# Fredholm Characterization for a Wave Diffraction Problem with Higher Order Boundary Conditions: Impedance Case

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*Abstract:* In this paper, a Fredholm characterization for operators related to a wave diffraction problem with higher order impedance boundary conditions is developed. We consider an impedance boundary transmission problem for the Helmholtz equation. The problem will be analysed in an operator theory viewpoint and is considered within a framework of Bessel potential spaces. Relations between operators and extension methods are built to deal with the problem and, as consequence, a transparent interpretation of the problem in an operator theory framework are associated to the problem. Different types of operator relations are exhibited for different types of operators acting between Lebesgue and Bessel potential spaces on a finite interval and on the positive half-line. At the end, we describe when the operators associated with the problem enjoy the Fredholm property with Fredholm index equal to zero in terms of the initial space order parameters. In addition, an operator normalization method is applied.

*Key–Words:* higher order impedance boundary condition, wave diffraction, Helmholtz equation, Bessel potential space, convolution type operator, Wiener-Hopf operator, Fredholm property, normalization.

## 1 Introduction

The boundary value problems, related with wave propagation and wave diffraction, are studied in differential and integral equations theory on several areas of mathematics, physics, mechanics and engineering with many methods, in different contexts and taking into account various points of view [1, 19, 21, 27, 38]. By using methods from operator theory, in this paper, inspired by the work [15], we will consider a boundary-transmission problem for the Helmholtz equation which arises within the context of wave diffraction theory [3], [5, 6], [9]–[18], [20], [23, 24] and [28]–[35] on a finite strip with impedance boundary conditions [9, 15].

Was A. Sommerfeld the first one to consider canonical boundary value problems for time-harmonic waves governed by the Helmholtz equation in the famous work entitled *Mathematische Theorie der Diffraction*, [36]. Since then, a great number of researchers have made such a study their priority and a great number of different approaches have been presented and developed in the applied mathematics literature for studying canonical problems of plane wave diffraction. The most known and efficient methods and procedures to study such kind of problems are based on the classical Wiener-Hopf technique and the Maliushinets method [24, 35].

In the present work we will consider a Sommerfeld type problem where the geometry comprises a strip facing higher order imperfect boundary conditions. We want to understand better what are the operators behind such a problem. Thus, one of the main goals of the present work is the use of an operator theoretical machinery that will translate the problem into the study of properties of certain known types of operators associated to the problem.

To be more concrete, we will consider Wiener-Hopf operators and convolution type operators on finite intervals with semi-almost periodic Fourier symbol matrices. Convolution type operators $\mathcal{W}$ on finite intervals $\mathcal{I}$,

$$\mathcal{W}\varphi(x) = c\varphi(x) + \int_{\mathcal{I}} K(x-y)\varphi(y)\,dy, \quad x \in \mathcal{I}.$$

are one-dimensional linear integral operators where the integration kernels $K$ depend on the difference of the arguments and the domain of integration as well as the range of the independent variable are given by the same interval. In a constructive way, we will obtain this type of operators in Bessel potential and Lebesgue spaces. This is because we will consider the problem formulated between Bessel potential spaces and defined with a complex wave number $k$ which also allows a certain freedom in the corresponding smoothness orders.

## 2 Notations and Formulation of the Problem

In this section we establish the notation and some preliminary concepts in view of presenting the mathematical formulation of the problem.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing functions and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. As mentioned in the previous section, we will develop our study in a framework of Bessel potential spaces $\mathcal{H}^s$ defined by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\varphi\|_{\mathcal{H}^s(\mathbb{R}^n)} := \left\| \mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi \right\|_{L^2(\mathbb{R}^n)} < +\infty,$$

with $s \in \mathbb{R}$ and where $\mathcal{F} = \mathcal{F}_{x \mapsto \xi}$ is the Fourier transformation in $\mathbb{R}^n$ defined by

$$(\mathcal{F}\phi)(\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x}\phi(x)dx, \qquad \xi \in \mathbb{R}^n.$$

