Finite groups with three or four conjugacy class sizes of primary and biprimary elements

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Abstract: Let G be a finite group. In this paper, we determine completely the structure of G with three or four conjugacy class sizes of elements of orders divisible by at most two primes.

Key–Words: Conjugacy class sizes; Finite groups; Nilpotent groups; Solvable groups; Sylow p-subgroup.

1 Introduction

Throughout this paper all groups are finite. We will denote by x^G the conjugacy class of x in G and (following Baer [1]) we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G(in some other papers, $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$ is called conjugacy class size or length of x in G, for example, [2],[3]). We will often refer to the index of an element, this is just the size of the conjugacy class containing the element. The benefit of this definition is entirely linguistic. We say that a group element has primary or biprimary order respectively if its order is divisible by at most one or two primes. The rest of our notation and terminology are standard. The reader may refer to ref.[4].

It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist many results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes. For instance, in [5], N. Itô shows that if the sizes of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. In [6], he shows that if the conjugacy class sizes of G are $\{1, n, m\}$, then G is solvable. Later, A.Beltrán and M.J.Felipe in [7] and [8] show that if the conjugacy class sizes of G are exactly $\{1, m, n, mn\}$ with $(m, n) = 1$, then G is nilpotent. Also in [9], they prove that if the conjugacy class sizes of G are $\{1, m, mn\}$, where $m, n > 1$ are coprime, then $m = p$ for some prime p dividing n-1 and G has an abelian normal p-complement, also if P is a Sylow p-subgroup of G, then $|P'| = p$ and $|P/Z(G)_p| = p^2$. Further, they obtain other properties and determine completely the structure of G.

Recently, we replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to determine completely the structure of G. We put our emphasis on conjugacy class sizes of all elements of primary or biprimary orders of G to analyze a new case of groups having four conjugacy class sizes of primary and biprimary orders of G. For example, Kong and Guo in [10] prove that let G be a group and assume that the conjugacy classes sizes of primary and biprimary orders of G are exactly $\{1, p^a, n, p^an\}$ with $(p, n) = 1$, where p is a prime and a and n are positive integers. If there is a p-element in G whose index is precisely p^a, then G is nilpotent and $n = q^b$ for some prime $q \neq p$. Also in [11], Kong proves that if the set of conjugacy class sizes of all elements of primary and biprimary orders of G is exactly $\{1, p^a, q^b, p^aq^b\}$, where p and q are two distinct primes and a and b are positive integers, then G is nilpotent.

In this paper, we go on studying the structure of a group under some arithmetical conditions on its conjugacy class sizes and will replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to determine completely the structure of G. That is, we use conjugacy class sizes of all elements of primary or biprimary orders of G to analyze the structure of groups having three or four conjugacy class sizes of primary and biprimary orders of G and generalize Theorem A in [9] and Theorem 3.1 in [10]. The first one of our results is the following: Let G be a finite p-solvable group and let G^* be the set of elements of primary and biprimary orders of

G. Suppose that the conjugacy class sizes of G^* are $\{1, m, mn\}$ with $(m, n) = 1$, then G is an F-group, $m = p$ for some prime p and G contains an abelian normal subgroup $M = H \times P_0$ of index p, where P_0 is a Sylow p-subgroup of M, and neither H nor P_0 is central in G. Furthermore, M is the set of all elements of G of index 1 or p, and if P is a Sylow p-subgroup of G then P/P_0 acts fixed-point-freely on $H/Z(G)_{p'}$ and $n = |H/Z(G)_{p'}|$. Also $|P'| = p$ and $|P/Z(G)_p| = p^2$. The second one is: Let G be a group and assume that the conjugacy class sizes of all elements of primary orders and $\{p, q\}$-elements of G are exactly $\{1, p^a, n, p^a n\}$ with $(p, n) = 1$, where p is a fixed prime, q is an arbitrary prime, a and n are positive integers. If there is a p-element in G whose index is precisely p^a, then G is nilpotent and $n = q^b$ for some prime $q \neq p$.

2 Basic definitions and preliminary results

In this section, we state the necessary results for the proof of our main theorem.

Lemma 1 *([11, Lemma 5]) Let G be a group. A prime p does not divide any conjugacy class length of any element of prime power order of G if and only if G has a central Sylow p-subgroup.*

Lemma 2 *([11, Theorem 14])Let G be a group such that p^a is the highest power of the prime p which divides the index of any element of primary and biprimary orders of G. Assume that there is a p-element in G whose index is precisely p^a. Then G has a normal p-complement.*

Lemma 3 *([12, Lemma 1]) Let G be a group and let x be an element of G whose index is p^a where p is a prime and a is a natural number. Then $[x^G, x^G] \subseteq O_p(G)$.*

This has an immediate consequence.

Lemma 4 *Let G a group and let x be an element of G of index p^a. Then $x \in O_{p,p'}(G)$.*

Lemma 5 *([13, Corollary B])Let G be a finite group and suppose that the class size of every element of prime power order of G is 1 or m. Then G is nilpotent. More precisely, $m = p^n$ for some prime p, and $G = P \times A$ with A abelian and P a p-group.*

Lemma 6 *([14, Corollary 2.2]) Let P be a p-group whose class sizes are $\{1, p^a\}$. Then P/Z(P) has exponent p.*

Lemma 7 *([15, Theorem 5]) Suppose that G is a group and let p be a prime such that every conjugacy class size of any $p'-$element of prime power order of G is a p'-number. Then $G = P \times H$ where P is a Sylow p-subgroup and H is a p-complement of G.*

Lemma 8 *([11, Lemma 8 (a)]) Suppose that G is a group and p a prime. Then all conjugacy class sizes of any $p'-$element of prime power order of G are powers of p if and only if G has an abelian p-complement.*

Lemma 9 *([16, Theorem A]) Suppose that G is a finite p-solvable group and that 1 and m are the conjugacy class sizes of p'-elements of prime power order. Then $m = p^a q^b$, with q a prime distinct from p and a, $b \geq 0$. If $b = 0$, then G has abelian p-complement. If $b \neq 0$, then $G = PQ \times A$, with P and Q a Sylow $p-$subgroup and a Sylow $q-$subgroup of G, respectively, and $A \leq Z(G)$. Furthermore, if $a = 0$, then $G = P \times Q \times A$.*

We need to introduce for an arbitrary set of primes π some new properties generalizing the ones given by Itô in [6] for ordinary conjugacy classes. We will say that G has the property F_π, or that it is an F_π-group, if every non-central $x \in G_\pi$ satisfies

(i) if $C_G(x) \subseteq C_G(a)$ for some $a \in G_\pi$, then $a \in Z(G)$ or $C_G(x) = C_G(a)$, and

(ii) if $C_G(a) \subseteq C_G(x)$ for some $a \in G_\pi$, then $C_G(x) = C_G(a)$.

