Square-Mean Almost Periodic Solutions of Neutral Stochastic Functional Differential Equations on Time Scales

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Abstract: In this paper, based on the properties of almost periodic function and exponential dichotomy of linear system on time scales as well as Krasnoselskii's fixed point theorem, some sufficient conditions are established for the existence of square-mean almost periodic solutions of neutral stochastic functional differential equations on time scales. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Neutral stochastic differential equation; Square-mean almost periodic solution; Exponential dichotomy; Krasnoselskii's fixed point theorem; Time scale.

1 Introduction

In the past few years, different types of neutral differential and difference equations with periodic coefficients have been studied extensively, see, for example, [1-5] and the references therein. However, upon considering long-term dynamical behaviors, the periodic parameters often turn out to experience certain perturbations, that is, parameters become periodic up to a small error, then one has to consider the systems to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Therefore, if we consider the effects of the environmental factors, the assumption of almost periodicity is more realistic, more important and more general.

On the other hand, in the real word, lots of dynamic systems have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of components, changes in the interconnections of subsystems, sudden environment changes, and so on [6-10]. Moreover, in applications, there are many systems whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

The theory of calculus on time scales (see [11] and references cited therein) was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [12] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since his foundational work, one may see [13-18]. Therefore, it is practicable to study that on time scales which can unify the continuous and discrete situations. However, to the best of the authors' knowledge, there are few papers published on the existence of almost periodic solution of neutral stochastic functional differential equations on time scales.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with the normal filtration $\{\mathcal{F}_t, t \in \mathbb{T}\}$, that is, a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$. Let $L^2(P, \mathbb{R}^n)$ be the set of all $\mathbb{R}^n$-valued random variables x on $\mathbb{T}$ with

$$\mathbf{E}\|x\|^2 = \int_\Omega \|x\|^2 \Delta P < \infty,$$

then, $L^2(P, \mathbb{R}^n)$ is a Banach space equipped with the norm $\|x\|_2 = (\int_\Omega \|x\|^2 \Delta P)^{\frac{1}{2}}$.

In the present paper, we focus on the following neutral stochastic functional differential equations on time scales:

$$\begin{aligned} \Delta x(t) &= A(t)x(t)\Delta t + \Delta Q(t, x(t)) \\ &\quad + G(t, x(t))\Delta W(t),\ t \in \mathbb{T}, \end{aligned} \tag{1}$$

where $\mathbb{T}$ is an almost periodic time scale, $A(t)$ is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements; the functions $Q : \mathbb{T} \times L^2(P, \mathbb{R}^n) \to L^2(P, \mathbb{R}^n)$ and $G : \mathbb{T} \times L^2(P, \mathbb{R}^n) \to L^2(P, \mathbb{R}^n)$ are jointly continuous functions; $\{W(t), t \in \mathbb{T}\}$ is a Brownian motion or a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$

generated by $W(t)$, and denote by $\mathcal{F}$ the associated σ-algebra generated by $W(t)$ with the probability measure P.

The purpose of this paper is to establish the existence of almost periodic solutions of (1) based on the properties of almost periodic function and exponential dichotomy of linear system on time scales as well as krasnoselskii's fixed point theorem.

In this paper, for each $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C(\mathbb{T}, L^2(P, \mathbb{R}^n))$, when it comes to that ϕ is continuous, delta derivative, delta integrable, and so forth, we mean that each element ϕ_i is continuous, delta derivative, delta integrable, and so forth.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\begin{aligned}\sigma(t) &= \inf\{s \in \mathbb{T} : s > t\},\\ \rho(t) &= \sup\{s \in \mathbb{T} : s < t\},\\ \mu(t) &= \sigma(t) - t.\end{aligned}$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R})$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous, if $Y^\Delta(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

The basic theories of calculus on time scales, one can see [9].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0,\\ z, & \text{if } h = 0.\end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p \oplus q = p + q + \mu pq, \ \ominus p = -\frac{p}{1 + \mu p}, \ p \ominus q = p \oplus (\ominus q).$$

Lemma 1. (see [9]) *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $(e_{\ominus p}(t, s))^\Delta = (\ominus p)(t)e_{\ominus p}(t, s)$.

Lemma 2. (see [9]) *If $p \in \mathcal{R}$ be an $n \times n$-matrix-valued function on $\mathbb{T}$ and $a, b, c \in \mathbb{T}$, then*

$$\begin{aligned}&[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma,\\ &\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).\end{aligned}$$

The following definitions of almost periodic function and uniformly almost periodic function on time scales can be found in [19,20].

A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of function f

$$E\{\varepsilon, f\} = \{\tau \in \Pi : |f(t + \tau) - f(t)| < \varepsilon, t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ and

$$|f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

τ is called the ε-translation number of f.

Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{R}^n)$ is called an almost periodic function in $\mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of function f

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t+\tau)-f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset S of D, where D denotes an open set in $\mathbb{R}^n$ or $D = \mathbb{R}^n$; that is for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \forall t \in \mathbb{T}, x \in S.$$

τ is called the ε-translation number of f.

