The Stability, Bifurcation and Chaos of a Duopoly Game in the Market of Complementary Products with Mixed Bundling Pricing

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Abstract: -This paper constructs a non-competitive duopoly price model with mixed bundling pricing in complementary product market. The system is modeled by nonlinear difference equations. The existence of Nash equilibrium point and its stability are studied. The effects of complementarity degree and discount of bundling on the stability are investigated. The effects of the price adjustment parameters on the producer surplus are also investigated, especially when the parameters are used with the discount of bundling. The results show: increasing the degree of complementarity or the discount of bundling can enlarge the stable region of the system. Firms can get higher producer surplus with suitable discount of bundling in stable state than in unstable state of the market.

Key–Words: - Complex Dynamics, Duopoly game, Complementary products, Mixed Bundling pricing

1 Introduction

In 1838, Cournot [1] proposed Cournot model and assumed that firms compete using quantity as their strategic variable and then take prices as determined by the market through an inverse demand function. In 1883, reworked Cournot model, Bertrand [2] constructed a game model using prices as the strategic variable and taking quantity as determined. In recent years, chaos and bifurcation theory based on difference equation has been applied in oligarchs economic model [3-6]; many scholars in various fields have the analysis about the complex characteristics of dynamic model [7-10], and their results have inspiring contributions to later researches. Ma and Ji [11] have constructed a Cournot model in electric power triopoly with nonlinear inverse demand, and the model is further studied by Ji [12] base on heterogeneous players. Chen [13] studied dynamic process of the triopoly games in Chinese 3G telecommunication market. Sun [14] studied the complexity of triopoly price game in Chinese cold rolled steel market and the model is further studied by Ma [15] with different decision-making rules. Xu [16] built a dynamic model of a duopoly price game in insurance market and investigated its complexity. Wang [17] proposed a multi-enterprise output game model under the circumstances of information asymmetry and analyzed its complex dynamics. Zhang [18] built a nonlinear price game of insurance market in which one of the two competitors in the market makes decision only with bounded rationality without delay, and the other competitor makes the delayed decision with one period and two periods. Tu [19] constructed a new dynamic Cournot duopoly game model in exploitation of a renewable resource with bounded rationality players, investigated its complex dynamics, and controlled the chaos. All above studies are limited in homogeneous or substitutable products, that is customers choose between the competing products depending on their preferences and the marketing strategy of the firms. Dynamic models in oligopoly market lack the research about the case of complementary products.

The situation of complementary products appears when customers have to buy more than one product at the same time to obtain the full utility of the products. The joint consumption of complementary products has a specific value for the consumer which exceed the mere addition of utilities when products are consumed in isolation: the higher this excess, the larger the complementarity between the products [20]. The complementary products is different from homogeneous and substitutable products in that complementary products benefit from each others sales rather than losing sales to the other firms. For examples, computer hardware and software, bed and mattress, baby bottle and nipple, a washer and a dryer, their demands are interrelated and coupled. Gabszewicz [21] considered a duopoly industry with two separate firms each selling an indivisible product and analyzed price equilibrium in this market as related to the degree of complementarity between the two products. Yue [20] analyzed a Bertrand model of complementary products under alliance-cooperative game, non-cooperation-Bayesian Bertrand game and information sharing-Bertrand game. Mukhopadhyay [22] studied a Stackelberg model of complementary products, and investigated the effects of information sharing. Yan [23] studied the profit benefits of bundle pricing of complementary products, and provided a framework that can help firms find optimum bundling product categories and pricing strategies that maximize their profits. Matsumoto [24] studied the statistical dynamics in chaotic Cournot model in which reactions functions are non-linear and products are complements.

In this paper, we consider a duopoly market with mixed bundling pricing where two separate firms offer complementary products. Our gaming system is described by nonlinear difference equations, which modify and extend the results of [20-23] in which the firms are considered with static expectations and linear equations. Matsumoto [24] thoroughly investigated the statistical dynamics of chaotic behavior in the market of complementary products, but in his paper, no study investigating the effect of the complementarity on the system, and no mixed bundling pricing involved in.

In real economic activities, bundling pricing refers to the sale of two or more separate products in a package at a discount. In order to increase sales, bundling pricing is frequently used by producers. Thus, the effectiveness of bundling pricing needs more research especially when it is used with price adjustment parameters that have good dynamic characteristics. Mixed bundling pricing means each product can be consumed separately but joint consumption is also an available alternative for the consumer. There, in our study, in order to extend the existing research, we consider the case in which products are complements and are sold by mixed bundling pricing.

The main novelty of our paper include: A dynamical model of complementary products which is non-competitive is constructed and the existence of chaos dynamics in this model is verified; the joint effects of the price adjustment parameter, the degree of complementarity and the discount of bundling on the stability of system and the producer surplus are studied.

This paper is organized as follows. In Section 2 a game model of complementary products with mixed bundling pricing is proposed. In Section 3 the local stability of the Nash equilibrium of the system is analyzed and investigated by numerical simulations under some change of parameters (such as price adjustment parameters, the degree of complementarity and the discount of bundling). In Section 4 the joint effects of price adjustment parameters, the degree of complementarity and the discount of bundling on the complexity of the model and the producer surplus are analyzed. Section 5 is the conclusion.

2 The model

2.1 Assumptions

Our model is based on following assumptions:

Assumption 1. Consider two separate firms offering complementary products to customers who have a need to buy both products and also to customers who have a need to buy only one product independently.

Assumption 2. There are three segments of consumers in the market. Segment 1 buys one product only, segment 3 buys another product, and segment 2 buys the both products. There is no price discrimination among the three segments.