This means that the centralizer of each non-central p-element is maximal and minimal among the centralizers of all non-central p-elements.

On the other hand, we will say that G has the property A_π if for all non-central $x \in G_\pi$ the centralizer factorizes as $C_G(x) = C_G(x)_\pi \times C_G(x)_{\pi'}$, with $C_G(x)_\pi$ an abelian p-subgroup and $C_G(x)_{\pi'}$ a π'-subgroup. It is easy to see that every group having the property A_π is an F_π-group. When p is the set of all primes, an F_π-group is trivially an F-group and if G has the property A_π we will say that G has the property A. The following theorem is one of the key results used in the proof of our main theorem and it extends Lemmas 5 and 9.

Lemma 10 *Let G be a group and π a set of primes. Suppose that G^* satisfies the property A_π and suppose that $|x^G|_\pi = m$ for any non-central $x \in G_\pi$, where $m > 1$ is a fixed number. Suppose further that the centralizers of non-central π-elements are not all conjugate. Then $m = p^a$ for some prime $p \in \pi$ and $P/Z(G)_p$ has exponent p for any Sylow p-subgroup P of G.*

Proof The proof is based on the one which we have cited for Lemma 5. We proceed in several claims.

Claim 1. *Let x and y be two non-central π-elements of primary orders. If $C_G(x) \neq C_G(y)$, then $(C_G(x) \bigcap C_G(y))_\pi = Z(G)_\pi$.*

Suppose that there exists a non-central element $a \in (C_G(x) \bigcap C_G(y))_\pi$ of primary order. Since G satisfies A_π, we have $C_G(x) \subseteq C_G(a)$ and $C_G(y) \subseteq C_G(a)$. Now, as G has the property A_π, it also has the property F_π, and since $C_G(a) \neq G$, we conclude $C_G(x) = C_G(a) = C_G(y)$, a contradiction.

In the following steps, we set $\overline{G} = G/Z(G)_\pi$ and use bars to work in the factor group.

Claim 2. *Let $\overline{x}$, $\overline{y} \neq 1$ be two π-elements of primary orders in G such that $\overline{xy} = \overline{yx}$ and $C_G(x) \neq C_G(y)$, Then $o(\overline{x}) = o(\overline{y})$ is a prime.*

Notice that x and y are π-elements of primary orders. Moreover, since x and y commute, then $\overline{xy} = \overline{yx}$ is a π-element and consequently, so is xy. Suppose first that $o(\overline{x}) < o(\overline{y})$, then $(\overline{xy})^{o(\overline{x})} = \overline{y}^{o(\overline{x})} \neq 1$. Furthermore,

$$1 \neq (\overline{xy})^{o(\overline{x})} = \overline{xy}^{o(\overline{x})} \in \overline{C_G(xy)} \cap \overline{C_G(y)}.$$

By applying Step 1, we deduce that $C_G(y) = C_G(xy)$, so in particular $x \in C_G(y)$. As G satisfies A_π then $C_G(x) \subseteq C_G(y)$, and since y is not central and G is an F_π-group we have equality, contradicting the hypothesis of this step. Therefore, $o(\overline{x}) = o(\overline{y})$.

On the other hand, if s is a prime divisor of $o(\overline{x})$ and $\overline{x}^s \neq 1$, then we have $C_G(x) \subseteq C_G(x^s) < G$, whence we obtain $C_G(x) = C_G(x^s)$. Moreover, $\overline{x}^s\overline{y} = \overline{yx}^s$. By the above paragraph it follows that $o(\overline{x}^s) = o(\overline{y}) = o(\overline{x})$, a contradiction.

Claim 3. *Let g be a non-central element of primary order in G_π and consider the conjugacy class of $\overline{g}$ in $\overline{G}$, $\overline{g}^{\overline{G}}$. Then there exists some non-central $x \in G_\pi$ of primary orders such that $\overline{g}^{\overline{G}} \cap \overline{C_G(x)} = \emptyset$.*

Suppose that this is false. Then for every non-central $x \in G_\pi$ of primary orders we have that $\overline{C_G(x)}$ must contain some conjugate of $\overline{g}$, say $\overline{g}^{\overline{n}}$ for some $\overline{n} \in \overline{G}$. Thus, $\overline{g^n} = \overline{g}^{\overline{n}} \in \overline{C_G(x)}$ and consequently $g^n \in C_G(x)_\pi$. As G satisfies $A\overline{\pi}$ we deduce that $C_G(x) \subseteq C_G(g^n)$ and hence equality holds because G is an F_π-group. It follows that the centralizers of any two non-central π-elements of primary orders of G are conjugate in G, contradicting the hypotheses of the theorem.

Claim 4. *The order of every non-trivial π-element in $\overline{G}$ is a prime.*

Suppose that $o(\overline{g})$ is composite for some π-element $\overline{g}$. Notice that g is a π-element of primary orders too. By Step 3, there exists a non-central element $x \in G_\pi$ of primary orders such that $\overline{g}^{\overline{G}} \cap \overline{C_G(x)} = \emptyset$. Write $\overline{C_\pi} = \overline{C_G(x)_\pi}$ and observe that $\overline{C_\pi}$ operates on $\overline{g}^{\overline{G}}$ by conjugation. Furthermore, by Step 2 no element in $\overline{C_\pi}$ distinct from 1 centralizes any element in $\overline{g}^{\overline{G}}$, and hence all orbits of $\overline{C_\pi}$ on $\overline{g}^{\overline{G}}$ have the same size, $|\overline{C_\pi}|$, which implies that $|\overline{C_\pi}|$ divides $|\overline{g}^{\overline{G}}|$.