Definition 3. *A stochastic process $x : \mathbb{T} \to L^2(P, \mathbb{R}^n)$ is said to be stochastically continuous whenever $\lim\limits_{t\to s} \mathbf{E}\|x(t) - x(s)\|^2 = 0$.*

Definition 4. *A stochastic process $x : \mathbb{T} \to L^2(P, \mathbb{R}^n)$ is said to be stochastically bounded whenever $\lim\limits_{N\to\infty} \sup\limits_{t\in\mathbb{T}}\{P\|x(t)\| > N\} = 0$.*

Definition 5. *A stochastically continuous process $x : \mathbb{T} \to L^2(P, \mathbb{R}^n)$ is called square-mean almost periodic on $\mathbb{T}$, if for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists at least a τ for which $\sup\limits_{t\in\mathbb{T}} \mathbf{E}\|x(t+\tau) - x(t)\|^2 < \varepsilon$, where τ is called the ε-translation number of x.*

Let $AP(\mathbb{T})$ be the set of all $\mathbb{R}^n$-valued square-mean almost periodic stochastic processes $x : \mathbb{T} \to L^2(P, \mathbb{R}^n)$ on almost time scales $\mathbb{T}$, then $(AP(\mathbb{T}), \|\cdot\|)$ is a Banach space equipped with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{T}} \|x(t)\|_2 = \sup_{t\in\mathbb{T}}(\mathbf{E}\|x(t)\|^2)^{\frac{1}{2}}.$$

In what follows, we need the following notation. For every real sequence $\alpha = (\alpha_n)$ and a stochastically continuous process $x : \mathbb{T} \to L^2(P, \mathbb{R}^n)$, define $T_\alpha x = \lim\limits_{n\to\infty} x(t+\alpha_n)$ if $\lim\limits_{n\to\infty} x(t+\alpha_n)$ exists.

Similar to the proof of Theorem 3.14 in [20], we have

Lemma 6. *A stochastically continuous process $x : \mathbb{T} \to \mathbb{R}^n$ is square-mean almost periodic if and only if f is continuous and for each $\alpha = (\alpha_n)$, there exists a subsequence α' of (α_n) such that $\mathbb{T}_{\alpha'} f = g$ uniformly on $\mathbb{T}$.*

Lemma 7. (see [21]) *If x belongs to $AP(\mathbb{T})$, then*
(i) *the mapping $t \to \mathbf{E}\|x(t)\|^2$ is uniformly continuous;*
(ii) *there exists a constant $M > 0$ such that $E\|x(t)\|^2 \leq M$, for all $t \in \mathbb{T}$;*
(iii) *x is stochastically bounded.*

Definition 8. *A function $f : \mathbb{T} \times L^2(P, \mathbb{R}^n) \to L^2(P, \mathbb{R}^n)$, which is jointly continuous, is said to be square-mean almost periodic on $\mathbb{T}$ uniformly in $x \in K$, where $K \subset L^2(P, \mathbb{R}^n)$ is compact, if for any $\varepsilon > 0$, there exists a constant $l(\varepsilon, K) > 0$, such that any interval of length $l(\varepsilon, K)$ contains at least a τ for which $\sup\limits_{t\in\mathbb{T}} \mathbf{E}\|f(t+\tau, x) - f(t, x)\|^2 < \varepsilon$, for each stochastic process $x : \mathbb{T} \to K$, where τ is called the ε-translation number of $f(t, x)$.*

Similar to the proof of Lemma 9 in [22], we can get

Lemma 9. *Let $f : \mathbb{T} \times L^2(P, \mathbb{R}^n) \to L^2(P, \mathbb{R}^n)$ be a square-mean almost periodic process in $t \in \mathbb{T}$ uniformly in $x \in K$, where $K \subset L^2(P, \mathbb{R}^n)$ is compact. Suppose that f is Lipschitz in the following sense:*

$$\mathbf{E}\|f(t, x) - f(t, y)\| \leq M\mathbf{E}\|x - y\|^2,$$

for all $x, y \in L^2(P, \mathbb{R}^n)$ and for each $t \in \mathbb{T}$, where $M > 0$. Then for any square-mean almost periodic process $\phi : \mathbb{T} \to L^2(P, \mathbb{R}^n)$, the stochastic process $t \to f(t, \phi(t))$ is square-mean almost periodic.

Definition 10. *Let $x \in \mathbb{R}^n$ and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system*

$$x^\Delta(t) = A(t)x(t) \tag{2}$$

is said to admit an exponential dichotomy on $\mathbb{T}$, if there exist positive constants $\alpha > 0, k \geq 1$, projection P and the fundamental solution matrix $X(t)$ of (2) satisfying

$$\begin{aligned} \|X(t)PX^{-1}(\sigma(s))\|^2 &\leq ke_{\ominus\alpha}(t, \sigma(s)) \\ s, t &\in \mathbb{T}, t \geq \sigma(s), \quad (3) \\ \|X(t)(I-P)X^{-1}(\sigma(s))\|^2 &\leq ke_{\ominus\alpha}(\sigma(s), t) \\ s, t &\in \mathbb{T}, t \leq \sigma(s), \quad (4) \end{aligned}$$

where $\|\cdot\|$ is a matrix norm on $\mathbb{T}$.

Remark 11. *In the case $A(t) \equiv A$, a constant matrix, (2) admits exponential dichotomy if and only if the eigenvalues of A have a nonzero real part.*

