New Periodic Exact Solutions of the Kuramoto-Sivashinsky Evolution Equation

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Abstract: In the present paper, three families of exact periodic localized solutions of the popular Kuramoto-Sivashinsky model evolution partial differential equation have been obtained. Similar exact solutions have not been published so far. The exact solutions found are cnoidal, sinusoidal and a solitary-wave one, which were established to be dynamically equivalent. To obtain them a spatial modification of the Hirota-Matsuno bilinear transformation method has been applied. The non-integrability of the evolution equation under consideration generates specific dynamic phenomena – the individual spatial displacements, defined exactly for each separate harmonics in the localized periodic solutions.

Key–Words: Hirota bilinear operators, Hirota-Matsuno bilinear transformation method, Weierstrass elliptic functions, Jacobi elliptic functions, Jacobi Theta functions, phase modulations of elliptic functions

1 Introduction

The Kuramoto-Sivashinsky nonlinear evolution equation (KSE) has the form

$$u_t + uu_x + \alpha\, u_{xx} + \beta\, u_{xxx} + \gamma\, u_{xxxx} = 0, \qquad (1)$$

and describes the elevation of a viscous fluid flowing down an inclined surface (when $\beta \neq 0$), and in the case $\beta = 0$, it describes the elevation of a vertically flowing film. The fluid viscosity, its surface tension and the gravitational force are reflected in the equation through the values of the real parameters α, β, γ. This equation was introduced by Kuramoto and Tsuzuki [1] and has been an object of study for the last 36 years, both theoretically and experimentally, by dozens, even hundreds of authors. In view of the nature of this paper, we will emphasize on those theoretic studies of equation (1) that have obtained and analyzed its exact solutions.

Kuramoto and Tsuzuki [1] have obtained for the first time a solitary-wave solution of KSE in the 'vertical' variant of the equation (i.e. for $\beta = 0$). Their solution has the form of a one-parametric family of shock waves growing from the superposition of solitary forms of the type $\sim\tanh^m \xi$, i.e.

$$\begin{aligned} u(x,t) &= 15\gamma[k\tanh(k\xi/2)]^3 \\ &+ \frac{k}{2}\left(\frac{60}{19}\alpha - 30\gamma\, k^2\right)\tanh(k\xi/2) + c \end{aligned}$$

where $\xi = x - ct - \delta$, $c = const$, $\delta = const$, and

$$k = \sqrt{11\alpha/19\gamma} \quad \text{or} \quad k = \sqrt{-\alpha/19\gamma}$$

Later, in 1988, Kudryashov [2], as well as R. Conte and M. Musette [3] used one and the same approach–the Penleve method and Beklund transformation of the unknown function $u(x,t)$ in the KSE equation, employing different variants of that same approach. Kudryashov [2] obtained an exact solitary solution for $\beta \neq 0$, while Conte and Musette [3] studied equation (1) in the vertical variant$\beta = 0$. As a generalization of the exact meromorphic solutions of the evolution equation (1) mentioned above, we can refer to the paper [4] where Eremenko pointed out that elliptic (Weierstrass) solutions of the KSE equation exist only if $\beta^2 = 16\alpha\gamma$, while the exponential solutions of this equation have the form $P_m(\tanh\xi)$, where P_m is a polynomial of degree at most three, which corresponds to the degree of singularity of equation (1). As regards the rational solutions, the author of [4] has proved that they exist only if $\beta = \alpha = C = 0$, where C is the integration constant obtained under the single integration of equation (1) along the phase variable $\xi = kx + \omega t + \delta$, and these rational solutions have the form $u(\xi) = 120\gamma(\xi - \xi_0)^{-3}, \xi_0 \in C$.

Regarding the periodic solutions of the evolution equation KSE, we can mention the periodic pulse solution obtained by N. Berloff and L. Howard [5], using the singular manifold method and the elliptic solution of Kudryashov [6] established by employing the mapping and deformation approach.

The bilinear transformation method of Matsuno [7] has not been applied so far for finding exact periodic solution of the model evolution equation KSE. The reason lies in the fact that equation (1) is non-integrable. Within the context of the bilinear transformations of non-linear partial differential equations, where they are applied to a non-integrable model equation, it is represented as a conjunction of one bilinear equation of the form $F(D_t, D_x)f.f = 0$ and one (or more) residual equation $G(f^2, D_x^2 f.f,\, D_x^4 f.f, ...) = 0$, where $F(X, Y)$ is a polynomial of two variables with respect to Hirota [8] bilinear operators. The difficulties arise from the fulfillment of the residual equation (or equations), as the index parity principle [9] is not applicable to equations not having a bilinear structure.

A spatial modification of the bilinear transformation method has been applied in the present paper, thus overcoming the difficulties related to the fulfillment of residual equations. The formal mathematical manipulations used here, have a clear and logical physical interpretation - in the case of non-integrable model equations, the spatial displacements generated are individual for each separate harmonics of the periodic solution. A similar spatial modification of the bilinear transformation method has been also applied to other non-integrable model equations, such as the sixth-order generalized Boussinesq equation (SGBE) [10], the convective fluid equation (CFE) [11] and the Kawahara equation (KE) [12].