On the other hand, again by applying Step 2, we deduce that $\overline{C_G(g)_\pi}$ operates without fixed points on $\overline{g}^{\overline{G}} - \overline{g}^{\overline{G}} \cap \overline{C_G(g)}$. As a result, $|\overline{C_G(g)_\pi}|$ divides $|\overline{g}^{\overline{G}}| - |\overline{g}^{\overline{G}} \cap \overline{C_G(g)}|$. As $|\overline{C_G(g)_\pi}| = |\overline{C_\pi}|$, we conclude that $|\overline{C_G(g)_\pi}|$ also divides $|\overline{g}^{\overline{G}} \cap \overline{C_G(g)}|$, which is a contradiction because

$$0 < |\overline{g}^{\overline{G}} \cap \overline{C_G(g)}| < |\overline{C_G(g)_\pi}|.$$

Claim 5. *Conclusion.*

As the subgroups $C_G(x)_\pi$ for non-central $x \in G_\pi$ of primary orders are abelian and have the same order, each $|\overline{C_G(x)_\pi}|$ is a power of some prime $p \in \pi$ by Step 4. Hence $\overline{G}$ is a $\{\pi' \bigcup \{p\}\}$-group and thus $m = p^a$.

Moreover, by Step 4, if $P \in \mathrm{Syl}_p$(G) then every element of $\overline{P}$ has prime order, and thus $\overline{P} \cong P/Z(G)_p$ has exponent p.

Finally, we will make use of two classical results on automorphism groups. The first is Thompson′s $A \times B$ Lemma and the second is due to Isaacs and Passman. □

Lemma 11 *([17, 24.2]) Let AB be a finite group represented as a group of automorphisms of a p-group G with $[A, B] = 1 = [A, C_G(B)]$, B a p-group and $A = O^p(A)$. Then $[A, G] = 1$.*

We recall that a permutation representation is half-transitive if all orbits have the same size.

Lemma 12 *([18, Theorem 1]) Let A be a group of automorphisms of G which acts half-transitively as a permutation group on G-$\{1\}$. If $|A| > 1$, then either A acts fixed-point-freely on G or G is elementary abelian q-group for some prime q and A acts irreducibly.*

Lemma 13 *(Wielandt′s theorem [1, Lemma 6]) $O_p(G)$ contains every element in G whose order and index are powers of p.*

Lemma 14 *([3, Lemma 1.1]) Let $N \trianglelefteq G, x \in N$, and $y \in G$. Then*

(i) $|x^N| \mid |x^G|$.

(ii) $|(yN)^{G/N}| \mid |y^G|$.

3 Main results

Theorem 15 *Let G be a finite p-solvable group and let G^* be the set of elements of primary and biprimary orders of G. Suppose that the conjugacy class sizes of G^* are $\{1, m, mn\}$ with $(m, n) = 1$, then G is an F-group, $m = p$ for some prime p and G contains an abelian normal subgroup $M = H \times P_0$ of index p, where P_0 is a Sylow p-subgroup of M, and neither H nor P_0 is central in G. Furthermore, M is the set of all elements of G of index 1 or p, and if P is a Sylow p-subgroup of G then P/P_0 acts fixed-point-freely on $H/Z(G)_{p'}$ and $n = |H/Z(G)_{p'}|$. Also $|P'| = p$ and $|P/Z(G)_p| = p^2$.*

Proof: We denote by π the set of primes dividing m and π' the set of primes dividing n. By Lemma 1, we can certainly assume that $\pi(G) = \pi \bigcup \pi'$. The proof splits into two cases, depending on whether there are π-elements of index m in G or not. The first case provides the structure described in the theorem and the second will lead to a contradiction. We will use several steps to finish the proof.

Case 1. We can assume that G has π-elements of index m. Suppose that x is such an element and observe that the maximality of $C_G(x)$ and the primary decomposition of x allow us to assume that x is a p-element for some $p \in \pi$. Now, if y is a p'-element of primary order of $C_G(x)$, then $C_G(xy) = C_G(x) \bigcap C_G(y) \subseteq C_G(x)$ and thus the hypotheses on class sizes imply that y may have index 1 or n in $C_G(x)$. Since n is a p'-number, by Lemma 7 we can write $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$. We will distinguish the cases when $C_G(x)_{p'}$ is abelian and when it is not. We will see first that the second case is not possible.

Case 1.1. Assume that $C_G(x)_{p'}$ is not abelian, which means that the class sizes of p'-elements of primary order in $C_G(x)$ are exactly $\{1, n\}$. As $C_G(x)$ is a p-solvable group, we may apply Lemma 9 to obtain that $n = p^a r^b$ for some prime $r \in \pi$. But since p does not divide n, we get $n = r^b$ and

$$C_G(x) = P_x \times R_x \times A_x,$$

where P_x and R_x are Sylow p and r-subgroups of $C_G(x)$ and A_x is abelian. Note that in fact R_x is a Sylow r-subgroup of G. We distinguish two cases and prove that both lead to a contradiction.

Case 1.1.a. Suppose that there are no r-elements of index m. Since a Sylow r-subgroup of G cannot be central in G, there must exist r-elements of index mn. Consider an element $w \in G^*$ of index mn and its decomposition $w = w_r w_{r'}$, where w_r and $w_{r'}$ are elements of primary orders. If w_r is central in G, then $C_G(w) = C_G(w_{r'})$ and it follows that every r-element of $C_G(w)$ must be central in $C_G(w)$ by its minimality. Therefore, we can write $C_G(w) = R_w \times T_w$, with R_w an abelian Sylow r-subgroup of $C_G(w)$. Moreover, R_w cannot be central in G, otherwise $R_w = Z(G)_r$, so $|G : Z(G)|_r = n$ and this certainly contradicts the existence of r-elements of index mn. Consequently, we can take some non-central $b \in R_w$, so $C_G(w) \subseteq C_G(b)$ and as no r-element has index m, we get $C_G(w) = C_G(b)$. If w_r is not central in G, then clearly $C_G(w_r) = C_G(w)$. Therefore, in any case we have $C_G(w) = C_G(b)$ for some b in some Sylow r-subgroup R_w of $C_G(w)$. Notice also that $R_w \subseteq R_x^g$ for some $g \in G$. Then $b \in R_x^g$ and as $C_G(x^g) = P_x^g \times A_x^g \times R_x^g$, we deduce that $P_x^g \times A_x^g \subseteq C_G(b)$, and this is a Hall r'-subgroup of $C_G(b)$. On the other hand, any r'-element of primary order of $C_G(b)$ is central in $C_G(b)$ by its minimality, so $C_G(w) = C_G(b) = R_w \times P_x^g \times A_x^g$. So we have shown that $w_r \in R_x^g$ and that $w_{r'} \in P_x^g \times A_x^g \subseteq C_G(R_x^g)$. Then for any $w \in G^*$ of index mn we conclude that $w \in R_x^g C_G(R_x^g)$ for some $g \in G$. Finally, if $w \in G^*$ has index m, then $C_G(w)$ contains some conjugate of R_x, say R_x^g for some $g \in G$, so $w \in C_G(R_x^g)$. We conclude that