Lemma 12. *Suppose (2) admits exponential dichotomy, that is there exist constants $\alpha > 0, k \geq 1$, such that (3), (4) hold. If $A(t + t_k)$ converges to $\overline{A}(t)$ uniformly on any compact subset* *of $\mathbb{T}$, then $\{X(t+t_k)PX^{-1}(\sigma(s)+t_k)\}$ and $\{X(t+t_k)(I-P)X^{-1}(\sigma(s)+t_k)\}$ converges to $\{\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\}$ and $\{\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s))\}$ uniformly on any compact subset $\mathbb{T}\times\mathbb{T}$, respectively. Furthermore, the following inequalities hold:*

$$\|\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\|^2 \le ke_{\ominus\alpha}(t,\sigma(s))\\ s,t\in\mathbb{T}, t\ge\sigma(s),$$
$$\|\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s))\|^2 \le ke_{\ominus\alpha}(\sigma(s),t)\\ s,t\in\mathbb{T}, t\le\sigma(s),$$

where $\overline{X}$ is the fundamental matrix solution of the following equation

$$x^{\Delta}(t) = \overline{A}(t)x. \tag{5}$$

Proof. we first prove that $\{X(t_k)PX^{-1}(t_k)\}$ is convergent. From (3), we see that

$$\|X(t_k)PX^{-1}(t_k)\|^2 \le k.$$

Suppose $\{X(t_k)PX^{-1}(t_k)\}$ is not convergent. Then we can find two subsequence:

$$\{X(t_{k_m})PX^{-1}(t_{k_m})\},\ \{X(t_{k'_m})PX^{-1}(t_{k'_m})\},$$

such that

$$\lim_{m\to\infty} X(t_{k_m})PX^{-1}(t_{k_m}) = \overline{P},\\ \lim_{m\to\infty} X(t_{k'_m})PX^{-1}(t_{k'_m}) = \underline{P},$$

and $\overline{P}\ne\underline{P}$.

Then from (3) we have

$$\|X(t+t_{k_m})PX^{-1}(\sigma(s)+t_{k_m})\|^2\\ \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s), \tag{6}$$

and

$$\|X(t+t_{k'_m})PX^{-1}(\sigma(s)+t_{k'_m})\|^2\\ \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s). \tag{7}$$

Assume that $X_{k_m}(t), X_{k'_m}(t)$ are the fundamental matrix solutions of systems

$$x^{\Delta}(t) = A(t+t_{k_m})x,\ x^{\Delta}(t) = A(t+t_{k'_m})x$$

respectively, then $X(t+t_{k_m}) = X_{k_m}(t)X(t_{k_m})$, $X(t+t_{k'_m}) = X_{k'_m}(t)X(t_{k'_m})$. Since $\{A(t+t_k\}$ converges to $\overline{A}(t)$ uniformly on any compact subset of $\mathbb{T}$, then $\{A(t+t_k)x\}$ converges to $\overline{A}(t)x$ uniformly on any compact subset of $\mathbb{T}\times\mathbb{R}^n$. It follows that $\{A(t+t_{k_m})x\}$ and $\{A(t+t_{k'_m})x\}$ converge to $\overline{A}(t)x$ uniformly on any compact subset of $\mathbb{T}\times\mathbb{R}^n$. So $X_{k_m}(t)$, $X_{k'_m}(t)$ converge to $\overline{X}(t)$ uniformly on any compact set of $\mathbb{T}$. Furthermore, it follows from (6), (7) that

$$\|X_{k_m}(t)X(t_{k_m})PX^{-1}(t_{k_m})X_{k_m}^{-1}(\sigma(s))\|^2\\ \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s)$$

and

$$\|X_{k'_m}(t)X(t_{k'_m})PX^{-1}(t_{k'_m})X_{k'_m}^{-1}(\sigma(s))\|^2\\ \le ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\ge\sigma(s).$$

Let $m\to\infty$, we have

$$\|\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\|^2 \le ke_{\ominus\alpha}(t,\sigma(s))\\ s,t\in\mathbb{T}, t\ge\sigma(s) \tag{8}$$

and

$$\|\overline{X}(t)\underline{P}\,\overline{X}^{-1}(\sigma(s))\|^2 \le ke_{\ominus\alpha}(t,\sigma(s))\\ s,t\in\mathbb{T}, t\ge\sigma(s). \tag{9}$$

Similarly, we can obtain

$$\|\overline{X}(t)(I-\overline{P})\,\overline{X}^{-1}(\sigma(s))\|^2 \le ke_{\ominus\alpha}(\sigma(s),t)\\ s,t\in\mathbb{T}, t\le\sigma(s) \tag{10}$$

and

$$\|\overline{X}(t)(I-\underline{P})\,\overline{X}^{-1}(\sigma(s))\|^2 \le ke_{\ominus\alpha}(\sigma(s),t)\\ s,t\in\mathbb{T}, t\le\sigma(s). \tag{11}$$

From (8)-(11), we see that (5) admits exponential dichotomy; both $\overline{P}$ and $\underline{P}$ are its projections. So $\overline{P}=\underline{P}$, which is a contradiction. Hence, $\{X(t_k)PX^{-1}(t_k)\}$ is convergent.

Let $\{X(t_k)PX^{-1}(t_k)\}\to\overline{P}$ as $k\to\infty$. Now assume that $X_k(t)$ is the fundamental matrix solution of the system $x^{\Delta}(t) = A(t+t_k)x$, then $X_k(t)$ converges to $\overline{X}(t)$ uniformly on any compact set of $\mathbb{T}$. It is easy to see that $\{X_k^{-1}(\sigma(s))\}$ converges to $\overline{X}^{-1}(\sigma(s))$ uniformly on any compact subset of $\mathbb{T}$. So $X(t+t_k)PX^{-1}(\sigma(s)+t_k)$ and $\{X(t+t_k)(I-P)X^{-1}(\sigma(s)+t_k)\}$ converges to $\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))$ and $\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s))$ uniformly on any compact subset $\mathbb{T}\times\mathbb{T}$, respectively. Furthermore, from (6) and (7) we have

$$\|X(t+t_k)PX^{-1}(\sigma(s)+t_k)\|^2 \le ke_{\ominus\alpha}(t,\sigma(s))\\ s,t\in\mathbb{T}, t\ge\sigma(s)$$

and

$$\|X(t+t_k)(I-P)X^{-1}(\sigma(s)+t_k)\|^2\\ \le ke_{\ominus\alpha}(\sigma(s),t) \quad s,t\in\mathbb{T}, t\le\sigma(s).$$