2 Biperiodic Solutions

Instead of equation (1) we will analyze the equation

$$u_t + uu_x + u_{xx} + \mu\, u_{xxx} + u_{xxxx} = 0, \quad (2)$$

which is an exact reduction of the initial equation (1) by rescaling the variables

$$t \to \left(\frac{\gamma}{\alpha^2}\right) t;\;\; x \to \sqrt{\frac{\gamma}{\alpha}}\, x;\;\; u \to \frac{1}{\alpha}\sqrt{\frac{\gamma}{\alpha}}\, u;$$

$$\mu = \alpha\beta\sqrt{\frac{\alpha}{\gamma}}\,.$$

Without limiting the generality we can assume $\gamma > 0$. In case that $\alpha < 0$ in equation (1), we use the rescaling

$$t \to \left(\frac{\gamma}{\alpha^2}\right) t;\;\; x \to \sqrt{-\frac{\gamma}{\alpha}}\, x;\;\; u \to -\frac{1}{\alpha}\sqrt{-\frac{\gamma}{\alpha}}\, u;$$

$$\mu = -\alpha\beta\sqrt{-\frac{\alpha}{\gamma}}$$

We will seek a classic, localized, periodic solution of the evolution equation (2), i.e., a solution $u(x, t)$, defined in a space with strong topology. Specifically in our case this means, that the functions $u(x, t)$, u_t, u_{xx}, u_{xxx}, u_{xxxx} have to be defined in an open two-dimensional domain: $\Omega = \{(x, t) \in \Re^2,\, -\infty < x < \infty,\, 0 < t < \infty\}$ and be continuous therein. Let us represent the solution of equation (2) through the Hirota-Satsuma [12] transformation

$$u(x, t) = 4a + \lambda(\ln \zeta)_{xx}, \quad (3)$$

where the spatial displacement a, as well as λ, are unknown parameters at this stage, while $\zeta = \zeta(x, t)$ is an unknown function of the class $C^6(\Omega)$. If we substitute $u(x, t)$ from equation (3) in (2),

$$(\ln\zeta)_{txx} + [4a + \lambda(\ln\zeta)_{xx}](\ln\zeta)_{xxx} + (\ln\zeta)_{xxxx} + \\ +\mu(\ln\zeta)_{xxxxx} + (\ln\zeta)_{xxxxxx} = 0,$$

after integrating once along the axial variable x and applying the bilinear logarithmic identities for Hirota operators [13], (See Appendix A), we will obtain the following bilinear equation

$$\begin{aligned}&\tfrac{1}{2\zeta^2}[D_tD_x + 4aD_x^2 + \mu D_x^4 - 8B]\zeta.\zeta \\ &+\left(\tfrac{\lambda}{2} - 6\right)\left(\tfrac{D_x^2\zeta.\zeta}{2\zeta^2}\right)^2 + \\ &+\tfrac{\partial}{\partial x}\left[\tfrac{D_x^2\zeta.\zeta}{2\zeta^2} + \tfrac{D_x^4\zeta.\zeta}{2\zeta^2} - 6\left(\tfrac{D_x^2\zeta.\zeta}{2\zeta^2}\right)^2 - C\right] = 0\end{aligned} \quad (4)$$

By D_x^m, $m = 1, 2, ...$ is denoted the Hirota bilinear operator, defined on the diagonals $x = x'$, $t = t'$ by the equality [14],

$$D_t^m D_x^n \varphi(x,t)\psi(x,t) \\ = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \varphi(x,t)\psi(x',t')|_{x=x';t=t'}$$

$m,\, n \in N$, where$B$ is a total integration constant (more precisely, $B = \frac{1}{8}\left(B_0 - \frac{2a^2}{3}\right)$, $B_0 = const$), and C will be called a bidifferential constant. As will be explained further, these two constants play a key role for generating a localized periodic solution of the equation under consideration. As we are free to choose the parameter λ, we will set $\lambda = 12$, for which the bidifferential equation (4) is represented as a conjunction of the following two equations:

$$\left(D_tD_x + 4aD_x^2 + \mu D_x^4 - 8B\right)\zeta.\zeta = 0 \quad (5)$$

$$\zeta^2(D_x^2\zeta.\zeta) + \zeta^2(D_x^4\zeta.\zeta) - 3(D_x^2\zeta.\zeta)^2 = 2C\zeta^4 \quad (6)$$

The first of these two equations (5) possesses all characteristic features of the bilinear equations, which gives us grounds to call it bilinear ([9], [15]), while equation (6), which does not have a bilinear structure, will be termed as residual. A sufficiently smooth function $\zeta(x,t)$ would be an exact localized solution of the evolution equation (2) according to the transformation (3), if it satisfies both the bilinear and the residual equations (5) and (6). We will parameterize the unknown function $\zeta(x,t)$ with the help of the fourth Jacoby θ- function [16], namely,

$$\zeta(x,t) = \theta_4(\xi, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2in\xi}, \qquad (7)$$

where $\xi = \xi(x,t)$ denotes the phase variable $\xi = kx + \omega t + \delta$. The wave number$k$, the phase velocity ω, and the phase shift δ could assume complex values in the general case, but they are unknown parameters at this stage. The function $\theta_4(\xi, q)$ is infinitely smooth and biperiodic – with period π (with respect to the variable ξ), and with period 2 (with respect to the perturbation parameter$q$). This function is well defined for each $\xi \in C$ if $Im\tau > 0,$ where $q = e^{i\pi\tau}$, i.e. if $0 < |q| < 1$. When substituting $\zeta(x,t)$ from equality (7) in the bilinear equation (5) and applying the property of the D-operators (See [14]),

$$D_x^m e^{k_1 x}.e^{k_2 x} = (k_1 - k_2)^m e^{(k_1+k_2)x}, \quad k_1 \neq k_2, \; m \in N$$

we will obtain the following infinite system $\sum_{m=-\infty}^{\infty} F(m)\, e^{2im\xi} = 0$, where

$$F(m) = \sum_{n=-\infty}^{\infty} q^{n^2+(n-m)^2}[-4k\omega(2n-m)^2 + 16\alpha k^2(2n-m)^2 + 16\mu k^4(2n-m)^4 - 8B] \qquad (8)$$