Assumption 3. Firms always make the optimal price decision for the maximal profit in every period.

2.2 A real word example

In an area, there are two firms 1 and 2. The firm 1 might be the monopolistic supplier of a sort of hardware, such as IBM. The firm 2 might be a monopolistic supplier of a sort of software, such as Microsoft. Segment 1 would buy the hardware but not buy the software. Segment 2 would buy the hardware and the software in the same time. Part 3 would buy the software but not buy the hardware. Firm 1 and firm 2 engage in jointly selling their products. In practice, IBM and Microsoft have decided to conduct strategic cooperation sales.

The demands of two firms are relative by the demand of segment 2 who wants to buy the hardware and the software. The price decision of one firm would affect the other firm's market performance and vice versa. We say there is a price game of complementary products between firms 1 and 2.

2.3 Nomenclature

The following is a list of nomenclatures that will be used throughout the paper:

Nomenclature 1. $p_{1_t}, p_{2_t}$are prices per unit product for firms 1 and 2 in period t , respectively. The demands of products 1 and 2 from segment 1 and segment 3 are as follows:

$$q_{1_t - Seg_1} = a_1 - b_1 p_{1_t}, \tag{1}$$

$$q_{2_t - Seg_3} = a_2 - b_2 p_{2_t}, \tag{2}$$

where a_1 is the potential demand if offered free of charge for product 1 from segment 1 , a_2 is the potential demand for product 2 from segment 3. b_1, b_2 are parameters of self-price sensitivity.

Nomenclature 2. Segment 2 buys the both products. Firms 1 and 2 define bundling pricing as the sale of two products in a package at a discount. They decide the price of the bundling after discount as $u_1 p_{1_t} + u_2 p_{2_t}, u_1, u_2 \in (0, 1]$ under the circumstance of collaboration. The intent of our research is to study the effect on the system exerted by the total discount rate. In order to research conveniently, we set $u_1 = u_2 = u.d = 1 - u$ is the discount rate. If $u = 1$, the discount rate of the bundling products is zero. The demands of products 1 and 2 after discount of bundling from segment 2 are as follows:

$$q_{1_t - Seg_2} = a - b_{12}(up_{1_t}) - b_{12} r(up_{2_t}) \tag{3}$$

$$q_{2_t - Seg_2} = a - b_{22}(up_{2_t}) - b_{22} r(up_{1_t}) \tag{4}$$

where a is the potential demand for product 1 and product 2 from segment 2. b_{12} and b_{22} represent self-price sensitivity. rb_{12} and rb_{22} are cross-price sensitivities, r reflects the degree of complementarity between two products, a larger r means a higher degree of complementarity. Because $r \in (0, 1)$, a product's self-price sensitivity is larger than its cross-price sensitivity. All parameters in this model are positive.

Nomenclature 3. The total demands of products 1 and 2 are the sum of their demands from two segments:

$$\begin{aligned} q_{1_t} &= q_{1_t - Seg_1} + q_{1_t - Seg_2} \\ &= a_1 - b_1 p_{1_t} + [a - b_{12}(up_{1_t}) - b_{12} r(up_{2_t})] \end{aligned} \tag{5}$$

$$\begin{aligned} q_{2_t} &= q_{2_t - Seg_3} + q_{2_t - Seg_2} \\ &= a_2 - b_2 p_{2_t} + [a - b_{22}(up_{2_t}) - b_{22} r(up_{1_t})] \end{aligned} \tag{6}$$

Nomenclature 4. The cost functions of firms 1 and 2 are$C_{1_t} = c_1 q_{1_t}$, $C_{2_t} = c_2 q_{2_t}$. So, (7) and (8) are the profits of firms 1 and 2 in period t, respectively.

$$\Pi_{1_t} = q_{1_t - Seg_1} p_{1_t} + q_{1_t - Seg_2}(up_{1_t}) - c_1 q_{1_t} \tag{7}$$

$$\Pi_{2_t} = q_{2_t - Seg_3} p_{2_t} + q_{2_t - Seg_2}(up_{2_t}) - c_2 q_{2_t} \tag{8}$$

The demand structure and functions are shown in Figure 1.

2.4 The model

The two firms use bounded rationality to make their price decisions based on the marginal profits $\frac{\partial \Pi_{i_t}}{\partial p_{i_t}}, i = 1, 2$. Each of them increase (decrease) its price in period $t+1$ if the marginal profit is positive (negative). The marginal profits of firms 1 and 2 at period t are given respectively by following nonlinear form:

$$\begin{aligned} \frac{\partial \Pi_{1_t}}{\partial p_{1_t}} = -2b_1 p_{1_t} + a_1 - b_{12} u^2 p_{1_t} + (a - b_{12} up_{1_t} \\ - b_{12} rup_{2_t})u - c_1(-b_1 - b_{12} u) \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial \Pi_{2_t}}{\partial p_{2_t}} = -2b_2 p_{2_t} + a_2 - b_{22} u^2 p_{2_t} + (a - b_{22} up_{2_t} \\ - b_{22} rup_{1_t})u - c_2(-b_2 - b_{22} u) \end{aligned} \tag{10}$$