That is

$$\|X_k(t)X(t_k)PX^{-1}(t_k)X_k^{-1}(\sigma(s))\|^2$$
$$\leq ke_{\ominus\alpha}(t,\sigma(s)) \quad s,t\in\mathbb{T}, t\geq\sigma(s)$$

and

$$\|X_k(t)X(t_k)(I-P)X^{-1}(t_k)X_k^{-1}(\sigma(s))\|^2$$
$$\leq ke_{\ominus\alpha}(\sigma(s),t) \quad s,t\in\mathbb{T}, t\leq\sigma(s).$$

Let $k\to\infty$, we obtain

$$\|\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))\|^2\leq ke_{\ominus\alpha}(t,\sigma(s))$$
$$s,t\in\mathbb{T}, t\geq\sigma(s)$$

and

$$\|\overline{X}(t)(I-\overline{P})\,\overline{X}^{-1}(\sigma(s))\|^2\leq ke_{\ominus\alpha}(\sigma(s),t)$$
$$s,t\in\mathbb{T}, t\leq\sigma(s).$$

The proof is completed. □

Lemma 13. *(see [23]) Let M be a closed convex nonempty subset of a Banach space $(B,\|\cdot\|)$. Suppose that B and C map M into B, such that*
(1) $x,y\in M$, implies $Bx+Cy\in M$,
(2) C is continuous and $C(M)$ is contained in a compact set,
(3) B is a contraction mapping.
Then there exists $z\in M$ with $z=Bz+Cz$.

3 Main results

In this section, we require the following assumptions:

(H_1) $A(t)$ is a square-mean almost periodic function, $Q(t,u)$ and $G(t,u)$ be two square-mean almost periodic functions in t uniformly for $u\in AP(\mathbb{T})$, respectively.

(H_2) The functions Q and G are Lipschitz, that is there exist two positive numbers L_Q and L_G such that

$$\mathbf{E}\|Q(t,u)-Q(t,v)\|^2\leq L_Q\mathbf{E}\|u-v\|^2, \quad (12)$$
$$\mathbf{E}\|G(t,u)-G(t,v)\|^2\leq L_G\mathbf{E}\|u-v\|^2, \quad (13)$$

for all $t\in\mathbb{T}$, $u,v\in AP(\mathbb{T})$.

(H_3) System (2) admits exponential dichotomy, that is there exist constants $\alpha>0, k\geq1$, such that (3) and (4) hold.

Define a mapping Φ by

$$\begin{aligned}(\Phi u)(t) &= Q(t,u(t))\\ &+\int_{-\infty}^{t}X(t)PX^{-1}(\sigma(s))G(s,u(s))\Delta W(s)\\ &-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(\sigma(s))G(s,u(s))\\ &\times\Delta W(s). \end{aligned} \quad (14)$$

Lemma 14. *If u is a square-mean almost periodic function, then Φu is a square-mean almost periodic function.*

Proof. For $u(t)$ is an almost periodic function, from (H_1) and Lemma 7 to Lemma 9, then $Q(t,u(t)),G(t,u(t))$ are square-mean almost periodic functions, so they are uniformly bounded on $\mathbb{T}$.

Now, we prove that $(\Phi u)(t)$ is a square-mean almost periodic function. First, it is clear that $(\Phi u)(t)$ is continuous on $\mathbb{T}$. For any sequence $\alpha=(\alpha_n)$, since $Q(t,u(t)),G(t,u(t))$ are square-mean almost periodic functions, combining with Lemma 6 and Lemma 12, we can find a common subsequence of (α_n), we still denote it as (α_n), such that

$$T_\alpha Q(t,u(t))=Q_1(t),\quad T_\alpha G(t,u(t))=G_1(t) \quad (15)$$

uniformly for $t\in\mathbb{T}$ and

$$\begin{aligned}&\lim_{k\to\infty}X(t+\alpha_k)PX^{-1}(\sigma(s)+\alpha_k)\\ &=\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s)), t\geq\sigma(s) \quad (16)\\ &\lim_{k\to\infty}X(t+\alpha_k)(I-P)X^{-1}(\sigma(s)+\alpha_k)\\ &=\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s)), t\leq\sigma(s). \quad (17)\end{aligned}$$

Let $\widetilde{W}(s):=W(s+\alpha_k)-W(\alpha_k)$ for each $s\in\mathbb{T}$. Note that $\widetilde{W}$ is also a Brown motion and has the same distribution as W.

From the above and (14), then

$$\begin{aligned}&(\Phi u)(t+\alpha_k)\\ =\;& Q(t+\alpha_k,u(t+\alpha_k))\\ &+\int_{-\infty}^{t+\alpha_k}X(t+\alpha_k)PX^{-1}(\sigma(s))G(s,u(s))\\ &\times\Delta W(s)\\ &-\int_{t+\alpha_k}^{+\infty}X(t+\alpha_k)(I-P)X^{-1}(\sigma(s))\\ &\times G(s,u(s))\Delta W(s)\\ =\;& Q(t+\alpha_k,u(t+\alpha_k))\\ &+\int_{-\infty}^{t}X(t+\alpha_k)PX^{-1}(\sigma(s)+\alpha_k)\\ &\times G(s+\alpha_k,u(s+\alpha_k))\Delta\widetilde{W}(s)\\ &-\int_{t}^{+\infty}X(t+\alpha_k)(I-P)X^{-1}(\sigma(s)+\alpha_k)\\ &\times G(s+\alpha_k,u(s+\alpha_k))\Delta\widetilde{W}(s).\end{aligned}$$

