1] and $\sum_{n=1}^{\infty} \gamma_n = \infty$,
2. either $\lim \sup_{n \to \infty} \sigma_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \sigma_n| < \infty$.
Then, $\lim_{n \to \infty} a_n = 0$.

We are now ready to prove our main result.

Theorem 13 Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a $k$-strictly pseudo-contractive mapping such that $0 \leq k < \frac{\beta_n}{1 - \alpha_n} < 1$ and $F(T) \neq \emptyset$. Let \{x$_n$\} be a sequence defined by (11). Suppose that \{a$_n$\}, \{b$_n$\} and \{\gamma$_n$\} satisfy the following conditions:

1. (C1) $\lim_{n \to \infty} \alpha_n = 0$,
2. (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
3. (C3) $\lim_{n \to \infty} \beta_n \neq k$ and $\lim_{n \to \infty} \gamma_n \neq 0$.

Then the sequence \{x$_n$\} converges strongly to a fixed point of $T$.

Proof: We divide the proof into three steps. In the first step we show that \{x$_n$\}, \{y$_n$\} and \{T x$_n$\} are bounded sequences. In the second step we show that $\lim_{n \to \infty} d(x_n, T x_n) = 0$. Finally, we show that \{x$_n$\} converges to a fixed point $z \in F(T)$ which is nearest to $u$.

First step: Take any $p \in F(T)$, then, from Lemma 1 and (11), we have
\[
d(y_n, p)^2 \leq \frac{\beta_n}{1 - \alpha_n} d(u, p)^2 + \frac{\gamma_n}{1 - \alpha_n} d(T x_n, p)^2
\]
\[
- \frac{\beta_n \gamma_n}{(1 - \alpha_n)^2} d(x_n, T x_n)^2
\]
\[
\leq \frac{\beta_n}{1 - \alpha_n} d(u, p)^2
\]
\[
+ \frac{\gamma_n}{1 - \alpha_n} \left( d(x_n, p)^2 + k d(x_n, T x_n)^2 \right)
\]
\[
- \frac{\beta_n \gamma_n}{(1 - \alpha_n)^2} d(x_n, T x_n)^2
\]
\[
d(x_n, p)^2 - \frac{\gamma_n}{1 - \alpha_n} \left( \frac{\beta_n}{1 - \alpha_n} - k \right) d(x_n, T x_n)^2
\]
\[
\leq d(x_n, p)^2.
\]
Also, we obtain
\[
d(x_{n+1}, p)^2 \leq \alpha_n d(u, p)^2 + (1 - \alpha_n) (d(y_n, p)^2
\]
\[
- \alpha_n (1 - \alpha_n) d(u, y_n)^2 \leq \alpha_n d(u, p)^2 + (1 - \alpha_n) \left\{ d(x_n, p)^2
\]
\[
- \frac{\gamma_n}{1 - \alpha_n} \left( \frac{\beta_n}{1 - \alpha_n} - k \right) d(x_n, T x_n)^2
\]
\[
- \alpha_n (1 - \alpha_n) d(u, y_n)^2 = \alpha_n d(u, p)^2 + (1 - \alpha_n) d(x_n, p)^2
\]
\[
- \gamma_n \left( \frac{\beta_n}{1 - \alpha_n} - k \right) d(x_n, T x_n)^2
\]
\[
- \alpha_n (1 - \alpha_n) d(u, y_n)^2 \leq \alpha_n d(u, p)^2 + (1 - \alpha_n) d(x_n, p)^2
\]
\[
\leq \max \left\{ d(u, p)^2, d(x_n, p)^2 \right\}.
\]
By induction,
\[
d(x_{n+1}, p)^2 \leq \max \left\{ d(u, p)^2, d(x_0, p)^2 \right\}.
\]
This proves the boundedness of the sequence \{x$_n$\}, which leads to the boundedness of \{T x$_n$\} and \{y$_n$\}.

Second step: In fact, we have from (13) (for some appropriate constant $M > 0$) that
\[
d(x_{n+1}, p)^2 \leq \alpha_n d(u, p)^2 + (1 - \alpha_n) d(x_n, p)^2
\]
\[
- \gamma_n \left( \frac{\beta_n}{1 - \alpha_n} - k \right) d(x_n, T x_n)^2
\]
\[
= \alpha_n d(u, p)^2 - d(x_n, p)^2 \leq d(x_n, p)^2
\]
\[
- \gamma_n \left( \frac{\beta_n}{1 - \alpha_n} - k \right) d(x_n, T x_n)^2
\]
Thus, following (14), we have
\[\leq \alpha_n M + d(x_n, p)^2\]
\[-\gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2,\]
which implies that
\[\gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M \leq d(x_n, p)^2 - d(x_{n+1}, p)^2.\] (14)

If \(\gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M \leq 0\), then
\[d(x_n, Tx_n)^2 \leq \frac{\alpha_n}{\gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right)} M,\]
and hence the desired result is obtained by the conditions (C1) and (C3).