Based on above analysis the dynamic adjustment equations can be written as follows:

$$\begin{cases} p_{1_{t+1}} = p_{1_t} + \alpha p_{1_t} \frac{\partial \Pi_{1_t}}{\partial p_{1_t}} \\ p_{2_{t+1}} = p_{2_t} + \beta p_{2_t} \frac{\partial \Pi_{2_t}}{\partial p_{2_t}} \end{cases} \quad \alpha, \beta > 0 \tag{11}$$

where α, β are price adjustment parameters of firms 1 and 2, respectively. They imply the fluctuation of two firm's price decisions, and they are very small. The game model with bounded rational players has the following nonlinear form:

$$\begin{cases} p_{1_{t+1}} = p_{1_t} + \alpha p_{1_t}[-2b_1 p_{1_t} + a_1 - b_{12} u^2 p_{1_t} \\ +(a - b_{12} up_{1_t} - b_{12} rup_{2_t})u - c_1(-b_1 - b_{12} u)] \\ p_{2_{t+1}} = p_{2_t} + \beta p_{2_t}[-2b_2 p_{2_t} + a_2 - b_{22} u^2 p_{2_t} \\ +(a - b_{22} up_{2_t} - b_{22} rup_{1_t})u - c_2(-b_2 - b_{22} u)] \end{cases} \tag{12}$$

3 Stability of fixed point

The Nash equilibrium of Bertrand game is a state of price decision in which it is impossible to make any one player better off without making at least one player worse off. Thus, the stable fixed points of system (12) which are called Nash equilibrium of Bertrand game are solutions of (13):

$$\begin{cases} \frac{\partial \Pi_{1_t}}{p_{1_t}} = -2b_1 p_{1_t} + a_1 - b_{12} u^2 p_{1_t} + (a - b_{12} up_{1_t} \\ -b_{12} rup_{2_t})u - c_1(-b_1 - b_{12} u) = 0, \\ \frac{\partial \Pi_{2_t}}{p_{2_t}} = -2b_2 p_{2_t} + a_2 - b_{22} u^2 p_{2_t} + (a - b_{22} up_{2_t} \\ -b_{22} rup_{1_t})u - c_2(-b_2 - b_{22} u) = 0. \end{cases} \tag{13}$$

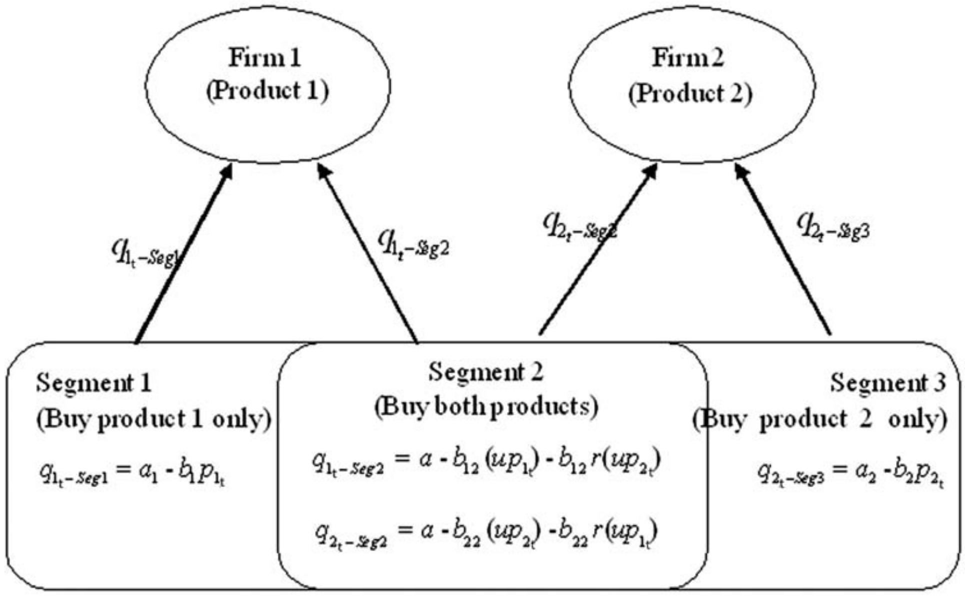

Figure 1: The demand structure and functions.

The system (13) does not depend on price adjustment parameters α and β . By simple computation, we found that only one positive Nash equilibrium of price game which is of practical significance $E^* = (p_1^*, p_2^*)$. Where,

$$p_1^* = [(b_{12}ra + b_{12}rc_2b_{22} - 2b_{12}c_1b_{22} - 2ab_{22})u^3 + (b_{12}ra_2 - 2a_1b_{22} - 2c_1b_{22}b_1 + b_{12}rc_2b_2)u^2 - 4b_2b_1 + (-2b_{12}b_2c_1 - 2b_2a)u - 2b_2a_1 - 2b_2c_1b_1] /(-4b_{22}b_{12} + b_{22}r^2b_{12})u^4 + (-4b_2b_{12} - 4b_{22}b_1)u^2 \tag{14}$$

and

$$p_2^* = [(-2ab_{12} - 2c_2b_{22}b_{12} + b_{22}ra + b_{22}rc_1b_{12})u^3 + (-2a_2b_{12} - 2c_2b_2b_{12} + b_{22}ra_1 + b_{22}rc_1b_1)u^2 - 4b_2b_1 + (-2ab_1 - 2c_2b_{22}b_1)u - 2b_1a_2 - 2b_1c_2b_2] /(-4b_{22}b_{12} + b_{22}r^2b_{12})u^4 + (-4b_2b_{12} - 4b_{22}b_1)u^2 \tag{15}$$

The Jacobian matrix of Nash equilibrium point E^* is as (16) described.