At first sight, the infinite algebraic system $F(m) = 0,\, m = 0, \pm 1, \pm 2, ...$ is indefinite since the unknown parameters therein are only four - $k,\, \omega,\, a,\, B$, but its bilinear structure allows us to apply the index parity principle. This means that if we apply in (8) a finite number of reductions of the type $n \to n+1$, we will obtain

$$F(m) = F(m-2)q^{2(m-1)} = F(m-4)q^{2(2m-4)} = \begin{cases} F(0)q^{m^2/2}, & \text{if } m \text{ is an even number;} \\ F(1)q^{(m^2-1)/2}, & \text{if } m \text{ is an odd number} \end{cases}$$

In other words, if in (8) we do the summing with respect to n, taking into account the above relations we will obtain the following compact form:

$$F(m) = F(0)\theta_3(2\xi, q^2) + F(1)\theta_2(2\xi, q^2)q^{-1/2} = 0,$$

where $\theta_2(z,q)$ and $\theta_3(z,q)$ are the second and third Jacobi θ–functions [16], respectively. They are defined by the equalities

$$\theta_2(z,q) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} e^{i(2n-1)z};$$

$$\theta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz};$$

i.e., the infinite algebraic system $F(m) = 0,\; m = 0,\, \pm 1,\, \pm 2, ...$ or the infinite algebraic system (8) is reduced to two equations: $F(0) = 0,\; F(1) = 0$. By the help of the functional identities for θ– functions (See Appendix B), they will assume the following form:

$$\begin{cases} (kq\theta_3')\omega + \theta_3 B = 8\mu q k^4(\theta_3' + q\theta_3'') - 4ak^2 q\theta_3'; \\ (kq\theta_2')\omega + \theta_2 B = 8\mu q k^4(\theta_2' + q\theta_2'') - 4ak^2 q\theta_2'. \end{cases} \qquad (9)$$

The algebraic system (9) is linear and non-homogeneous with respect to the unknown parameters $\omega,\, a,\, B$, while $k,\, q,\, a$ will be base parameters. We have used the denotation $\theta_j = \theta_j(0, q^2), \quad j = 2,\, 3$ and the symbols for differentiation in the θ– functions are with respect to the perturbation parameter q. The algebraic system (9) is definite and compatible because if $\Delta(k,q)$ is its determinant, then obviously we have:

$$\Delta(k,q) = kq(\theta_2\theta_3' - \theta_2'\theta_3) = kqW(\theta_2, \theta_3) \neq 0$$

since $k \neq 0,\; |q| \in (0,1)$, and $W(\theta_2, \theta_3)$ is the Wronski determinant for the linearly independent functions

$$\theta_2(0, q^2) = 2\sqrt{q}(1 + q^4 + q^6 + ...),$$

$$\theta_3(0, q^2) = 1 + 2q^2 + 2q^8 + 2q^{18} + ..$$

Its only solutions are as follows:

$$\omega(k,q) = 8\mu k^3\left[1 + q\frac{W'(\theta_2, \theta_3)}{W(\theta_2, \theta_3)}\right] - 4ak \qquad (10)$$

$$B(k,q) = 8\mu q^2 k^4 \frac{W(\theta_2', \theta_3')}{W(\theta_2, \theta_3)}. \qquad (11)$$

Let us note that the dispersion relation (10) specifies the dynamics of dispersive waves, on one hand, because $\omega''(k) \neq 0$ and on the other hand – that these periodic dispersive waves are generated only if

$$a \neq 2\mu\, k^2\left[1 + q\frac{W'(\theta_2, \theta_3)}{W(\theta_2, \theta_3)}\right],$$

The integration constant $B(k,q)$, which is determined by the equality (11), even not having dynamic characteristics, possesses a structural determining role with respect to the seeking localized periodic solutions of equation (2). In case that $B(k,q) \neq 0$ for the eligible values of the parameters k and q, then the Kuramoto-Sivashinsky equation could have such periodic solutions. Otherwise a possibility arises for generation of solitary or some other kind of localized solutions. It is obvious from (11) that for $k > 0$ and $|q| \in (0,1)$, the integration constant $B(k,q) \neq 0$, since $W(\theta_2', \theta_3') \neq 0$.