$$G = \bigcup_{g \in G} R_x^g C_G(R_x^g),$$

and as a result, $G = R_x C_G(R_x)$ that is, R_x is a direct factor of G. But this cannot happen since the class sizes of G^* do not allow this situation.

Case 1.1.b. There are r-elements of index m. Let us fix some r-element y of index m, which up to conjugacy can be assumed to centralize R_x, so $y \in Z(R_x)$ and thus $C_G(x) \subseteq C_G(y)$. As these subgroups have the same order then $C_G(x) \subseteq C_G(y)$, whence every r'-element of primary order of $C_G(x)$ must have index 1 or r^b in $C_G(x)$. Lemma 8 asserts that the r-complement of $C_G(x)$, that is, $P_x \times A_x$, is abelian. Now we observe that there must exist r'-elements of index mn since if every r'-element of primary order of G has index 1 or m, then Lemma 7 implies that the Sylow r-subgroup of G is a direct factor of G, which is a contradiction. Therefore, we may take an r'-element w of index mn and assert that every r-element in $C_G(w)$ is central by the minimality of $C_G(w)$, so we write $C_G(w) = R_w \times T_w$ with R_w an abelian Sylow r-subgroup of $C_G(w)$. We distinguish two cases: R_w central or non-central in G. We prove that both provide a contradiction.

Suppose first that $R_w \nsubseteq Z(G)$ and take some non-central $b \in R_w$. It is clear that $C_G(w) \subseteq C_G(b)$.

Assume that b has index m. Then $C_G(b)$ must contain some Sylow r-subgroup of G, say R_x^g for some $g \in G$. So in particular $b \in Z(R_x^g)$ and thus

$$C_G(x^g) = (P_x \times A_x \times R_x)^g \subseteq C_G(b).$$

Since they have the same order these subgroups are equal. Hence $(P_x \times A_x)^g$ is the only Hall r'-subgroup of $C_G(b)$, so it coincides with T_w. As $w \in (P_x \times A_x)^g$ then $R_x^g \subseteq C_G(w)$ which is a contradiction. Thus, any non-central element b of R_w has index mn and accordingly, $C_G(w) = C_G(b)$. From this we easily obtain that T_w is abelian and therefore $C_G(w)$ is abelian. But $R_w \subseteq R_x^g$ for some $g \in G$, and since $y \in Z(R_x)$, we get $y^g \in C_G(b) = C_G(w)$. This cannot happen as we have proved that there are no r-elements of index m in $C_G(w)$.

Suppose finally that $R_w \subseteq Z(G)$. This implies that $|G|r/|Z(G)|r = r^b$ and hence there are no r-elements of index mn in G, so all r-elements have index 1 or m. Now if we take $b \in R_x$ of index m then $b \in Z(R_x^g)$ for some $g \in G$. Hence

$$C_G(x^g) = P_x^g \times A_x^g \times R_x^g \subseteq C_G(b)$$

and these subgroups coincide because they have the same order. On the other hand, since $b \in R_x$ then $P_x \times A_x \times R_x^g \subseteq C_G(b)$, so $R_x^g \subseteq C_G(P_x) \subseteq C_G(x)$ and $R_x = R_x^g$. Consequently, $b \in Z(R_x)$ and R_x is abelian. But this shows that $C_G(x)$ is abelian, which contradicts the assumption of this case.

Case 1.2. Assume that $C_G(x)_{p'}$ is abelian. In this case, we can write

$$C_G(x) = P_x \times S_x \times H_x$$

where P_x is a p-subgroup, S_x is an abelian $(\pi - \{p\})$-subgroup and H_x is an abelian Hall π'-subgroup of G. We will prove that P_x, and hence $C_G(x)$ is abelian. Observe that Hall π'-subgroups exist and they are all conjugate in G by a well-known theorem of Wielandt. Also, notice that H_x cannot be central in G. So, if we take some non-central $b \in H_x$ of primary order, then we have $C_G(x) \subseteq C_G(b)$ and by maximality we get $C_G(x) = C_G(b)$. Now for any p-element $w \in P_x$ we have $C_G(wb) = C_G(w) \bigcap C_G(b) \subseteq C_G(b)$. Then the index of w in $C_G(b)$ may be 1 or n and necessarily must be 1 because $H_x \subseteq C_G(w)$. So P_x is central in $C_G(x)$ and hence $C_G(x)$ is abelian, as wanted.

We claim that the centralizers of all elements of index m are abelian. If $w \in G$ has index m, then there exists a Hall π'-subgroup, say H_x^g for some $g \in G$, such that $H_x^g \subseteq C_G(w)$ and if we choose some non-central $b \in H_x^g$, then

$$C_G(x^g) = P_x^g \times S_x^g \times H_x^g \subseteq C_G(b^g).$$

By maximality, $C_G(x^g) = C_G(b^g)$, and then $w \in CG(x^g)$. As this is abelian, we have $C_G(x^g) \subseteq C_G(w)$. Since these subgroups have the same order they are equal, and in particular $C_G(w)$ is abelian as claimed.

We prove now that G is an F-group. Suppose first that $w \in G$ has index m. Clearly $C_G(w)$ is maximal among the centralizers. On the other hand, if $C_G(g) \subseteq C_G(w)$ then equality also holds since $C_G(w)$ is abelian. Suppose then that w has index mn. It is obvious that $C_G(w)$ is minimal among the centralizers and if $C_G(w) \subseteq C_G(g)$ for some $g \in G$, then necessarily $C_G(w) = C_G(g)$. Otherwise g would have index m and by the above paragraph $C_G(g)$ would be abelian, which would imply that $C_G(g) \subseteq C_G(w)$ a contradiction.