$$J(E^*) = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \tag{16}$$

$$D_1 = 1 + \alpha[-2b_1p_{1_t} + a_1 - b_{12}u^2p_{1_t} + (a - b_{12}up_{1_t} - b_{12}rup_{2_t})u - c_1(-b_1 - b_{12}u)] + \alpha p_{1_t}(-2b_1 - 2b_{12}u^2) \tag{17}$$

$$D_2 = -\alpha p_{1_t}b_{12}ru^2 \tag{18}$$

$$D_3 = -\beta p_{2_t}b_{22}ru^2 \tag{19}$$

$$D_4 = 1 + \beta[-2b_2p_{2_t} + a_2 - b_{22}u^2p_{2_t} + (a - b_{22}up_{2_t} - b_{22}rup_{1_t})u - c_2(-b_2 - b_{22}u)] + \beta p_{2_t}(-2b_2 - 2b_{22}u^2) \tag{20}$$

The trace and determinant of $J(E^*)$ are denoted $Tr(E^*)$ and $Det(E^*)$. The characteristic polynomial of (16) as (21) shows:

$$F(\lambda) = \lambda^2 - Tr(J(E^*)) \times \lambda + Det(J(E^*)) \tag{21}$$

where

$$Tr(J(E^*)) = -(-\alpha a_1 - 2 + \beta b_{22}ru^2p_{1_t} - \alpha c_1b_1 + 4\beta b_2p_{2_t} - \beta c_2b_2 + 4\alpha b_1p_{1_t} - \beta a^2 - \alpha ua - \beta ua + 4\alpha b_{12}u^2p_{1_t} + \alpha b_{12}ru^2p_{2_t} - \alpha c_1b_{12}u + 4\beta b_{22}u^2p_{2_t} - \beta c_2b_{22}u),$$

$$Det(E^*) = 1 + \beta ua + \beta c_2b_2 + (4\alpha b_{12}ru^2\beta b_2 + 4\alpha b_{12}u^4\beta b_{22}r)p_{1_t}^2 + \alpha c_1b_1 + \alpha a_1 + \beta a_2\alpha a_1\beta a_2 + \alpha a_1\beta c_2b_2 + \alpha c_1b_1\beta a_2 + \alpha c_1b_1\beta c_2b_2 + (4\alpha b_1\beta b_{22}ru^4 + 4\alpha b_{12}u^4\beta b_{22}r)p_{1_t}^2 + \alpha ua + \alpha c_1b_{12}u + \beta c_2b_{22}u + \alpha u^2a^2\beta + \alpha a_1\beta ua + \alpha a_1\beta c_2b_{22}u + \alpha ua\beta a_2 + \alpha ua\beta c_2b_2 + \alpha u^2a\beta c_2b_{22} + \alpha c_1b_1\beta ua + \alpha c_1b_1\beta c_2b_{22}u + \beta c_1b_{12}u\beta a_2 + \alpha c_1b_{12}u^2\beta a + \alpha c_1b_{12}u\beta c_2b_2 + \alpha c_1b_{12}u^2\beta c_2b_{22} + (-4\alpha b_1\beta c_2b_2 - 4\alpha b_1\beta ua - \beta b_{22}ru^2 - 4\alpha b_1\beta c_2b_{22}u - \alpha c_1b_1\beta b_{22}ru^2 - 4\alpha b_{12}u^2\beta a_2 - 4\alpha b_{12}u^3\beta c_2b_{22} - 4\alpha b_{12}u^3\beta a$$

$$
\begin{aligned}
&-4\alpha b_{12}u^2 - 4\alpha b_{12}u^2\beta c_2 b_2 - \alpha u^3 a\beta b_{22} r \\
&-\alpha a_1 \beta b_{22} r u^2 - 4\alpha b_1 \beta a_2 - 4\alpha b_1 - \alpha c_1 b_{12} u^3 \beta b_{22} r) p_{1_t} \\
&+[-4\alpha a_1 \beta b_2 + (16\alpha b_1 \beta b_2 + 16\alpha b_1 \beta b_{22} u^2 \\
&+16\alpha b_{12} u^2 \beta b_2 + 16\alpha b_{12} u^4 \beta b_{22}) p_{1_t} - 4\alpha c_1 b_1 \beta b_2 \\
&-4\alpha a_1 \beta b_{22} u^2 - 4\beta b_{22} u^2 - 4\alpha c_1 b_1 \beta b_{22} u^2 - \alpha b_{12} r u^2 \\
&-4\beta b_2 - \alpha b_{12} r u^3 \beta c_2 b_{22} - 4\alpha u^3 a \beta b_{22} - \alpha b_{12} r u^3 \beta a \\
&-\alpha b_{12} r u^2 \beta c_2 b_2 - 4\alpha c_1 b_{12} u \beta b_2 - \alpha b_{12} r u^2 \beta a_2 \\
&-4\alpha c_1 b_{12} u^3 \beta b_{22} - 4\alpha u a \beta b_2] p_{2_t}
\end{aligned}
.
$$

By the Jury stability criterion, we can evaluate the locally stable region $\Omega_{E^*}(r,u)$ with respect to parameters (r,u) as follows:

$$
\begin{aligned}
&\Omega_{E^*}(r,u) = \{(r,u) : \\
&1 + Tr(J(E^*)) + Det(J(E^*)) > 0, \\
&1 - Tr(J(E^*)) + Det(J(E^*)) > 0, \\
&1 - Det(J(E^*)) > 0\}.
\end{aligned}
\tag{22}
$$

The conditions (22) define a stable region where Nash equilibrium point E^* is local stable. In this region, the price will reach E^* by adjusting limited times with random initial prices.