To satisfy the residual equation (6) with the biperiodic function $\zeta(x,t) = \theta_4(\xi, q)$, under the condition of the equality (10), it is necessary to restructure the "differential" constant C since it is not directly related to the alleged localized solution (3). The freedom in the choice of C, allows binding this artificially included constant with the spatial displacement a, and for convenience we will use the simplest relation – the linear, setting $2C = a$. Here a is represented in the formal functional series

$$a = \sum_{m=-\infty}^{\infty} a_m(k,q), \tag{12}$$

as at this stage $a_m(k,q)$ are unknown. After applying the Cauchy formula for multiplication of infinite series

$$\left(\sum_{s=-\infty}^{\infty} A_s\right)\left(\sum_{\nu=-\infty}^{\infty} B_\nu\right) = \sum_{m,n=-\infty}^{\infty} A_{n-m}B_m,$$

the residual equation (6), in which we have substituted a from (12), will assume the following structure:

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\left(a_m \sum_{n=-\infty}^{\infty} q^{2(n-m)^2+(2n-3m)^2}\right) e^{2im\xi} \\ &= \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\{4k^4[(2n-m)^4 - 3m^2(2n-3m)^2] \right. \\ &\left. -k^2(2n-m)^2\}\, q^{2n^2+(2n-m)^2}\right) e^{2im\xi}\end{aligned} \tag{13}$$

The natural limitation for the perturbation parameter q is $0 < |q| < 1$, which assures the uniform convergence of $\theta_j(\xi,q)$, $j = 1, 2, 3, 4$, for every $\xi \in C$. But for given limitation on q, the coefficients of the two infinite series on both sides of equality (13), are absolutely convergent (the internal sums with respect to n), which gives grounds to draw the next conclusion. Namely, if we define the spatial displacements by the equality

$$\begin{aligned}&a_m(k,q) = \\ &\frac{\sum_{n=-\infty}^{\infty} 4k^4[(2n-m)^4 - 3m^2(2n-3m)^2]q^{2n^2+(2n-m)^2}}{\sum_{n=-\infty}^{\infty} q^{2(n-m)^2+(2n-3m)^2}} \\ &- \frac{\sum_{n=-\infty}^{\infty} k^2(2n-m)^2 q^{2n^2+(2n-m)^2}}{\sum_{n=-\infty}^{\infty} q^{2(n-m)^2+(2n-3m)^2}}\end{aligned} \tag{14}$$

$m = 0, \pm1, \pm2$. then the residual equation (6) is satisfied by the function $\zeta(x,t) = \theta_4(\xi,q)$, moreover the parameters $a_m(k,q)$ defined by equality (14) are members of an absolutely convergent series for every $k \neq 0$, which sum is a, i.e.

$$a = ...a_{-1}(k,q) + a_0(k,q) + a_1(k,q) + ...$$

Satisfying the bilinear equation (5) and the residual equation (6) with the parameterized function $\zeta(x,t) = \theta_4(\xi,q)$, under the conditions of the dispersion relation (10) and the condition for the integration constant (11), allows us to conclude that the function

$$u(x,t) = 4\sum_{m=-\infty}^{\infty} a_m(k,q) + 12\frac{\partial^2}{\partial x^2}[\ln\theta_4(\xi,q)] \tag{15}$$

is an exact localized periodic solution of the evolution non-linear KSE equation (2) provided the spatial displacements $a_m(k,q)$, $m = 0, \pm1, \pm2, ...$ are defined as in equality (14).

3 Criteria for Analyticity and Real Periodic Solutions

The solution (15), which is an exact localized periodic solution of the KS equation, is a meromorphic complex function with twofold poles in the lattice $\xi_{mn} = m + i(n + 1/2)Im\tau$. These circumstances make the considered periodic solution with small practical applicability. However, we could choose the free parameters k, q, δ so that the phase variable ξ to be real, together with the spatial displacements $a_m(k,q)$, $m = 0, \pm1, \pm2, ...$

For this purpose, we will assume that

$$\tau = i\varepsilon,\ \varepsilon > 0,\ \text{i.e.,} q = e^{-\pi\varepsilon} < 1 \tag{16}$$

which, according to (14), provides $a_m(k,q)$ to be real numbers if the wave number is real, i.e. $k > 0$ and for a real phase shift δ, the phase variable $\xi = kx + \omega t + \delta$ is also real (See (10)). Having these limitations for the parameters, if we restrict the phase variable within the horizontal stripe

$$-\pi\varepsilon < Im(\xi) < \pi\varepsilon \tag{17}$$

then this domain (for arbitrary k, q, δ) would be a domain of analicity for the localized periodic solution (15), since for the hypothesis (17) the poles ξ_{mn} will be avoided. Within the conditions of (16) and (17), if we choose real values for the parameters k, $k > 0$ and δ, and apply the formula (Appendix **C**)

$$\frac{\partial^2}{\partial x^2}[\ln\theta_4(\xi,q)] = k^2\mu_0^2\mathrm{cn}^2(\xi_0,\mu_0) - b(\varepsilon), \tag{18}$$

where

$$\xi_0 = kx + (\omega\pi\theta_3^2)\, t + \delta;\ \ \mu_0 = \theta_2^2/\theta_3^2;$$

$$b(\varepsilon) = \frac{1}{3}\left(2\mu^2 - 1 + \frac{\theta_1^{'''}}{\pi^2\theta_3^4\theta_1^{'}}\right),$$

then from (15) we will obtain the following real, conoidal, localized solution of equation (1):

$$u(x,t) = 4\left[a(k,\varepsilon) - 3b(\varepsilon)\right] + 12k^2\mu_0^2 cn^2(\xi_0, \mu_0) \tag{19}$$

where

$$a(k,\varepsilon) = \sum_{m=-\infty}^{\infty} a_m(k,\varepsilon).$$

The period of the conoidal wave (referring to the real period) is $4K_1(\mu_0)$, thus coinciding with the waves' length, and their crests are at the points $0,\ \pm 4K_1\varepsilon,\ \pm 8K_1\varepsilon, ...$