From (15)-(17) and Lebesgue's control convergence theorem, we see that $(\Phi u)(t+\alpha_k)$ converges to

$$\begin{aligned} &\Psi(t)\\ =\ & Q_1(t)+\int_{-\infty}^{t}\overline{X}(t)\overline{P}\,\overline{X}^{-1}(\sigma(s))G_1(s)\Delta\widetilde{W}(s)\\ &-\int_{t}^{+\infty}\overline{X}(t)(I-\overline{P})\overline{X}^{-1}(\sigma(s))G_1(s)\Delta\widetilde{W}(s) \end{aligned}$$

uniformly for $t\in\mathbb{T}$.

By the continuous dependence of the integrands of the deterministic and stochastic integrals [24], then

$$\lim_{k\to\infty}\mathbf{E}\|(\Phi u)(t+\alpha_k)-\Psi(t)\|^2=0,\forall t\in\mathbb{T}.$$

It follows from Lemma 6 that $(\Phi u)(t)$ is a square-mean almost periodic function. The proof is completed. □

In order to apply Krasnoselskii's theorem, we need to construct two mappings, one is a contraction and the other is compact. Let

$$(\Phi u)(t)=(Bu)(t)+(Cu)(t),$$

where $B,C:AP(\mathbb{T})\to AP(\mathbb{T})$ are given by

$$(Bu)(t)=Q(t,u(t)), \tag{18}$$

$$\begin{aligned} &(Cu)(t)\\ =\ &\int_{-\infty}^{t}X(t)PX^{-1}(\sigma(s))G(s,u(s))\Delta W(s)\\ &-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(\sigma(s))\\ &\times G(s,u(s))\Delta W(s). \end{aligned}\tag{19}$$

Lemma 15. *(see [25]) The operator B is a contraction provided $L_Q<1$.*

Lemma 16. *The operator C is continuous and the image $C(\mathbb{M})$ is contained in a compact set, where $\mathbb{M}=\{u\in AP(\mathbb{T}):\mathbf{E}\|u\|^2\le R\}$, R is a fixed constant.*

Proof. First, by (19), we have

$$\begin{aligned} &\mathbf{E}\|(Cu)(t)\|^2\\ =\ &\mathbf{E}\|\int_{-\infty}^{t}X(t)PX^{-1}(\sigma(s))G(s,u(s))\Delta W(s)\\ &-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(\sigma(s))G(s,u(s))\\ &\times\Delta W(s)\|^2\\ \le\ & 2\mathbf{E}\|\int_{-\infty}^{t}X(t)PX^{-1}(\sigma(s))G(s,u(s))\\ &\times\Delta W(s)\|^2\\ &+2\mathbf{E}\|\int_{t}^{+\infty}X(t)(I-P)X^{-1}(\sigma(s))\\ &\times G(s,u(s))\Delta W(s)\|^2\\ \le\ & 2\int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\|^2\mathbf{E}\|G(s,u(s)\|^2\Delta s\\ &+2\int_{t}^{+\infty}\|X(t)(I-P)X^{-1}(\sigma(s))\|^2\\ &\times\mathbf{E}\|G(s,u(s))\|^2\Delta s\\ \le\ & 2\sup_{t\in\mathbb{T}}\mathbf{E}\|G(t,u(t))\|^2\bigg(\int_{-\infty}^{t}ke_{\ominus\alpha}(t,\sigma(s))\Delta s\\ &+\int_{t}^{+\infty}ke_{\ominus\alpha}(\sigma(s),t)\Delta s\bigg). \end{aligned}$$

By Lemma 2, we can get

$$\begin{aligned} &\int_{-\infty}^{t}ke_{\ominus\alpha}(t,\sigma(s))\Delta s+\int_{t}^{+\infty}ke_{\ominus\alpha}(\sigma(s),t)\Delta s\\ &\le k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}). \end{aligned}$$

Therefore,

$$\mathbf{E}\|(Cu)(t)\|^2\le k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})\sup_{t\in\mathbb{T}}\mathbf{E}\|G(t,u(t))\|^2. \tag{20}$$

Now, we show that C is continuous. In fact, let $u,v\in AP(\mathbb{T})$, for any $\varepsilon>0$, take $\delta=\frac{\varepsilon}{2kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})+1}$, whenever $\mathbf{E}\|u-v\|^2<\delta$, we have

$$\begin{aligned} &\mathbf{E}\|(Cu)(t)-(Cv)(t)\|^2\\ \le\ & 2\mathbf{E}\|\int_{-\infty}^{t}X(t)PX^{-1}(\sigma(s))[G(s,u(s))\\ &-G(s,v(s))]\Delta W(s)\|^2\\ &+2\mathbf{E}\|\int_{t}^{+\infty}X(t)(I-P)X^{-1}(\sigma(s))\\ &\times[G(s,u(s))-G(s,v(s))]\Delta W(s)\|^2\\ \le\ & 2\int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\|^2\\ &\times\mathbf{E}\|G(s,u(s))-G(s,v(s))\|^2\Delta s\\ &+2\int_{t}^{+\infty}\|X(t)(I-P)X^{-1}(\sigma(s))\|^2\\ &\times\mathbf{E}\|G(s,u(s))-G(s,v(s))\|^2\Delta s \end{aligned}$$

$$
\begin{aligned}
&\leq 2L_G \sup_{s\in\mathbb{T}} \mathbf{E}\|u(s)-v(s)\|^2 \\
&\quad \times\Big(\int_{-\infty}^{t} ke_{\ominus\alpha}(t,\sigma(s))\Delta s \\
&\quad + \int_{t}^{+\infty} ke_{\ominus\alpha}(\sigma(s),t)\Delta s\Big) \\
&\leq 2kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})\sup_{s\in\mathbb{T}}\mathbf{E}\|u(s)-v(s)\|^2 \\
&< \varepsilon.
\end{aligned}
$$

This proves that C is continuous.