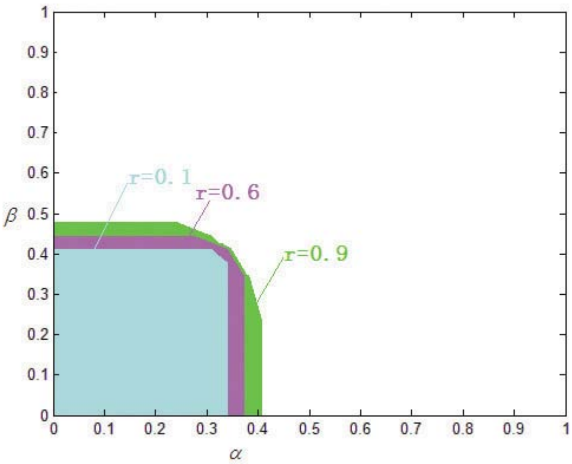

Figure 2: The stability with the change of r where $d = 0.3$.

4 The numerical simulation and analysis

In order to do further research on the dynamics of system (12), it is convenient to take some parameters' values as follows:
$a = 3.5; b = 0.5; a_1 = 3; b_1 = 0.5; a_2 = 2.5; b_2 = 0.5; c_1 = 0.1; c_2 = 0.04; b_{12} = 1.05; b_{22} = 1.05.$

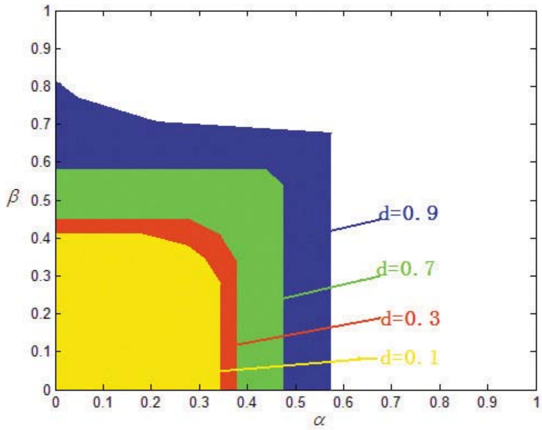

Figure 3: The stability with the change of d where $d = 0.6$.

4.1 The stable region

In this Section, some experiments will be done to investigate the effect of r and d on the stability of the model.

Experiment 1. We get the stable region of Nash equilibrium point in the phase plane of price adjustment speed parameters with different degree of complementarity between the two products where $d = 0.3$:

(1) $r = 0.1$ (The cyan area),
(2) $r = 0.6$ (The mauve areas),
(3) $r = 0.9$ (The green area).

From Figure 2, we can find that with the increase of the degree of complementarity between the two products, the stable region increases obviously, and the sensitivity of the market on price adjustment parameters decreases. A larger stable region implies better market stability, so larger degree of complementarity between the two products can lead to better stability in the market of complementary products.

Experiment 2. Figure 3 shows the stable regions with different discount rate of bundling where $r = 0.6$:

(1) $d = 0.1$ (The yellow area),
(2) $d = 0.3$ (The red area),
(3) $d = 0.7$ (The green area),
(4) $d = 0.9$ (The blue area).

We can see that with the increase of the discount rate of bundling, the stable region becomes larger. So, with fixed degree of complementarity, larger discount rate of bundling can make the market more stable, and can decrease the sensitivity of the market to price adjustment parameters.

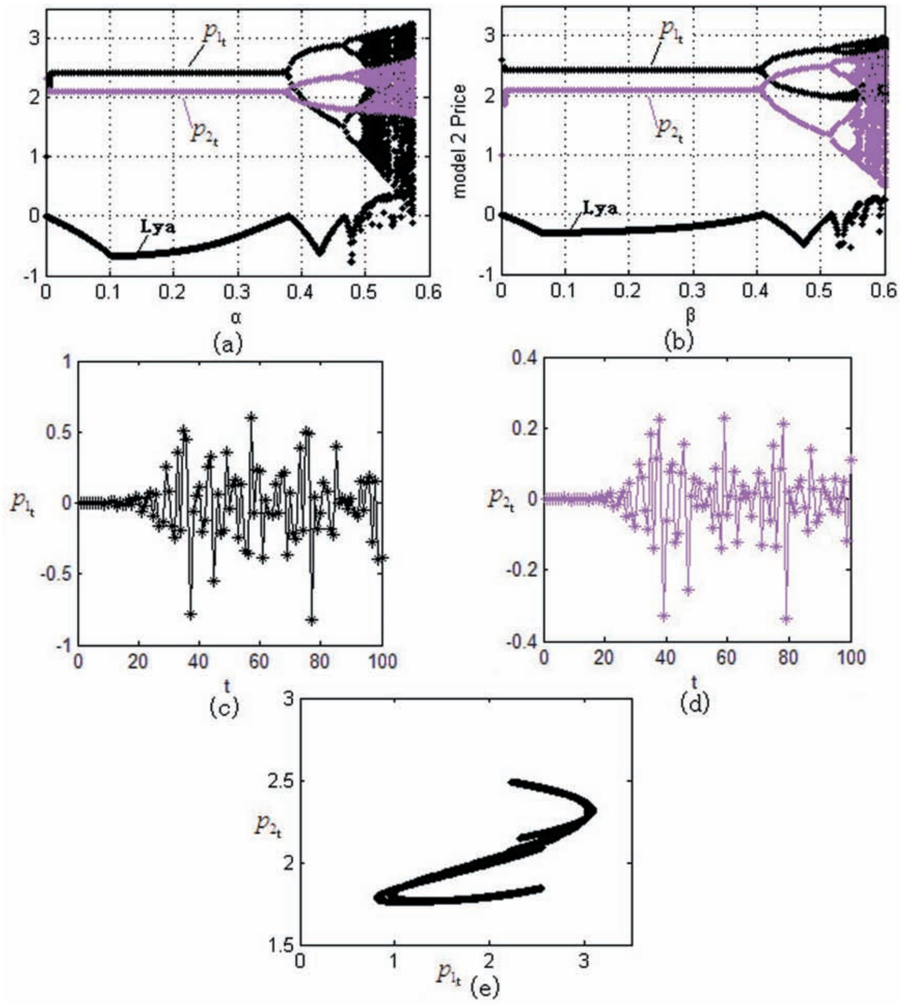

Figure 4: The complexity aroused by price adjustment parameters. (a): the bifurcation with $\beta = 0.35, u = 0.7, r = 0.6$.(b): the bifurcation with $\alpha = 0.35, u = 0.7, r = 0.6$. (c)(d): the initial condition sensitiveness with $\alpha = 0.47, \beta = 0.47, u = 0.7, r = 0.6$. (e) : the chaotic attractor with $\alpha = 0.52, \beta = 0.35, u = 0.7, r = 0.6$.