We show now that m is a power of p. We assume that m is not a prime power and we will prove first that the centralizers of elements of index mn are abelian. First of all, notice that if g has index mn and write $g = g_\pi g_{\pi'}$, where g_π and $g_{\pi'}$ are elements of primary orders, then $C_G(g) \subseteq C_G(g_{\pi'})$. However, $g_{\pi'}$ has index 1 or m because the Hall π'-subgroups are abelian, so since G is an F-group $g_{\pi'}$ is central and g can be assumed to be a p-element. Furthermore, by using the primary decomposition, we can also assume g to be an s-element for some prime $s \in \pi$ and by the minimality of the centralizer we can write $C_G(g) = C_G(g)_s \times C_G(g)_{s'}$ with $C_G(g)_{s'}$ abelian. As m is not a prime power, let us take another prime $l \in \pi$ distinct from s. Observe that l must divide $|C_G(g)|$ because a Sylow l-subgroup cannot be central in G, and if t is a non-central l-element, then l divides $|C_G(t)|_l = |C_G(g)|_l$. Also, for such t we have $C_G(g) \subseteq C_G(t)$. If t has index m we know then that $C_G(t)$ is abelian and $C_G(g)$ is abelian too, as we wanted to prove. If t has index mn then $C_G(g) = C_G(t)$ and by arguing with t as with g, it follows that $C_G(g)$ is also abelian. In particular, we have shown that G has the property A. Moreover, the centralizers of non-central π-elements are clearly not all conjugate because of the existence of π-elements of index m and index mn. So we can apply Lemma 10 to get that m is a prime power, which is a contradiction.

Therefore, for the rest of this case we have $m = p^a$. As we have assumed the existence of p-elements of index p^a throughout Case 1, we may apply Lemma 2 to obtain that G has an (abelian) normal p-complement H. We are ready to show that G has the structure described in the statement of the theorem.

Let M be the set of elements in G whose index is 1 or p^a. Note that such elements are exactly those elements whose centralizer contains H, so $M = C_G(H)$, whence M is a normal subgroup of G. Also if we take some non-central $h \in H$, then $C_G(H) \subseteq C_G(h)$ and as $C_G(h)$ and H are abelian we deduce that $C_G(h) = C_G(H)$. As a consequence, M is abelian and we can write $M = H \times P_0$, with P_0 a p-subgroup (the set of p-elements in G of index p^a or 1), which is trivially normal in G. Also, P_0 is non-central in G by the assumption of Case 1.

Let P be a Sylow p-subgroup of G and consider the coprime action of P/P_0 on H defined by $h^{\overline{g}} = h^g$ for all $h \in H$ and all $g \in P$. As H is abelian, we can write $H = [H, P/P_0] \times C_H(P_0)$. Moreover, if $h \in C_H(P_0)$ then $h^g = h$ for all $g \in P$, so $h \in Z(G)$ and this shows that $C_H(P_0) = Z(G)_{p'}$. We assert that P/P_0 acts fixed-point-freely on $[H, P/P_0]$. To see this it is enough to notice that any $h \in [H, P/P_0] - \{1\}$ is non-central and we know that $C_G(h) = M = H \times P_0$ by the above paragraph, so h cannot be centralized by any element of $P - P_0$. Then, by [19, Theorem 16.12], P/P_0 must be cyclic or generalized quaternion. On the other hand, we prove that the class sizes of P are $\{1, p^a\}$. As $G = HP$ with H normal in G, it is easy to see that $C_G(g) = C_H(g)C_P(g)$ for each $g \in P$. This implies that

$$|G : C_G(g)| = |H : C_H(g)||P : C_P(g)|,$$

and this index may be 1, p^a or $p^a n$. This forces $|P : C_P(g)|$ to be 1 or p^a, as claimed. Then we can apply Lemma 6 and $P/Z(P)$ has exponent p. But note that the class sizes of G imply that $Z(P) = Z(G)_p \subseteq P_0$ and then, by the results obtained above, the only possibility for P/P_0 is to be cyclic of order p, and thus $a = 1$ and M has index p in G. Finally, observe that if $g \in P - P_0$ then

$$p^a n = |G : C_G(g)| = |H : C_H(g)||P : C_P(g)|,$$

so $n = |H : C_H(g)| = |H/Z(G)_{p'}|$.

Finally the structure stated in the theorem will be completely established when we prove that $|P'| = p$ and $|P/Z(G)_p| = p^2$. The first claim follows easily from the fact that the class sizes of P are $\{1, p\}$ (see [20], for instance). On the other hand, P_0 is an abelian normal subgroup of P of index p, so we have $P = P_0\langle y \rangle = P_0 C_G(y)$ for any $y \in P - P_0$. It follows that $C_{P_0}(y) = Z(P)$ and then

$$|P : Z(P)| = |P : P_0||P_0 : Z(P)| = p|P : C_P(y)| = p^2.$$

We have shown above that $Z(G)_p = Z(P)$ and thus G has all properties stated in the theorem.

Case 2. Suppose that every π-element of G has class size 1 or mn. We will prove that this case is impossible.

For the rest of the proof, let us fix a q-element x of index m for some prime $q \in \pi'$. By the existence of π-elements of index mn, we have $|C_G(x)|_\pi > |Z(G)|_\pi$, so we can choose then a π-element $g \in C_G(x)$ of index mn. The minimality of $C_G(g)$ yields that $C_G(g) = C_G(g)_\pi \times C_G(g)_{\pi'}$, where $C_G(g)_{\pi'}$ is abelian. Hence $x \in C_G(g)_{\pi'}$ and thus $C_G(g) \subseteq C_G(x)$. We will distinguish two subcases depending on whether n is a prime power or not.