We could obtain much more interesting localized periodic solutions from 15 under the hypothesis of (16) and (17) and for real$k,\ k > 0$ and δ, if we represent the following logarithmic derivative in the form of Fourier series [16]:

$$\begin{aligned}\tfrac{\theta_4'(\xi,q)}{\theta_4(\xi,q)} &= 4\sum_{m=1}^{\infty} \tfrac{q^m}{1-q^{2m}} \sin\, 2m\xi \\ &= 2\sum_{m=1}^{\infty} \cos\, ech(\varepsilon\pi\, m).\sin(2m\xi)\end{aligned}$$

which allows us to obtain a periodic real function well-defined in the stripe 17:

$$\begin{aligned}u(x,t) = 4 \sum_{m=-\infty}^{\infty} &\left[a_m\ (k,\varepsilon)\right. \\ &\left.+6k^2 m\cos ech(\varepsilon\pi\ m).\cos(2m\xi)\right]\end{aligned} \tag{20}$$

This localized periodic solution represents a non-linear superposition of sinusoidal harmonics having individual spatial displacements $a_m(k,\varepsilon)$ defined by equalities (14). We can say that the solution (20) covers the case "small amplitude", i.e. for $\varepsilon \to 0$ (or $q \to 1$). These areas are dominated by linear phenomena, and non-linear effects are weak.

To achieve validity of the Fourier representation for the logarithmic derivative in (15), in the strongly nonlinear areas $\varepsilon \to \infty$ (i.e.$q \to 0$), we will define under the condition (16), a new perturbation parameter $q_0 = e^{i\pi\tau_0}$, where $\tau_0 = -1/\tau$ or $q_0 = e^{-\pi/\varepsilon}$. This is done to make the boundary transition $q \to 1$ equivalent to the transition $q_0 \to 0$. After a first degree transformation [16] of $\theta_4(\xi,q)$:

$$\theta_4(\xi,q) = (-i\tau_0)^{1/2} e^{i\tau_0\,\xi^2/\tau}\theta_2(\tau_0\xi, q_0)$$

and in conjunction with the formula for the logarithmic derivative (See Appendix D):

$$\frac{d^2}{dz^2}\left[\ln\theta_2(z,q)\right] = -\sum_{m=-\infty}^{\infty} \operatorname{sech}^2\left[i(z - m\pi\tau)\right], \tag{21}$$

we reduce the solution (15), after applying the linear rescaling: $k \to k\varepsilon$, $\omega \to \omega\varepsilon$, $\delta \to \delta\varepsilon$, (i.e., $\xi \to \xi\varepsilon$) to the form

$$\begin{aligned}u(x,t) = &-\tfrac{24k^2\varepsilon}{\pi} + \\ &+4\sum_{m=-\infty}^{\infty}\left[a_m(k,\varepsilon) + 3k^2 \sec h^2(\xi - m\pi/\varepsilon)\right]\end{aligned} \tag{22}$$

where the spatial displacements$a_m(k,\varepsilon)$, $m = 0,\ \pm 1,\ \pm 2, ...$ are real again and defined by (14). The function$u(x,t)$, defined by 22, is an infinite sum of solitary-wave profiles whose crests are at the points$\xi_m = 0,\ \pm\pi/\varepsilon,\ \pm 2\pi/\varepsilon ... 2m\pi/\varepsilon, ..,$ and the wave lengths are $\pi/k\varepsilon$. The solution (22) is valid in the areas where non-linear effects are most evident and sensible, i.e. for $\varepsilon \to \infty$. These non-linear effects generate a process of localization because the overlapping zones are decreasing.

The solitary-wave solution (22) reveals two important dynamic features of the phenomenon, described by the Kuramoto-Sivashinsky equation. The first one is that the solitary wave is manifested as a "model" function, which by repeating its values at equal intervals, generates the periodic wave for the whole spectrum of the perturbation parameter q in the interval (0, 1). The second dynamic feature of the solitary-wave solution is related to the possibility the sum of all individual spatial displacements$a_m(k,\varepsilon)$, $m = 0,\ \pm 1,\ \pm 2, ...$ to compensate for the difference in the velocities of the cnoidal and the solitary wave.

4 Conclusion

In our analysis of the Kuramoto-Sivashinsky equation, striving to find exact localized periodic solutions, we have used a spatially modified variant of the bilinear transformation method of Matsuno [7]. Previous experience shows ([10, 11, 12]) that the non-linear partial differential equations that are non-integrable have individual spatial displacements for each harmonics in their periodic solutions. As a rule, the non-integrable equations have a residual equation (equations), which does not possess a bilinear structure of the form $F(D_t, D_x)\zeta.\zeta = 0$, where F is a polynomial of two variables or an exponential function. In essence, these residual equations represent an infinite system of algebraic equations having a limited resource of unknown parameters. However, the problem of satisfying the residual equations does not consist in the formal representation of infinite systems into a closed form, but rather lies in the interpretations of the results obtained, from the physics perspective. For example, in the residual equation of the model equation under consideration (6) – there is only one "differential" constant C figuring as a potential resource for its satisfaction. If this constant was formally represented into the form of an infinite numerical series with unknown members, inevitably the question would arise, what physical interpretation is needed for a parameter which moreover does not participate in the solution. In such cases it is necessary to seek a linear or non-linear dependence between the differential parameter and the spatial one, and here, in particular, we have chosen the simple dependency $2C = a$. Yet it is reassuring that not all model non-integrable equations necessitate the introduction of similar differential constants. They have to be introduced only in the cases when the model partial differential equation has at least one member with an even derivative, with respect to the spatial variable. In the case of equation (1), this is the term γu_{xxxx}.