4.2 The effects of price adjustment parameters

Figure 4 (a) shows the prices bifurcation and the largest Lyapunov exponent where $\beta = 0.35, u = 0.7, r = 0.6$. We can see that with the increase of the price adjustment parameter α, the first bifurcation occurs at $\alpha = 0.37$, the second at $\alpha = 0.46$, and the third at $\alpha = 0.49$, ...chaos occurs. Thus, the greater the speed of price adjustment, the faster the price decision making appear to the change, and the more likely the chaos occurs in the market of complementary products. The similar situation occurs with changing β as shown in Figure 4(b).

The initial condition sensitiveness is proved by Figure 4 (c) and (d). The butterfly effect, namely initial condition sensitiveness is one of the important characteristics of chaos. A slight variation of initial condition will result in great difference of the ultimate system. The two initial values are (1, 1) and (1+Δx, 1), $\Delta x = 0.001$. For the original system (12), at the beginning the time series are near to zero, but with the increase of game times, the difference becomes increasingly evident and unpredictable as shown in Figure 4(c) and (d). Figure 4 (e) is the chaos attractor. So, economic chaos means a small change in the initial condition, which causes a chain of events leading to completely different economic outcomes, which is random and unpredictable. Thereby, in chaotic market, two firms face great risk, and it is difficult to give an accurate assess of possible circumstances. The risk of price fluctuation is a major risk in our economic model.

4.3 The effects of discount rate of bundling

4.3.1 The bifurcation with changing discount rate

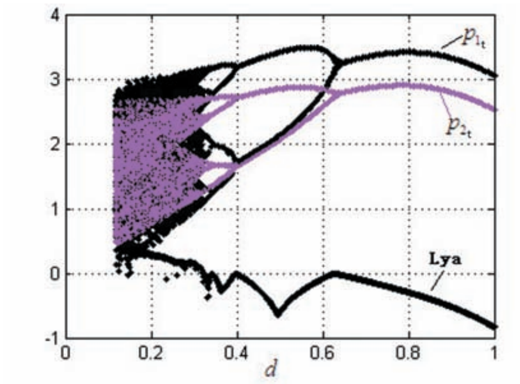

Figure 5: The bifurcation and the largest Lyapunov exponent with changing discount rate where $\alpha = 0.47, \beta = 0.47, r = 0.6$.

Figure 5 is the price bifurcation with changing discount rate of bundling and the largest Lyapunov exponent. When $d \in (0.65, 1]$, the system is in the state of equilibrium, and has positive largest Lyapunov exponent, we also can find the equilibrium prices are variational with the change of d; With decreasing d, the system enters into doubling period bifurcation; then when $d < 0.32$, the system enters into chaos, and the largest Lyapunov exponent is greater than zero.

We can gain two conclusions:

(1) Suitable discount rate can make the model have a higher chance of reaching stable state.

(2) The change of discount rate can cause the change of equilibrium prices.

4.3.2 The average producer surplus

How does the change of equilibrium prices which is caused by changing discount rate influence the producer surplus? It will be investigated in detail by numerical simulations.

The key point of the research on average producer surplus (APS) is how to deal with firms' profits in chaos. The average profit of chaotic trajectory [24] can be got by formula (23):

$$\Pi_i = (1/T) \sum_{t=0}^{T-1} \Pi_i(x_t, y_t) \qquad (23)$$
$$i = X, Y. \ Take\, T = 3 \times 10^3.$$

Producer surplus is the amount that producers benefit by selling at a market price that is higher than the least that they would be willing to sell for. In this paper, the average producer surplus (APS) with the change of discount rate can be obtained as Figure 6 shows:

Figure 6 (a)-(f) show the bifurcation diagrams of system (12) and average producer surplus with changing discount rate where price adjustment parameters are given different values. Table 1 gives the data of the maximal average producer surplus when they occurred in the stable state, period-doubling bifurcation and chaos, respectively.

Form these bifurcation diagrams and data, we can draw some conclusions:

(1) Changing discount rate of bundling can lead to the change of equilibrium prices, thereby affect the average producer surplus in the market of complementary products.

(2) When the discount rate reaches certain value, the average producer surplus can reach the maximum with fixed price adjustment parameters, which is called maximal average producer surplus (MAPS) in this paper.

(3) From Figure 6(a)(b)(c), when $\alpha = \beta = 0.3$, $\alpha = \beta = 0.35$ and $\alpha = \beta = 0.45$, respectively, the average producer surplus reach the maximum 10.78 with $d = 0.625$ .The three cases share one thing in common: The maximal average producer surplus (MAPS) occurs in stable state of the model; From Figure 6(d), when $\alpha = \beta = 0.55$, the maximal average producer surplus (MAPS) is 10.25 with $d = 0.71$. This MAPS occurs in period doubling bifurcation, and is lesser than in stable state; From Figure 6 (e)(f), when $\alpha = \beta = 0.65$ and $\alpha = \beta = 0.7$, the MAPS is 9.883 and 9.826, respectively, and declined once again. They all occur in chaotic state of the model. Thereby, we can find: The MAPS which occurs in stable state of the system is larger than the MAPS which occurs in unstable state of the system (including period doubling bifurcation and chaos).