Let $\mathbb{M}=\{u\in AP(\mathbb{T}):\mathbf{E}\|u\|^2\leq R\}$. Now, we show that the image of $C(\mathbb{M})$ is contained in a compact set. In fact, let u_n be a sequence in $\mathbb{M}$. In view of (13), we have

$$
\begin{aligned}
&\mathbf{E}\|G(t,u(t))\|^2 \\
\leq\ & 2\mathbf{E}\|G(t,u(t))-G(t,0)\|^2+2\mathbf{E}\|G(t,0)\|^2 \\
\leq\ & 2L_G\mathbf{E}\|u(t)\|^2+2a \\
\leq\ & 2L_GR+2a, \qquad (21)
\end{aligned}
$$

where $a=\sup_{t\in\mathbb{T}}\mathbf{E}\|G(t,0)\|^2$. From (20) and (21), we have

$$
\begin{aligned}
&\mathbf{E}\|(Cu_n)(t)\|^2 \\
&\leq k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})(2L_GR+2a):=L. \qquad (22)
\end{aligned}
$$

Next, we calculate $(Cu_n)^{\Delta}(t)$ and show that it is uniformly bounded. By a direct calculate, we have

$$
\begin{aligned}
&(Cu_n)^{\Delta}(t) \\
=\ & A(t)(Cu_n)(t)+X(t)PX^{-1}(\sigma(s))G(t,u_n(t)) \\
& -X(t)(I-P)X^{-1}(\sigma(s))G(t,u_n(t)). \qquad (23)
\end{aligned}
$$

Since $A(t)$ is a square-mean almost periodic function, then $A(t)$ is bounded. So, there exists a positive constant A_0, such that $\mathbf{E}\|A(t)\|^2\leq A_0$.

Together with (21), (22) and (23), then

$$
\begin{aligned}
&\mathbf{E}\|(Cu_n)^{\Delta}(t)\|^2 \\
\leq\ & 3[A_0L+(ke_{\ominus\alpha}(t,\sigma(s))+ke_{\ominus\alpha}(\sigma(s),t)) \\
& \times\mathbf{E}\|G(t,u_n(t))\|^2] \\
\leq\ & 3[A_0L+(k+k)(2RL_G+2a)] \\
\leq\ & 3[A_0L+2k(2RL_G+2a)].
\end{aligned}
$$

Thus the sequence (Cu_n) is uniformly bounded and equi-continuous. Hence, by the Arzela-Ascoli theorem, $C(\mathbb{M})$ is compact. The proof is completed. □

Theorem 17. *Assume that* $(H_1)-(H_3)$ *hold. Let* $a=\sup_{t\in\mathbb{T}}\mathbf{E}\|G(t,0)\|^2, b=\sup_{t\in\mathbb{T}}\mathbf{E}\|Q(t,0)\|^2$. *Let* R_0 *be a positive constant satisfies*

$$
\begin{aligned}
&4[L_QR_0+b+k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})(2L_GR_0+2a)] \\
&\leq R_0. \qquad (24)
\end{aligned}
$$

Then (1) has an almost periodic solution in $\mathbb{M}=\{u\in AP(\mathbb{T}):\mathbf{E}\|u\|^2\leq R_0\}$.

Proof. Define $\mathbb{M}=\{u\in AP(\mathbb{T}):\mathbf{E}\|u\|^2\leq R_0\}$. By Lemma 16, the mapping C defined by (19) is continuous and $C(\mathbb{M})$ is contained in a compact set. By Lemma 15, the mapping B defined by (18) is a contraction and it is clear that $B:AP(\mathbb{T})\to AP(\mathbb{T})$.

Next, we show that if $u,v\in\mathbb{M}$, we have $\mathbf{E}\|Bu+Cv\|^2\leq R_0$. In fact, let $u,v\in\mathbb{M}$ with $\mathbf{E}\|u\|^2,\mathbf{E}\|v\|^2\leq R_0$. Then

$$
\begin{aligned}
&\mathbf{E}\|Bu+Cv\|^2 \\
\leq\ & 4\mathbf{E}\|Q(t,u(t))-Q(t,0)\|^2+4\mathbf{E}\|Q(t,0)\|^2 \\
& +4\int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\| \\
& \times\mathbf{E}\|G(s,v(s))\|^2\Delta s \\
& +4\int_{t}^{+\infty}\|X(t)(I-P)X^{-1}(\sigma(s))\| \\
& \times\mathbf{E}\|G(s,v(s))\|^2\Delta s \\
\leq\ & 4L_Q\mathbf{E}\|u(t)\|^2+4b \\
& +4k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})(2L_GR+2a) \\
\leq\ & 4[L_QR_0+b+k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})(2L_GR_0+2a)] \\
\leq\ & R_0.
\end{aligned}
$$

Thus $Bu+Cv\in\mathbb{M}$. Hence all the conditions of Krasnoselskii's theorem are satisfied. Hence there exists a fixed point $z\in\mathbb{M}$, such that z=Bz+Cz. By Lemma 13, (1) has an almost periodic solution. The proof is completed. □

Theorem 18. *Assume that* $(H_1)-(H_3)$ *hold. If*

$$
[3(L_Q+kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))]^{\frac{1}{2}}<1, \qquad (25)
$$

then (1) has a unique almost periodic solution.