The obtained localized, exact periodic solutions of the non-integrable evolution Kuramoto-Sivashinsky equation, namely, the cnoidal (18), sinusoidal (20), and the solitary-wave (22) one, are dynamically equivalent since they are derived from one generalized complex meromorphic solution (15). However, there is one additional dynamic circumstance between the sinusoidal and the solitary-wave solution, related to the possibility these waves to achieve velocity, coinciding with the velocity of the soliton or solitary impulse. Provided this coincidence can occur, the infinite sum of solitary-wave profiles (of the type$\sec h^2$) in (22), with their individual spatial displacements, can be interpreted as a real linear superposition of solitary waves. In other words the soliton (or solitary) impulse in this case acts as a wrapping of the periodic solitary-wave forms. Since the KS equation is non-integrable, it does not have a one-soliton solution. Hence we can use the velocity $V_S = const$

of the solitary wave obtained analytically by Kuramoto and Tsuzuki [1]. It is obvious that if we choose this constant so that it coincides with the velocity $V_0$of the solitary-wave profiles (22), i.e.,

$$V_S = V_0 = \delta\mu k^2 \left[1 + qW'(\theta_2,\theta_4)/W(\theta_2,\theta_4)\right] - 4a.$$

(See (10)), the solitary wave will be found to be a wrapping of the solitary-wave profiles (22).

Appendix A

Logarithmic derivatives expressed by the Hirota's bilinear differential operators D_t, D_x

$$(\ln\zeta)_{tt} = \frac{D_t^2\zeta.\zeta}{2\zeta^2};\quad (\ln\zeta)_{xx} = \frac{D_x^2\zeta.\zeta}{2\zeta^2};$$

$$(\ln\zeta)_{xxxx} = \frac{D_x^4\zeta.\zeta}{2\zeta^2} - 6\left(\frac{D_x^2\zeta.\zeta}{2\zeta^2}\right)^2;$$

Formula for the sixth logarithmic derivative in x follows from the differential identities given below: $\zeta = \zeta(x,t) \in C^6(\Omega)$:

$$(\ln\zeta)_x = \frac{\zeta_x}{\zeta} \tag{23}$$

$$(\ln\zeta)_{xx} = \frac{\zeta_{xx}}{\zeta} - \frac{\zeta_x^2}{\zeta^2} \tag{24}$$

$$(\ln\zeta)_{xxx} = \frac{\zeta_{xxx}}{\zeta} - 3\frac{\zeta_x\zeta_{xx}}{\zeta^2} + 2\frac{\zeta_x^2}{\zeta^3}; \tag{25}$$

$$\begin{aligned}(\ln\zeta)_{xxxx} = &\tfrac{\zeta_{xxxx}}{\zeta} - \tfrac{1}{\zeta^2}(4\zeta_x\zeta_{xxx} + 3\zeta_{xx}^2)\\ &+12\tfrac{\zeta_x^2\zeta_{xx}}{\zeta^3} - 6\tfrac{\zeta_x^4}{\zeta^4}\end{aligned} \tag{26}$$

$$\begin{aligned}&(\ln\zeta)_{xxxxx}\\ &= \tfrac{\zeta_{xxxxx}}{\zeta} - \tfrac{5}{\zeta^2}(\zeta_x\zeta_{xxxx} + 2\zeta_{xx}\zeta_{xxx})\\ &+\tfrac{10}{\zeta^3}(2\zeta_x^2\zeta_{xxx} + 3\zeta_x\zeta_{xx}^2) - 60\tfrac{\zeta_x^3\zeta_{xx}}{\zeta^4}\\ &+24\tfrac{\zeta_x^5}{\zeta^5}\end{aligned} \tag{27}$$

$$\begin{aligned}&(\ln\zeta)_{xxxxxx} =\\ &\tfrac{\zeta_{xxxxxx}}{\zeta} - \tfrac{1}{\zeta^2}(6\zeta_x\zeta_{xxxxx} + 15\zeta_{xx}\zeta_{xxxx} + 10\zeta_{xxx}^2)\\ &+\tfrac{30}{\zeta^3}(\zeta_x^2\zeta_{xxxx} + 4\zeta_x\zeta_{xx}\zeta_{xxx} + \zeta_{xxx}^3)\\ &-\tfrac{30}{\zeta^4}(4\zeta_x^3\zeta_{xxx} + 9\zeta_x^2\zeta_{xx}^2) + 360\tfrac{\zeta_x^4\zeta_{xx}}{\zeta^5} - 120\tfrac{\zeta_x^6}{\zeta^6}\end{aligned} \tag{28}$$

After grouping the similar terms in formula (28), we obtain

$$\begin{aligned}&(\ln\zeta)_{xxxxxx} =\\ &\tfrac{1}{2\zeta^2}(2\zeta\zeta_{xxxxxx} - 12\zeta_x\zeta_{xxxxx} + 30\zeta_{xx}\zeta_{xxxx} - 20\zeta_{xxx}^2)\\ &-\tfrac{30}{\zeta^6}[\zeta^3\zeta_{xx}(\zeta\zeta_{xxxx} - 4\zeta_x\zeta_{xxx} + 3\ \zeta_{xx}^2)\\ &-(4\zeta^3\zeta_{xx}^3 - 12\zeta^2\zeta_x^2\zeta_{xx}^2 + 12\zeta\zeta_x^4\zeta_{xx} - 4\zeta_x^6)\\ &-(3\zeta^2\zeta_x^2\zeta_{xx}^2 - 4\zeta^2\zeta_x^3\zeta_{xxx} + \zeta^3\zeta_x^2\zeta_{xxxx})]\end{aligned} \tag{29}$$