Now, a new problem comes to draw our attention: From Figure 6 (a)(b)(c), the similar phenomena happen: their MAPS equal to 10.78 when $d = 0.625$. Is there only one point $(d, MAPS) = (0.625, 10.78)$ in this example of numerical simulation?

4.3.3 How to get higher MAP?

For further research, more comprehensive numerical simulations will be carried out: The scope of the test parameters is $(\alpha, \beta) \in \{(\alpha, \beta) \mid \alpha \in (0, 1), \beta \in (0, 1)\}$. In Figure 7 (a), the red zone represents the $(d, MAPS)$ which occurs in stable state of the system, the yellow point for $(d, MAPS)$ which occurs in period-doubling bifurcation, and the green point for

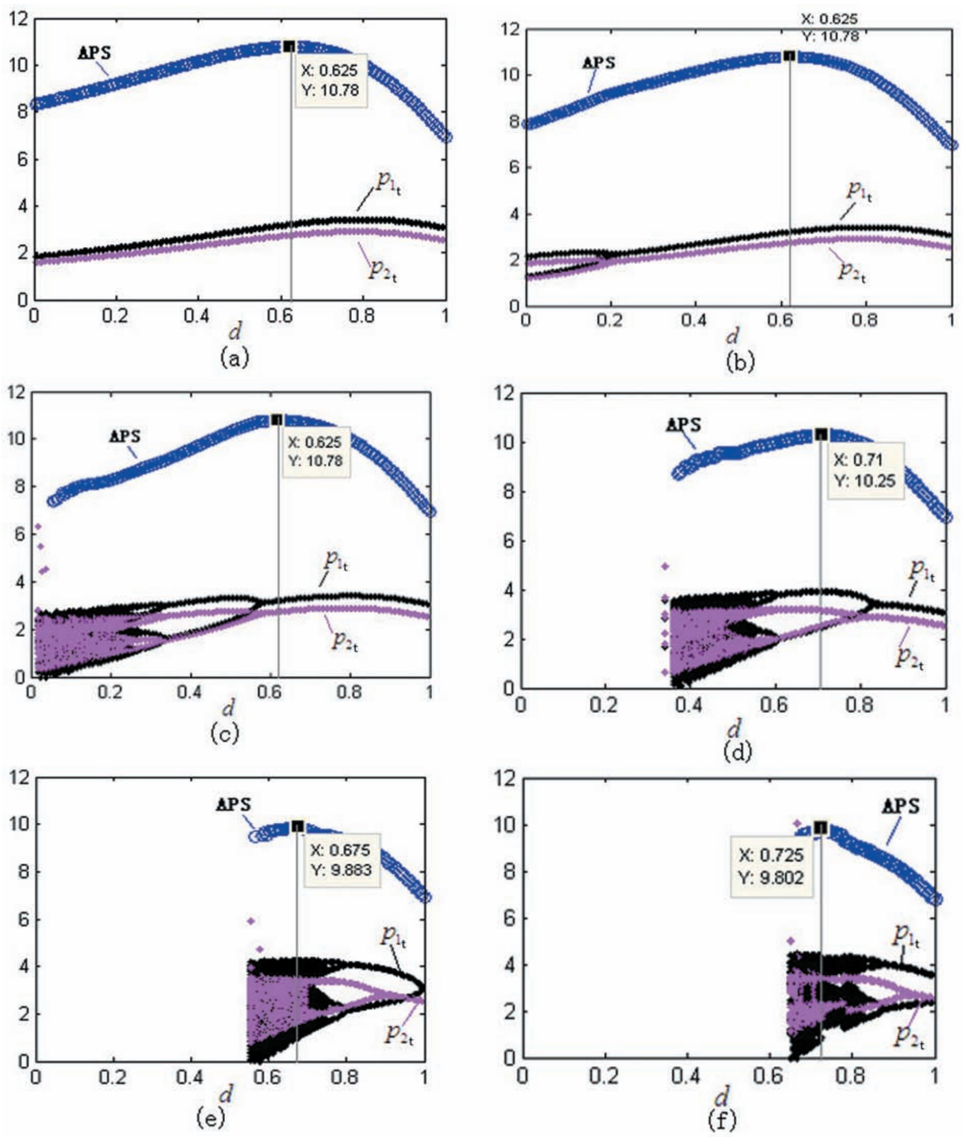

Figure 6: The bifurcation and the average producer surplus (APS) with changing discount rate. (a)$\alpha = \beta = 0.3$; (b)$\alpha = \beta = 0.35$; (c)$\alpha = \beta = 0.45$;(d)$\alpha = \beta = 0.55$;(e)$\alpha = \beta = 0.65$; (f)$\alpha = \beta = 0.7$

Table 1: The maximal average producer surplus (MAPS) happened in different state of the model.

α and β	$\alpha = 0.3$ $\beta = 0.3$	$\alpha = 0.35$ $\beta = 0.35$	$\alpha = 0.45$ $\beta = 0.45$	$\alpha = 0.55$ $\beta = 0.55$	$\alpha = 0.65$ $\beta = 0.65$	$\alpha = 0.7$ $\beta = 0.7$
d	0.625	0.625	0.625	0.71	0.675	0.725
State of system	Stable	Stable	Stable	Period-doubling bifurcation	Chaos	Chaos
Maximal average producer surplus=$\overline{\Pi_1} + \overline{\Pi_2}$	10.78	10.78	10.78	10.25	9.883	9.802

$(d, MAPS)$ which occurs in chaos. Figure 7 (b) is the red part of Figure 7(a) magnified 50 times.