For a given Lipschitz domain $\mathcal{D}$, on $\mathbb{R}^n$, by $\widetilde{\mathcal{H}}^s(\mathcal{D})$ we mean the closed subspace of $\mathcal{H}^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and by $\mathcal{H}^s(\mathcal{D})$ the space of distributions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ belonging to $\mathcal{H}^s(\mathbb{R}^n)$. The space $\widetilde{\mathcal{H}}^s(\mathcal{D})$ is endowed with the subspace topology, and on $\mathcal{H}^s(\mathcal{D})$ we introduce the norm of the quotient space $\mathcal{H}^s(\mathbb{R}^n)/\widetilde{\mathcal{H}}^s(\mathbb{R}^n \backslash \overline{\mathcal{D}})$. Throughout the paper we will use the notation

$$\mathbb{R}^n_\pm := \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : \pm x_n > 0\}.$$

Adopting cartesian axes $Oxyz$ with the $y$-axis vertically upwards, we will consider a perpendicular time-harmonic electromagnetic plane wave incident on a strip $\Sigma$ in $\mathbb{R}^3$ where the material is considered to be invariant under the $z$-axis direction. Thus, the geometry of the problem is two dimensional and the strip will be therefore represented by

$$\Sigma := ]0, a[ \quad \text{for} \quad 0 < a < \infty.$$

We are now in position to formulate our impedance boundary conditions problem.

For $\Omega := \mathbb{R}^2 \backslash \overline{\Sigma}$ and given $n \in \mathbb{N}_0$, we are interested in studying the properties of an element $u \in \mathcal{H}^{1+\varepsilon}(\Omega)$, for some $\varepsilon \geq 0$, which satisfies the Helmholtz equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) u = 0 \qquad \text{in} \quad \Omega,$$

together with the impedance boundary condition

$$\begin{cases} p^+ u^+_{n+1} + q^+ u^+_n = h^+ \\ p^- u^-_{n+1} + q^- u^-_n = h^- \end{cases} \quad \text{on} \quad \Sigma, \qquad (1)$$

where $k$ is a given complex wave number with $\Re e(k) > 0$ and $\Im m(k) > 0$ due to a dissipative medium, as well as the impedance parameters $p^\pm \in \mathbb{C} \setminus \{0\}$ and $q^\pm \in \mathbb{C}$,

$$u^\pm_n := \left( \frac{\partial^n u}{\partial y^n} \right)_{|y=\pm 0}$$

denote the traces of $u$ on the upper and lower banks of $\Sigma$, respectively, and $h^\pm \in \mathcal{H}^{-1/2-n+\varepsilon}(\Sigma)$ are arbitrarily given elements. For instance, is well known that for $n = 0$ and $n = 1$ we have $u^\pm_n$ as the traditional Dirichlet and Neumann traces, respectively.

## 3 Reduction of the Problem to a System of Convolution type Operators

In this section we will use operator techniques in view of a characterization of the problem by means of finite interval convolution type operators. In the next section, such characterization of the problem, will be used to present certain extension methods in view to obtain corresponding operator relations, between the operator related to the problem and new Wiener-Hopf operators.

We will consider the densities $\vartheta$ and $\varphi$ defined by

$$\begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} u^+_1 - u^-_1 \\ u^+_0 - u^-_0 \end{bmatrix} \in \widetilde{\mathcal{H}}^{-1/2+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{1/2+\varepsilon}(\Sigma).$$

For an integer $j$, it follows

$$u^+_j = (-1)^j \mathcal{F}^{-1} t^j \cdot \mathcal{F} u^+_0$$

and

$$u^-_j = \mathcal{F}^{-1} t^j \cdot \mathcal{F} u^-_0,$$

where

$$t(\xi) = (\xi^2 - k^2)^{1/2} = t_+(\xi) t_-(\xi)$$

with $t_\pm$ the squareroot functions

$$t_\pm(\xi) = (\xi \pm k)^{1/2} = |\xi \pm k|^{1/2} e^{1/2 i \arg(\xi \pm k)},$$