Case 2.1. Suppose that $n = q^b$ and thus $\pi' = \{q\}$. We are going to prove first that $C_G(z)$ is abelian for any non-central $z \in G_\pi$ of primary order. For such z, the minimality of $C_G(z)$ implies that any q-element of $C_G(z)$ is central in $C_G(z)$, that is, we have $C_G(z) = C_G(z)_\pi \times C_G(z)_q$ with $C_G(z)_q$ abelian. Since $|C_G(z)|_q = |C_G(x)|_q > |Z(G)|_q$ we can choose some non-central $w \in C_G(z)_q$ of primary order and get that either $C_G(z)$ is equal to or is strictly contained in $C_G(w)$. In the first case, $C_G(z) = C_G(w)$ be abelian. In the second case, w is a q-element of index m and every q'-element of primary order of $C_G(w)$ has index 1 or q^b in $C_G(w)$, so that by Lemma 2, the q-complement of $C_G(w)$ is abelian, and consequently $C_G(z)$ is abelian too. In particular, G has property A_π and so has property F_π. We consider the following subcases in order to apply Lemma 10 in the second one.

Case 2.1.a. Suppose that the centralizers of non-central elements in G_π are all conjugate. We will prove that every element $w \in G$ lies in a conjugate of $C_G(V)$ where $V = C_G(g)$. This will imply that $V \subseteq Z(G)$, which is a contradiction because g is not central in G.

If w has index m, then as $|C_G(w)| > |Z(G)|_\pi$, there is some non-central π-element $z \in C_G(w)$ of primary order, so $C_G(z) \subseteq C_G(w)$. By hypothesis, $C_G(z) = C_G(g^h)$ with $h \in G$, whence $w \in C_G(V)^h$. Now, if w has index mq^b, again as $|C_G(w)|_\pi > |Z(G)|_\pi$, there exists some non-central π-element $t \in C_G(w)$ of primary order. Since $C_G(t)$ is abelian we have $C_G(t) \subseteq C_G(w)$ and by orders, $C_G(w) = C_G(t)$. However, we are assuming that $C_G(t) = C_G(g)^h$ for some $h \in G$, so w belongs to $C_G(V)^h$, as wanted.

Case 2.1.b. Suppose that the centralizers of non-central elements in G_π are not all conjugate. Since $|z^G|_\pi = m$ for all $z \in G_\pi - Z(G)$, we can apply Lemma 10 and obtain that $m = p^a$ for some prime p and that $P/Z(G)_p$ has exponent p for a Sylow p-subgroup P of G. In particular, G is a $\{p, q\}$-group.

Now we show that $O_p(G)$ is central in G. Assume first that w is a q-element of index $m = p^a$. By the assumption of Case 2, there exists a p-element t

such that $C_G(t) \subseteq C_G(w)$ By applying Lemma 11, we obtain that $w \in C_G(O_p(G))$. Assume now that w is a q-element of index $p^a q^b$. Notice that $C_G(w)$ must be equal to the centralizer of some p-element. By Lemma 11 again, we have $w \in C_G(O_p(G))$. So any $z \in O_p(G)$ is centralized by any q-element of G and since the index of z is 1 or $p^a q^b$, we conclude that z must be central in G. Therefore $O_p(G) = Z(G)_p$, and thus $O_{p,q}(G) = Z(G)_p \times O_q(G)$.

We prove now that G has a normal abelian Sylow q-subgroup. Suppose that G has a q-element w of index $p^a q^b$. Then G will have a p-element t such that $C_G(t) = C_G(w)$ and this centralizer is abelian. Moreover, by Lemma 11, we have $O_q(G) \subseteq C_G(t) = C_G(w)$, so $O_q(G)$ is also abelian. Hence

$$w \in C_G(O_q(G)) = C_G(O_{p,q}(G)) \subseteq O_{p,q}(G)$$

and so $w \in O_q(G)$. On the other hand, if w is a q-element of index p^a, by Lemma 6 we have $w \in O_{p,q}(G)$, so $w \in O_q(G)$ too. We conclude that $Q = O_q(G)$ is a Sylow q-subgroup of G. Furthermore, if there is a q-element of index $p^a q^b$ we have proved that Q is abelian, and if every q-element has index 1 or p^a, by Lemma 2 we get that Q is abelian too.

Let M be the set of elements in G whose index is 1 or p^a. It follows that $M = C_G(Q)$, whence M is a normal subgroup of G. Moreover, by the assumption of Case 2, if z is a p-element of M then $z \in Z(G)$, so $M = Q \times Z(G)_p$. Let P be a Sylow p-subgroup of G. Observe that $Z(G)_p = Z(P)$ Write $P_0 = Z(P)$ and $\overline{P} = P//P_0$ (which we know has exponent p).

The group $\overline{P}$ acts coprimely on the abelian group Q, so we can write $Q = [Q, \overline{P}] \times C_Q(\overline{P})$. Also, observe that $C_Q(\overline{P}) = C_Q(P) = Z(G)_q$ and $[Q, \overline{P}] = [Q, P]$. We claim that the action of $\overline{P}$ on $[Q, P] - \{1\}$ is half-transitive, that is, all the orbits have the same size. Indeed, if $x \in [Q, P] - \{1\}$ then its class size is p^a and the size of its orbit is

$$|\overline{P} : C_{\overline{P}}(x)| = |P : C_P(x)| = |G : C_G(x)| = p^a$$

where the first equality holds since $P_0 = Z(G)_p$ and the second follows from the fact that $G = PC_G(x)$. By applying Lemma 12, either $\overline{P}$ acts fixed-point-freely on $[Q, P]$ or $\overline{P}$ acts irreducibly. We will see that this second possibility also yields to a fix-point-free action. Suppose that $\overline{P}$ acts irreducibly on $[Q, P]$ and take $\overline{z} \in Z(\overline{P})$. Then $C_{[Q,P]}(\overline{z})$ is certainly a $\overline{P}$-invariant subgroup, so either $C_{[Q,P]}(\overline{z})$=1 or $C_{[Q,P]}(\overline{z}) = [Q, P]$. In the latter case, as $Q = Z(G)_q \times [Q, P]$, it follows that z lies in $C_P(Q) = P_0$, so $\overline{z} = 1$. Therefore, we conclude that $Z(\overline{P})$ acts fixed-point-freely on $[Q, P]$. On the other hand, as $G = QP$ with Q normal in G, it is easy to see that $C_G(g) = C_Q(g)C_P(g)$ for each $g \in P$. In particular, if $\overline{z} \in Z(\overline{P}) - \{1\}$, then

$$p^a q^b = |G : C_G(z)| = |Q : C_Q(z)||P : C_P(z)|.$$

So $|Q : C_Q(z)| = q^b$. But notice that

$$C_Q(z) = C_Q(\overline{z}) = Z(G)_q \times C_{[Q,P]}(z) = Z(G)_q,$$

so $|Q : Z(G)_q| = q^b$. This implies that $\overline{P}$ acts fixed-point-freely. If $\overline{t} \in Z(\overline{P}) - \{1\}$, then

$$p^a q^b = |G : C_G(t)| = |Q : C_Q(t)||P : C_P(t)|.$$

Thus $|Q : C_Q(t)| = q^b$ and consequently we have $C_Q(t) = Z(G)_q$. This proves that $C_{[Q,P]}(\overline{t}) = 1$, as we wanted to show. Now we can apply [19, Theorem 16.12] again. So $\overline{P}$ must be cyclic or generalized quaternion; but as $\overline{P}$ has exponent p it is cyclic of order p. This forces P to be abelian, which leads to a contradiction.