We can conclude:

(1) From Figure 7 (a), the red area is always in the above of the yellow and green areas with fixed d. This means that MAPS which occurs in stable state is

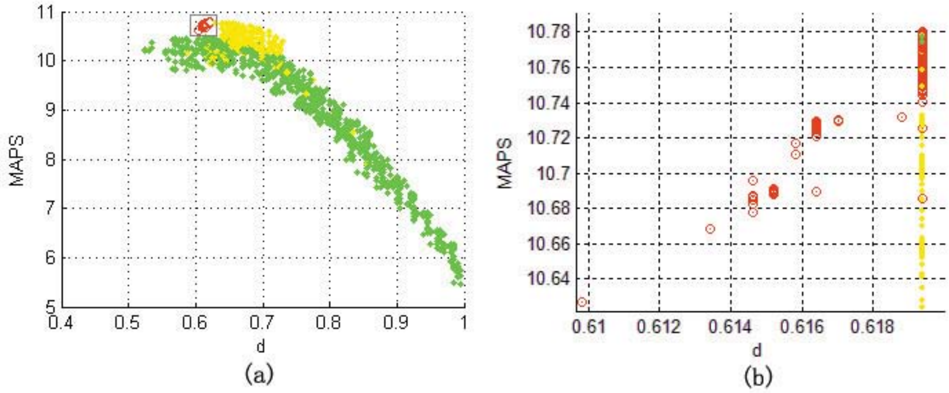

Figure 7: (a): The maximal average producer surplus where $\alpha \in (0, 1), \beta \in (0, 1)$. (b): The part of diagram (a) is magnified 50 times.

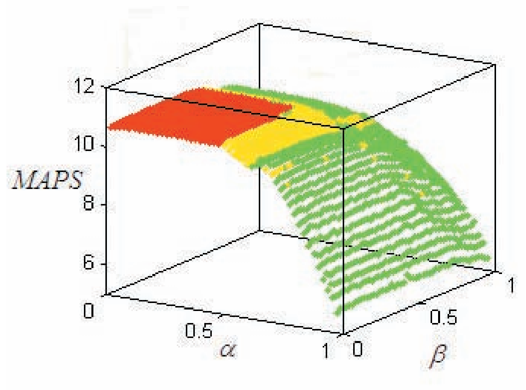

Figure 8: The maximal average producer surplus (MAPS) with changing (α, β).

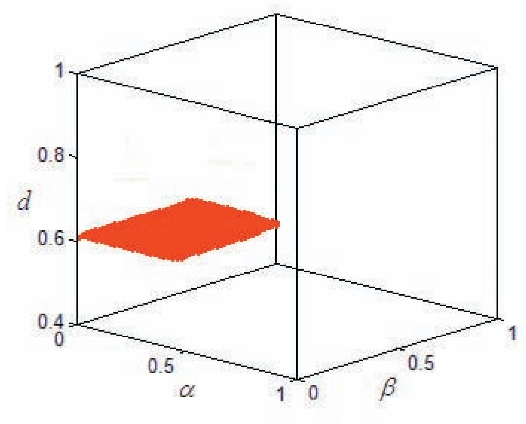

Figure 9: The discount rate with which producers can get MAPS in stable state of the system.

higher than the MAPS which occurs in unstable state with fixed discount rate.

(2) From Figure 7 (b), in the red space $MAPS \in [10.631, 10.78]$ and $d \in [0.61, 0.628]$. On the whole, the range of $(d, MAPS)$ is very small. The red area in the small range shows the characters of concentrative and dense point.

Through above experiments and analysis, we find that the stability of the model has important significance to maximizing producer surplus, and the joint research on price adjustment parameter and discount parameter is necessary in the market of complementary products with the strategy of mixed bundling pricing.

In Figure 8, red area implies that we get the MAPS in stable state of the system, yellow area implies that we get the MAPS in periodic bifurcation, and green area implies the MAPS happens in chaotic state from the view of three-dimensional visual. In red area, firms can get higher producer surplus; when the system enters into period doubling bifurcation and chaos, the producer surplus gradually decreases. Thus, it is more likely to obtain higher producer surplus in the stable market with suitable discount of bundling.

The discount rate of bundling has an important influence upon the stability and the producer surplus in the market of complementary products. Selecting suitable discount rate can help firms obtain higher producer surplus. Figure 9 describes the price adjustment parameters of both firms and the discount rate when MAPS is got in the stable state from the three-dimensional visual. Using the parameters of the red area are given, firms can have the chance to pursue higher producer surplus.

5 Conclusions

In this paper, we propose a duopoly game model in the market of complementary products with mixed bundling pricing. We analyze the effects of the price adjustment parameters on the system stability and the producer surplus, especially when the parameters are used with the degree of complementarity and the discount rate of bundling. We can get some useful results: Firstly, this complementary model which is non-competitive also contains abundant complex dynamical phenomena. Secondly, enlarging the degree of complementarity and the discount rate of bundling can increase the stability of the market. Thirdly, the discount rate of bundling can change the equilibrium prices, thereby, change the producer surplus. Finally, stable market and suitable discount rate of bundling are beneficial to obtain higher producer surplus.

Our study contributions are both theoretical and practical. Companies of related market can use the insights from our research to improve the market decisions and thus improve producer surplus. The complementary product models under more players and multi-products need to be considered, and they will be significant topics for further research.

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