Proof. Let the mapping Φ be given by (14). For $u,v\in AP(\mathbb{T})$, in view of (14), we have

$$
\begin{aligned}
&\mathbf{E}\|(\Phi u)(t)-(\Phi v)(t)\|^2 \\
\leq\ & 3\mathbf{E}\|Q(t,u(t))-Q(t,v(t))\|^2 \\
& +3\int_{-\infty}^{t}\|X(t)PX^{-1}(\sigma(s))\|^2 \\
& \times\mathbf{E}\|G(s,u(s))-G(s,v(s))\|^2\Delta s
\end{aligned}
$$

$$\begin{aligned}
&+3\int_t^{-\infty}\|X(t)(I-P)X^{-1}(\sigma(s))\|^2\\
&\times\mathbf{E}\|G(s,u(s))-G(s,v(s))\|^2\Delta s\\
\le\;& 3L_Q\mathbf{E}\|u(t)-v(t)\|^2\\
&+3L_G\sup_{t\in\mathbb{T}}\mathbf{E}\|u(t)-v(t)\|^2\\
&\times\Big(\int_{-\infty}^t ke_{\ominus\alpha}(t,\sigma(s))\Delta s\\
&+\int_t^{+\infty} ke_{\ominus\alpha}(\sigma(s),t)\Delta s\Big)\\
\le\;& 3(L_Q+kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))\sup_{t\in\mathbb{T}}\mathbf{E}\|u(t)-v(t)\|^2,
\end{aligned}$$

that is

$$\begin{aligned}
&\|(\Phi u)(t)-(\Phi v)(t)\|_2^2\\
&\le 3(L_Q+kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))\sup_{t\in\mathbb{T}}\|u(t)-v(t)\|_2^2.
\end{aligned}$$

Note that

$$\sup_{t\in\mathbb{T}}\|u(t)-v(t)\|_2^2\le(\sup_{t\in\mathbb{T}}\|u(t)-v(t)\|_2)^2.$$

Then

$$\begin{aligned}
&\|(\Phi u)(t)-(\Phi v)(t)\|_2\\
\le\;& [3(L_Q+kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))]^{\frac{1}{2}}\sup_{t\in\mathbb{T}}\|u(t)-v(t)\|_2\\
=\;& [3(L_Q+kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))]^{\frac{1}{2}}\|u-v\|_\infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\Phi u-\Phi v\|_\infty\\
&\le[3(L_Q+kL_G(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))]^{\frac{1}{2}}\|u-v\|_\infty.
\end{aligned}$$

This completes the proof by invoking the contraction mapping principle. □

4 Examples

Example 1. For small positive ε_1 and ε_2, we consider the stochastic Van Der Pol equation

$$\begin{aligned}
&x^{\Delta\Delta}+(\varepsilon_2x^2-1)x^\Delta+x-\varepsilon_1(x^2\sin t)^\Delta\\
&-\varepsilon_2\cos tW^\Delta(t)=0,\ t\in\mathbb{T}. \qquad (26)
\end{aligned}$$

Using the transformation $x_1^\Delta=x_2$, we can transform the equation (26) to

$$\begin{aligned}
\begin{pmatrix}x_1\\x_2\end{pmatrix}^\Delta =\;& \begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}0\\\varepsilon_1x_1^2\sin t\end{pmatrix}^\Delta\\
&+\begin{pmatrix}0\\\varepsilon_2\cos t-\varepsilon_2x_2x_1^2\end{pmatrix}W^\Delta(t),
\end{aligned}$$

that is $A=\begin{pmatrix}0&1\\-1&1\end{pmatrix}$, $Q(t,x(t))=\begin{pmatrix}0\\\varepsilon_1x_1^2\sin t\end{pmatrix}$, $G(t,x(t))=\begin{pmatrix}0\\\varepsilon_2\cos t-\varepsilon_2x_2x_1^2\end{pmatrix}$.

Since the real part of the eigenvalues of A is nonzero, by Remark 11, we see that $x^\Delta(t)=A(t)x(t)$ admits exponential dichotomy. Let $\phi(t)=(\phi_1(t),\phi_2(t))$, $\varphi(t)=(\varphi_1(t),\varphi_2(t))$. Define $\mathbb{M}=\{u\in AP(\mathbb{T}):\mathbf{E}\|u\|^2\le R_0\}$, where R_0 is a positive constant.

Then for $\phi,\varphi\in\mathbb{M}$, we have

$$\mathbf{E}\|Q(t,\phi(t))-Q(t,\varphi(t))\|^2\le 2\varepsilon_1R_0\mathbf{E}\|\phi(t)-\varphi(t)\|^2,$$

and

$$\begin{aligned}
&\mathbf{E}\|G(t,\phi(t))-G(t,\varphi(t))\|^2\\
=\;&\varepsilon_2\mathbf{E}\Big\|(\phi_2(t)(\phi_1(t)+\varphi_1(t)),\varphi_1^2(t))\\
&\times\begin{pmatrix}\phi_1(t)-\varphi_1(t)\\\phi_2(t)-\varphi_2(t)\end{pmatrix}\Big\|^2\\
\le\;&2\varepsilon_2R_0^2\mathbf{E}\|\phi(t)-\varphi(t)\|^2.
\end{aligned}$$

Let

$$\begin{aligned}
&L_Q=2\varepsilon_1R_0, L_G=2\varepsilon_2R_0^2,\\
&a=\sup_{t\in\mathbb{T}}\mathbf{E}\|G(t,0)\|^2=\varepsilon_2,\\
&b=\sup_{t\in\mathbb{T}}\mathbf{E}\|Q(t,0)\|^2=0,
\end{aligned}$$

then, inequality (24) becomes

$$4[2\varepsilon_1R_0^2+k(\frac{1}{\alpha}-\frac{1}{\ominus\alpha})(4R_0^3+2)\varepsilon_2]\le R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 17, (26) has an almost periodic solution.