Case 2.2. We assume that n is not a prime power and distinguish two cases depending on whether there are q'-elements of primary orders of index m or not.

Case 2.2.a. Suppose that every q'-element of primary order of G has index 1 or mn. Fix a prime $r \in \pi' - \{q\}$. For every r-element w of index mn we can certainly write $C_G(w) = C_G(w)_\pi \times C_G(w)_{\pi'}$ with $C_G(w)_\pi$ an abelian p-subgroup. Since

$$|C_G(w)|_\pi = |C_G(g)|_\pi > |Z(G)|_\pi$$

there exists a non-central π-element $t \in C_G(w)$ of primary order. As t has index mn too, we have $C_G(w) = C_G(t)$ and hence this subgroup is abelian. In general, if z is a non-central q'-element of primary order of G then r divides $|C_G(z)|$, and so $C_G(z)$ must coincide with the centralizer of some non-central r-element. However, we have seen that such centralizers are abelian, so all the centralizers of non-central q'-elements of primary orders of G are abelian. Now, if all centralizers of non-central elements of primary orders in $G_{q'}$ are conjugate, using the argument of Case 2.1.a, we arrive at a contradiction. If the centralizers of the non-central elements of primary orders in $G_{q'}$ are not all conjugate, by the remark made after Lemma 9, we can apply Lemma 9 although G is not q-solvable, to get $mn = p^a q^b$, for some prime p. This contradicts the hypothesis of Case 2.2.

Case 2.2.b. Suppose now that G has q'-elements of index m. We will prove that every element of G lies in a conjugate of $C_G(V)$ where $V = C_G(g)_\pi$, which is the Hall π-subgroup of $C_G(g)$ and a Hall

π-subgroup of $C_G(x)$ and where g and x are the elements fixed at the beginning of Case 2. Then $V \subseteq Z(G)$ and this is a contradiction because g is not central in G. We study separately the elements of index m and the elements of index mn in order to see this.

Let w be an element in G^* of index m. By considering the primary decomposition of w and by the assumption of Case 2, we can replace w so that its order is a power of some prime in π'.

Suppose first that w is an r-element where $r \neq q$, and let Q be a Sylow q-subgroup of G such that $Q \subseteq C_G(x)$. There exists $h \in G$ such that $x^h \in Q^h \subseteq C_G(w)$ so $C_G(wx^h) = C_G(w) \bigcap C_G(x^h) \subseteq C_G(w)$. We have two possibilities according to whether these centralizers are equal or not. Suppose first that $C_G(wx^h) = C_G(w)$ which implies that $C_G(wx^h) = C_G(w) = C_G(x^h)$. We deduce in this situation that every element of primary order of $C_G(w)$ has index 1 or n in $C_G(w)$, so by Lemma 5 we get that n is a prime power, which is a contradiction. Since the centralizer of the q'-element w of primary order coincides with the centralizer of the q-element x^h, it easily follows that any q-element and any q'-element of primary order of $C_G(w)$ must have index 1 or n in $C_G(w)$. Now take an arbitrary element z of $C_G(w)$ and consider its decomposition $z = z_q z_{q'}$, where z_q and $z_{q'}$ are elements of primary orders. If z_q or $z_{q'}$ has index mn in G, then $C_G(z)$ is equal to $C_G(z_q)$ or $C_G(z_{q'})$ and thus z has again index 1 or n in $C_G(w)$. So we can assume that z_q and $z_{q'}$ have index m and that z has index mn in G. Also it can be assumed without loss that z is a π'-element of primary order, by the assumption of Case 2. The existence of π-elements of index mn implies that $|C_G(z)|_\pi > |Z(G)|_\pi$. Therefore, there is a non-central π-element $k \in C_G(z)$ of primary order, but since k has index mn in G, we have $C_G(z) = C_G(k)$ and this subgroup is abelian. Thus $C_G(z) \subseteq C_G(w)$ and z also has index n in $C_G(w)$, so this case is finished. We assume now the second possibility, that is, $C_G(wx^h) \subset C_G(w)$. Again the existence of π-elements of index mn implies that $|C_G(wx^h)|_\pi > |Z(G)|_\pi$ and arguing similarly we get that $C_G(wx^h)$ coincides with the centralizer of some π-element. In particular, this centralizer is abelian, whence $C_G(wx^h)_\pi$ is an abelian Hall π-subgroup of $C_G(x^h)$ which, by Wielandt's theorem, is conjugate to V^h. We conclude that w belongs to some conjugate of $C_G(V)$ as wanted, and also that V is abelian.

Suppose now that w is a q-element. We are assuming that there are r-elements of index m for some $r \in \pi - \{q\}$, so we can take without loss such an element $v \in C_G(w)$.

Let Q be a Sylow q-subgroup of G such that $Q \subseteq C_G(x)$. Then there exists $h \in G$ such that $Q^h \subseteq C_G(v)$. Since x^h and w are q-elements of $C_G(v)$, we can replace w by a conjugate in $C_G(v)$ and assume that $w \in Q^h$ and thus $w \in C_G(vx^h)$. Arguing as in the above paragraph for the r-element v, we have that $C_G(vx^h)$ is strictly contained in $C_G(v)$. It follows that $C_G(vx^h)$ is an abelian subgroup strictly contained in $C_G(x^h)$. Hence the Hall π-subgroup of $C_G(vx^h)$ is conjugate to V^h and also this subgroup is abelian. As $w \in C_G(vx^h)$ we conclude that w centralizes the Hall π-subgroup of $C_G(vx^h)$, and consequently w centralizes some conjugate of V, as wanted.