Let us remind the basic identities of Hirota [14]:

$$\frac{1}{2}D_x^2\zeta.\zeta = \zeta\zeta_{xx} - \zeta_x^2;$$

$$\frac{1}{8}(D_x^2\zeta.\zeta)^3 = (\zeta\zeta_{xx} - \zeta_x^2)^3;$$

$$\frac{1}{2}D_x^4\zeta.\zeta = \zeta\zeta_{xxxx} - 4\zeta_x\zeta_{xxx} + 3\zeta_{xx}^2;$$
$$\begin{aligned} D_x^6\zeta.\zeta = \; & 2\zeta\zeta_{xxxxxx} - 12\zeta_x\zeta_{xxxxx} \\ & +30\zeta_{xx}\zeta_{xxxx} - 20\zeta_{xxx}^2; \end{aligned}$$

by means of which we can present (29) in the form:

$$\begin{aligned} & (\ln\zeta)_{xxxxxx} = \\ & = \tfrac{D_x^6\zeta.\zeta}{2\zeta^2} - \tfrac{30}{\zeta^6}\left[\zeta^3\zeta_{xx}\left(\tfrac{D_x^4\zeta.\zeta}{2}\right) - \right. \\ & \qquad \left. -\tfrac{1}{2}(D_x^2\zeta.\zeta)^3 - \zeta^2\zeta_x^2\left(\tfrac{D_x^4\zeta.\zeta}{2}\right)\right] \\ & = \tfrac{D_x^6\zeta.\zeta}{2\zeta^2} - 30\left[\left(\tfrac{\zeta\zeta_{xx}-\zeta_x^2}{\zeta^2}\right)\left(\tfrac{D_x^4\zeta.\zeta}{2\zeta^2}\right) - \tfrac{1}{2}\left(\tfrac{D_x^2\zeta.\zeta}{\zeta^2}\right)^3\right] \\ & = \tfrac{D_x^6\zeta.\zeta}{2\zeta^2} - 30\left(\tfrac{D_x^2\zeta.\zeta}{2\zeta^2}\right)\left(\tfrac{D_x^4\zeta.\zeta}{2\zeta^2}\right) + 120\left(\tfrac{D_x^2\zeta.\zeta}{2\zeta^2}\right)^3 \end{aligned}$$

or finally for the sixth logarithmic derivative we obtain:

$$\begin{aligned} (\ln\zeta)_{xxxxxx} = \tfrac{D_x^6\zeta.\zeta}{2\zeta^2} - 30\left(\tfrac{D_x^2\zeta.\zeta}{2\zeta^2}\right)\left(\tfrac{D_x^4\zeta.\zeta}{2\zeta^2}\right) + \\ +120\left(\tfrac{D_x^2\zeta.\zeta}{2\zeta^2}\right)^3 . \end{aligned}$$

Appendix B

The Jacobi θ-functions identities

$$\left\{ \begin{array}{l} \sum\limits_{n=-\infty}^{\infty} q^{2n^2} = \theta_3(0,q^2) = \theta_3; \\ \sum\limits_{n=-\infty}^{\infty} n^2 q^{2n^2} = q\theta_3'/2; \\ \sum\limits_{n=-\infty}^{\infty} n^4 q^{2n^2} = q(\theta_3' + q\theta_3'')/4; \\ \sum\limits_{n=-\infty}^{\infty} n^6 q^{2n^2} = q(\theta_3' + 3q\theta_3'' + q^2\theta_3''')/8; \end{array} \right.$$

$$\left\{ \begin{array}{l} \sum\limits_{n=-\infty}^{\infty} q^{n^2+(n-1)^2} = q^{1/2}\theta_2(0,q^2) = q^{1/2}\theta_2; \\ \sum\limits_{n=-\infty}^{\infty} (2n-1)^2 q^{n^2+(n-1)^2} = 2q^{3/2}\theta_2'; \\ \sum\limits_{n=-\infty}^{\infty} (2n-1)^4 q^{n^2+(n-1)^2} = 4q^{3/2}(\theta_2' + q\theta_2''); \\ \sum\limits_{n=-\infty}^{\infty} (2n-1)^6 q^{n^2+(n-1)^2} \\ = 8q^{3/2}(\theta_2' + 3q\theta_2'' + q^2\theta_2'''). \end{array} \right.$$

Appendix C

Logarithmic derivative formula of the fourth Jacobi θ-function $\theta_4(z,q)$

$$\frac{\partial^2}{\partial x^2}[\ln\theta_4(z,q)] = k^2\alpha^2\mathrm{cn}^2(z_0,\alpha) - b(\varepsilon), \quad (30)$$

where$z = (k/\pi\theta_3^2)x + \omega t + \delta$, $z_0 = kx + (\omega\pi\theta_3^2)t + \delta$, $\alpha = \theta_2^2/\theta_3^2$; $\theta_j = \theta_j(0,q)$, $j = 1,2,3$ $q = e^{-\varepsilon\pi}$,

$$\varepsilon > 0; b(\varepsilon) = \frac{1}{3}\left(2\alpha^2 - 1 + \frac{\theta_1'''}{\pi^2\theta_3^4\theta_1'}\right).$$