If \(\gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M > 0\), then following (14), we have
\[\sum_{n=0}^{m} \gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M \leq d(x_0, p)^2 - d(x_{m+1}, p)^2 \leq d(x_0, p)^2.\]

That is
\[\sum_{n=0}^{\infty} \gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M < \infty.\]
Thus
\[\lim_{n \to \infty} \gamma_n \left(\frac{\beta_n}{1 - \alpha_n} - k\right) d(x_n, Tx_n)^2 - \alpha_n M = 0.\]
Then we get
\[\lim_{n \to \infty} d(x_n, Tx_n) = 0.\] (15)

Third step: Using the condition (C1) and (15), we obtain
\[d(x_{n+1}, x_n) \leq d(x_{n+1}, Tx_n) + d(Tx_n, x_n) \leq \alpha_n d(u, Tx_n) + (1 - \alpha_n) d(y_n, Tx_n) + d(Tx_n, x_n) \leq \alpha_n d(u, Tx_n) + (1 - \alpha_n) \left(\frac{\beta_n}{1 - \alpha_n} d(x_n, Tx_n)\right) + d(Tx_n, x_n) = \alpha_n d(u, Tx_n) + (\beta_n + 1) d(x_n, Tx_n) \to 0, \text{ as } n \to \infty.\]

Also, from (15), we have
\[d(x_n, y_n) \leq \frac{\gamma_n}{1 - \alpha_n} d(x_n, Tx_n) \to 0, \text{ as } n \to \infty.\] (16)

Let \(z = \lim_{t \to 0} z_t\), where \(z_t\) is given by (12) in Lemma 10. Then, \(z\) is the point of \(F(T)\) which is nearest to \(u\). By Lemma 11 (2), we have \(\mu (d(u, z)^2 - d(u, x_n)^2) \leq 0\) for all Banach limits \(\mu\). Let \(\alpha_n = d(u, z)^2 - d(u, x_n)^2\). Moreover, since \(\lim_{n \to \infty} d(x_{n+1}, x_n) = 0\), we get
\[\lim_{n \to \infty} \sup (\alpha_{n+1} - \alpha_n) = 0.\]

By Lemma 9, we obtain
\[\lim_{n \to \infty} \sup \left(d(u, z)^2 - d(u, x_n)^2\right) \leq 0.\] (17)
It follows from the condition (C1) and (16) that
\[\lim_{n \to \infty} \sup \left(d(u, z)^2 - (1 - \alpha_n) d(u, y_n)^2\right) = \lim_{n \to \infty} \sup \left(d(u, z)^2 - d(u, x_n)^2\right).\] (18)

By (17) and (18), we have
\[\lim_{n \to \infty} \sup \left(d(u, z)^2 - (1 - \alpha_n) d(u, y_n)^2\right) \leq 0.\] (19)

We observe that
\[d(x_{n+1}, z)^2 \leq \alpha_n d(u, z)^2 + (1 - \alpha_n) d(y_n, z)^2 - \alpha_n (1 - \alpha_n) d(u, y_n)^2 \leq \alpha_n d(u, z)^2 + (1 - \alpha_n) d(x_n, z)^2 - \alpha_n (1 - \alpha_n) d(u, y_n)^2 = (1 - \alpha_n) d(x_n, z)^2 + \alpha_n \left(d(u, z)^2 - (1 - \alpha_n) d(u, y_n)^2\right).\]

It follows from the condition (C2) and (19), using Lemma 12, that \(\lim_{n \to \infty} d(x_n, z) = 0\). This completes the proof of Theorem 13. \(\square\)

We obtain the following corollary as a direct consequence of Theorem 13.

Corollary 14 Let \(X, C\) and \(T\) be as Theorem 13. Let \(\{\alpha_n\}\) be a real sequence in \((0, 1)\) satisfying the conditions (C1) and (C2). For a constant \(\delta \in (k, 1)\), an arbitrary initial value \(x_0 \in C\) and a fixed anchor \(u \in C\), let the sequence \(\{x_n\}\) be defined by
\[x_{n+1} = \alpha_n u + (1 - \alpha_n) (\delta x_n + (1 - \delta) T x_n),\] (20)
for all \(n \geq 0\). Then the sequence \(\{x_n\}\) is strongly convergent to a fixed point of \(T\).
Proof: If, in the proof of Theorem 13, we take $\beta_n = (1 - \alpha_n) \delta$ and $\gamma_n = (1 - \alpha_n)(1 - \delta)$, then we get the desired conclusion. □

Remark 15 The results in this section contain the strong convergence theorems of the iterative sequences (11) and (20) for nonexpansive mappings in a CAT(0) space. Also, our results contain the corresponding theorems proved for these iterative sequences in a Hilbert space.

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