Moreover,

$$[3(2\varepsilon_1R_0+2k\varepsilon_2R_0^2(\frac{1}{\alpha}-\frac{1}{\ominus\alpha}))]^{\frac{1}{2}}<1$$

is also satisfied for small ε_1 and ε_2. By Theorem 18, (26) has a unique almost periodic solution.

Example 2. For small positive ε_1 and ε_2, we consider the stochastic integro equation

$$\begin{aligned}
x^\Delta =\;& x-\int_{-\infty}^t e_{-0.5}(t,\sigma(s))x(s)\Delta s\\
&+\varepsilon_1[(\int_{-\infty}^t e_{-0.5}(t,\sigma(s))x(s)\Delta s)^2\sin t]^\Delta\\
&+\varepsilon_2[\cos t-x(\int_{-\infty}^t e_{-0.5}(t,\sigma(s))x(s)\Delta s)^2]\\
&\times W^\Delta(t),\ t\in\mathbb{T}. \qquad (27)
\end{aligned}$$

Now, we define a new variable

$$x_1(t) = \int_{-\infty}^{t} e_{-0.5}(t,\sigma(s))x_2(s)\Delta s, t \in \mathbb{T}, \quad (28)$$

then by Lemma 1, system (27) can be transformed into the following system

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\Delta} &= \begin{pmatrix} -0.5 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 x_1^2 \sin t \end{pmatrix}^{\Delta} \\ &\quad + \begin{pmatrix} 0 \\ \varepsilon_2 \cos t - \varepsilon_2 x_2 x_1^2 \end{pmatrix} W^{\Delta}(t),\end{aligned}$$

that is $A = \begin{pmatrix} -0.5 & 1 \\ -1 & 1 \end{pmatrix}$, $Q(t,x(t)) = \begin{pmatrix} 0 \\ \varepsilon_1 x_1^2 \sin t \end{pmatrix}$, $G(t,x(t)) = \begin{pmatrix} 0 \\ \varepsilon_2 \cos t - \varepsilon_2 x_2 x_1^2 \end{pmatrix}$.

Since the real part of the eigenvalues of A is nonzero, by Remark 11, we see that $x^{\Delta}(t) = A(t)x(t)$ admits exponential dichotomy. Let $\phi(t) = (\phi_1(t), \phi_2(t))$, $\varphi(t) = (\varphi_1(t), \varphi_2(t))$. Define $\mathbb{M} = \{u \in AP(\mathbb{T}) : \mathbf{E}\|u\|^2 \le R_0\}$, where R_0 is a positive constant.

Then for $\phi, \varphi \in \mathbb{M}$, we have

$$\mathbf{E}\|Q(t,\phi(t)) - Q(t,\varphi(t))\|^2 \le 2\varepsilon_1 R_0 \mathbf{E}\|\phi(t) - \varphi(t)\|^2,$$

and

$$\begin{aligned} & \mathbf{E}\|G(t,\phi(t)) - G(t,\varphi(t))\|^2 \\ = & \ \varepsilon_2 \mathbf{E}\left\| (\phi_2(t)(\phi_1(t) + \varphi_1(t)), \varphi_1^2(t)) \right. \\ & \left. \times \begin{pmatrix} \phi_1(t) - \varphi_1(t) \\ \phi_2(t) - \varphi_2(t) \end{pmatrix} \right\|^2 \\ \le & \ 2\varepsilon_2 R_0^2 \mathbf{E}\|\phi(t) - \varphi(t)\|^2. \end{aligned}$$

Let

$$\begin{aligned} L_Q &= 2\varepsilon_1 R_0, L_G = 2\varepsilon_2 R_0^2, \\ a &= \sup_{t\in\mathbb{T}} \mathbf{E}\|G(t,0)\|^2 = \varepsilon_2, \\ b &= \sup_{t\in\mathbb{T}} \mathbf{E}\|Q(t,0)\|^2 = 0, \end{aligned}$$

then, inequality (24) becomes

$$4[2\varepsilon_1 R_0^2 + k(\frac{1}{\alpha} - \frac{1}{\ominus\alpha})(4R_0^3 + 2)\varepsilon_2] \le R_0,$$

which is satisfied for small ε_1 and ε_2. By Theorem 17, (27) has an almost periodic solution.

Moreover,

$$[3(2\varepsilon_1 R_0 + 2k\varepsilon_2 R_0^2(\frac{1}{\alpha} - \frac{1}{\ominus\alpha}))]^{\frac{1}{2}} < 1$$

is also satisfied for small ε_1 and ε_2. By Theorem 18, (27) has a unique almost periodic solution.

5 Conclusion

This paper is focused on the existence of square-mean almost periodic solutions of neutral stochastic functional differential equations on time scales. Based on the properties of almost periodic function and exponential dichotomy of linear system on time scales as well as Krasnoselskii's fixed point theorem, some sufficient conditions are obtained.

The results obtained in this paper can be applied to the analysis of many other periodic and almost periodic dynamical systems, one may consider the systems which have been studied in [26-28].

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