If $\omega_1 = 2K_1(\eta)/\sqrt{e_1 - e_3}$ and $\omega_2 = 2iK_2(\eta)/\sqrt{e_1 - e_3}$ are the primitive periods of the basic Weierstrass elliptic function $\wp(u,\omega_1,\omega_2)$ ($Im(\omega_2/\omega_1) > 0$), then the entire complex-valued function $\sigma_3(u,\omega_1,\omega_2)$ can be defined by the equalities (See [16])

$$\sigma_3(z\omega_1) = e^{\eta_2\,\omega_1\,z/2}\sigma(\omega_2/2 - z\omega_1)\sigma^{-1}(\omega_2/2);$$

and

$$\sigma_3(z\omega_1) = e^{\eta\,\omega_1\,z^2}\theta_4(z,q)\theta_4^{-1},$$

where $d^2[\ln\sigma(u)]/du^2 = -\wp(u)$, $\eta = \eta_1 + \eta_2$ are complex parameters, such that $\eta\omega_1 = -\theta_1'''/6\theta_1'$. If we equalize the right sides of the last two equalities, take a logarithm and double differentiate with respect to z we will obtain

$$\frac{\partial^2}{\partial z^2}[\ln\theta_4(z,q)] = -\omega_1^2\wp(\omega_2 - z\omega_1) + 2\eta\omega_1. \quad (31)$$

Taking into account the evenness of the function $\wp(u)$ and the phase modulations:

$$\wp(u) = e_3 + (e_1 - e_3)sn^{-2}(u\sqrt{e_1 - e_3},\alpha);$$

$$sn(u + iK_2,\alpha) = \alpha\, sn(u,\alpha),$$

where $\alpha^2 = (e_2 - e_3)/(e_1 - e_3)$, and at the same time for given $\tau = \omega_2/\omega_1$ if we choose ω_1 so that

$$e_1 - e_3 = 1; \omega_1 = \pi\theta_3^2; e_1 - e_2 = \theta_4^4/\theta_3^4, ;$$

$$e_2 - e_3 = \theta_2^4/\theta_3^4 = \alpha^2,$$

then (31) takes the following more compact form:

$$\begin{aligned} \tfrac{\partial^2}{\partial x^2}[\ln\theta_4(z,q)] = \omega_1^2 k^2 cn^2(z\omega_1,\alpha) - \\ -\omega_1^2 k^2(e_2 - 2\eta/\omega) \end{aligned}, \quad (32)$$

($z = kx + \omega t + \delta$). After applying the rescaling $k \to k/\pi\theta_3^2$ and setting $\tau = i\varepsilon$, $\varepsilon > 0$ (i.e., $q = e^{-\varepsilon\pi}$), the residual term in (31) is

$$b(\varepsilon) = \omega_1^2 k^2(e_2 - 2\eta/\omega_1) = \frac{1}{3}\left(2\alpha^2 - 1 + \frac{\theta_1'''}{\pi^2\theta_3^4\theta_1'}\right),$$

and this confirms the equality (30).

Appendix D

The correctness of identity (21) follows from the presentation of the second theta function of Jacobi [16] in the form of an infinite product:

$$\begin{aligned} \theta_2(z,q) = \; & 2H_0(q)\sqrt[4]{q}\,\cos z. \\ & \textstyle\prod_{n=1}^{\infty}(1 + 2q^{2n}\cos 2z + q^{4n}) \end{aligned} \quad (33)$$

where $H_0(q) = \prod_{n=1}^{\infty}(1 - q^{2k})$, $q = e^{i\pi\tau}$, $Im\tau > 0$.

Differentiating (33) with respect to$z$, we obtain for the logarithmic derivative:

$$
\begin{aligned}
\frac{\theta_2'(z,q)}{\theta_2(z,q)} &= -tg\, z - 4\sum_{n=1}^{\infty} \frac{q^{2n}\sin 2z}{1+2q^{2n}\cos 2z+q^{4n}} \\
&= -tg\, z - 4\sum_{n=1}^{\infty} \frac{\sin 2z}{(q^{2n}+q^{-2n})+2\cos 2z} \\
&= -tg\, z - 2\sum_{n=-\infty}^{\infty} \frac{\sin 2z}{\cos 2n\pi\tau+\cos 2z} \\
&= -tg\, z - \sum_{n=-\infty}^{\infty} \frac{\sin 2z}{\cos(z+n\pi\tau)\cos(z-n\pi\tau)} \\
&= -tg\, z - \sum_{n=-\infty}^{\infty} \frac{\sin[(z+n\pi\tau)+(z-n\pi\tau)]}{\cos(z+n\pi\tau)\cos(z-n\pi\tau)} \\
&= -tg\, z - \sum_{n=1}^{\infty} [tg(z+n\pi\tau) + tg(z-n\pi\tau)] \\
&= -\sum_{n=-\infty}^{\infty} tg(z-n\pi\tau) \\
&= i\sum_{n=-\infty}^{\infty} \tanh[i(z-n\pi\tau)].
\end{aligned}
$$

Taking into account that$\frac{d}{dw}\tanh w = \sec h^2 w$, then we obtain (21)

$$\frac{d^2}{dz^2}[\ln\theta_2(z,q)] = -\sum_{m=-\infty}^{\infty} \mathrm{sech}^2[i(z-n\pi\tau)].$$

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