Finally, assume that w has index mn and write $w = w_\pi w_{\pi'}$, where w_π and $w_{\pi'}$ are elements of primary order. We observe that $|C_G(w)|_{\pi'} = |C_G(g)|_{\pi'} > |Z(G)|_{\pi'}$ because x is a π'-element of primary order in $C_G(g)$. If w_π is noncentral, it follows that $C_G(w) = C_G(w_\pi) = C_G(w_\pi)_\pi \times C_G(w_\pi)_{\pi'}$, and $C_G(w_\pi)_{\pi'}$ is abelian. Then there exists $k \in C_G(w)_{\pi'}$, which may be assumed of order r with $r \in \pi'$, such that $C_G(w) \subseteq C_G(k)$. If w_π is central, then $C_G(w) = C_G(w_{\pi'})$ and by the primary composition of $w_{\pi'}$ we can choose again an r-element $k \in C_G(w)$ with $r \in \pi'$, such that $C_G(w) \subseteq C_G(k)$. In both cases we study two possibilities for the index of k in G. If k has index m, the above paragraphs show that k centralizes V^h for some $h \in G$, and V^h is an abelian Hall π-subgroup of $C_G(k)$. Hence $C_G(w)_\pi = V^t$ for some $t \in G$, and w belongs to $C_G(V)^t$. On the other hand, if k has index mn, then $C_G(k) = C_G(w)$. As $|C_G(g)|_\pi > |Z(G)|_\pi$ by the existence of π-elements of index mn, then $C_G(k)$ coincides with the centralizer of a π-element and therefore it is abelian. As $C_G(x)$ contains a Sylow r-subgroup we can take an element $h \in G$ such that $k \in C_G(x^h)$. It follows that $C_G(k) \subseteq C_G(x^h)$. Therefore the Hall π-subgroups of $C_G(k)$ are abelian Hall π-subgroups of $C_G(x^h)$ and so are conjugate to V^h. We conclude that w also lies in a conjugate of $C_G(V)$, as wanted. Now the proof of the theorem is finished. □

Remark 16 *$n-1$ must be divisible by p as a consequence of the fixed-point-free action appearing in the structure of the group. For any prime p the situation described in Theorem 15 does exist. For instance, let*

$$P = \langle x, y | x^{p^2} = y^p = 1, x^y = x^{p+1} \rangle$$

be a non-abelian p-group of order p^3 and exponent p^2 and take $P_0 = \langle x \rangle$. Let n be any integer such that p divides $q-1$ for any prime factor q dividing n(accordingly p divides $n-1$) and let H be a cyclic group of order n. We consider the action of P on H defined in the following way: x acts trivially on H and y acts as a fixed-point-free automorphism of order p on each direct factor of prime-power order of H. Then $G = HP$ is an example of group with class sizes $\{1, p, pn\}$.

The following theorem can be seen as a generalization of [10, Theorem 3.1].

Theorem 17 *Let G be a group and assume that the conjugacy class sizes of all elements of primary orders and $\{p, q\}$-elements of G are exactly $\{1, p^a, n, p^a n\}$ with $(p, n) = 1$, where p is a fixed prime, q is an arbitrary prime, a and n are positive integers. If there is a p-element in G whose index is precisely p^a, then G is nilpotent and $n = q^b$ for some prime $q \neq p$.*

Proof The proof has been divided into two steps.

Step 1. G is p-nilpotent.

Let $H = O_p(G)$ and $Z = C_G(O_p(G))$. By hypothesis there is a p-element $x \in G$ such that $[G : C_G(x)] = p^a$. By Lemma 13, we know $x \in O_p(G)$. Let $y \in C_G(x)$ be an r-element for some prime r with $r \neq p$. Then xy is biprimary and, as x and y commute, $C_G(xy) = C_G(x) \cap C_G(y)$. The main hypothesis implies that $|y^{C_G(x)}| = |C_G(x) : C_G(xy)|$ is coprime to p. By Lemma 7,

$$C_G(x) = O_{p'}(C_G(x))O_p(C_G(x)).$$

In particular, y centralizes $O_p(C_G(x)) \geq H \cap C_G(x) = C_H(x)$. By Lemma 11, $y \in Z$. Since $x \in H$, we have $Z \leq C_G(x)$ and so Z has $O_{p'}(C_G(x))$ as a normal p-complement. Furthermore, $|G : Z| = |G : C_G(x)||C_G(x) : Z|$ is a power of p. Since Z is normal in G, we now have that $O_{p'}(C_G(x))$ is a normal p-complement in G. Hence G is p-nilpotent.

Step 2. G is nilpotent and $n = q^b$ for some prime $q \neq p$.

By Step 1 we have that G has a normal p-complement, say H. For every $p'-$element x of primary order of H we have that

$$|G : H||H : C_H(x)| = |G : C_G(x)||C_G(x) : C_H(x)|.$$

If $|x^G| = 1$ or p^a, then $H \subseteq C_G(x)$ and thus $|x^H| = 1$. If $|x^G| = n$ or $p^a n$, then the above equality along with the fact that $|x^H|$ divides $|x^G|$ (by Lemma 13) imply that $|x^H| = n$. Therefore, every conjugacy class of $p'-$elements of primary order of G has size 1 or n. By Lemma 9 we have that $n = p^c q^b$ for some prime $q \neq p$. Since (p, n)=1, then $n = q^b$ and again by Lemma 9, we conclude that G is nilpotent, the theorem is proved. □

4 Conclusion

The results explained in the previous sections show that the method that we replace conditions for all conjugacy classes by conditions referring to only some of the classes in order to investigate the structure of a finite group is very useful. Results of this type are interesting since they can be used to simplify the proofs of new or known properties related to conjugacy classes. Recently, in [21] and [22] the second author characterized the structure of a finite group by using the method. In addition, according to the parallel property of conjugacy class sizes and character degrees in [23] and [24], we may consider using the character degrees to characterize the structure of finite groups. As an application, we can investigate the structure of a finite group when its character degrees of G are $\{1, m, mn\}$ or $\{1, m, n, mn\}$, where m and n are integers with (m, n)=1.

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