$\xi \in \mathbb{R}$, with branch cuts $\Gamma_\mp = \{\pm k \pm it, t \geq 0\}$, respectively,

$$\arg(\xi - k) \in \left] -\frac{3\pi}{2}, \frac{\pi}{2} \right[$$

and

$$\arg(\xi + k) \in \left] -\frac{\pi}{2}, \frac{3\pi}{2} \right[.$$

Using these formulas, we can define an invertible convolution operator

$$B_{\Phi_B,\Sigma} := \mathcal{F}^{-1}\Phi_B \cdot \mathcal{F}$$

which maps $\widetilde{\mathcal{H}}^{-1/2+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{1/2+\varepsilon}(\Sigma)$ into $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\Sigma)$ as

$$B_{\Phi_B,\Sigma}\begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} u^+_{n+1} - u^-_{n+1} \\ u^+_n - u^-_n \end{bmatrix}, \qquad (2)$$

with Fourier symbol

$$\Phi_B = \frac{1}{2}\begin{bmatrix} (1+(-1)^n)t^n & (1-(-1)^n)t^{n+1} \\ (1-(-1)^n)t^{n-1} & (1+(-1)^n)t^n \end{bmatrix}.$$

Now, by the use of (2), it is possible to rewrite the boundary condition (1) as

$$C_{\Phi_C,\Sigma}\begin{bmatrix} u^+_{n+1} - u^-_{n+1} \\ u^+_n - u^-_n \end{bmatrix} = \begin{bmatrix} h^+ \\ h^- \end{bmatrix} \qquad (3)$$

where we define a convolution type operator

$$C_{\Phi_C,\Sigma} := r_\Sigma \mathcal{F}^{-1}\Phi_C \cdot \mathcal{F}$$

which maps the spaces $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\Sigma)$ into the spaces $\mathcal{H}^{-1/2-n+\varepsilon}(\Sigma) \times \mathcal{H}^{-1/2-n+\varepsilon}(\Sigma)$ with Fourier symbol

$$\Phi_C = \frac{1}{2}\begin{bmatrix} p^+ - q^+t^{-1} & -p^+t + q^+ \\ -p^- - q^-t^{-1} & -p^-t - q^- \end{bmatrix}. \qquad (4)$$

Throughout the paper, we are using $r_\Sigma$ to denote the restriction operator to $\Sigma \subset \mathbb{R}$ and in the particular case of $r_{\mathbb{R}_+}$ we will simply write $r_+$ for this restriction.

From (2) and (3), we obtain

$$C_{\Phi_C,\Sigma}B_{\Phi_B,\Sigma}\begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} h^+ \\ h^- \end{bmatrix}.$$

Our immediate goal will be to extend this last convolution type operator on a finite interval into a convolution type operator on the half-line. In view of this, we will need to consider some extension operator relations.

# 4 Extension Methods and Relations Between Operators

We will now perform some operator extension procedures in view of obtaining corresponding operator relations between the operators presented in the last section and new Wiener- Hopf operators. These operator relations will be used in the next section to study the Fredholm property of the operators associated with the problem.

**Definition 1** *[16] Let us consider two operators*

$$A : X_1 \to Y_1$$

*and*

$$B : X_2 \to Y_2,$$

*acting between Banach spaces.*

*(i) The operators $A$ and $B$ are said to be algebraically equivalent after extension if there exist additional Banach spaces $Z_1$ and $Z_2$ and invertible linear operators*

$$E : Y_2 \times Z_2 \to Y_1 \times Z_1$$

*and*

$$F : X_1 \times Z_1 \to X_2 \times Z_2$$

*such that*

$$\begin{bmatrix} A & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} B & 0 \\ 0 & I_{Z_2} \end{bmatrix} F. \qquad (5)$$

*(ii) If, in addition to (i), the invertible and linear operators $E$ and $F$ in (5) are bounded, then we will say that $A$ and $B$ are topologically equivalent after extension operators, or simply say that $A$ and $B$ are equivalent after extension operators, [2].*

*(iii) $A$ and $B$ are said to be equivalent operators in the particular case when*

$$A = E\,B\,F,$$

*for some bounded invertible linear operators*

$$E : Y_2 \to Y_1$$

*and*

$$F : X_1 \to X_2.$$

The above notion of topological equivalence after extension relation is equivalente to the concept of *matricial coupling* [2]. We refer to [6], [7] and [16] for a discussion on the differences between algebraic and topological equivalence after extension relations between convolution type operators.

We will now apply some results of [7] to our convolution type operator $C_{\Phi_C,\Sigma}$.

**Theorem 2** *The convolution type operator $C_{\Phi_C,\Sigma}$ with Fourier symbol (4) is algebraically equivalent after extension to the Wiener-Hopf operator $\mathcal{C}_{\Phi_\mathcal{C},\mathbb{R}_+}$ which maps $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)$*

*into* $\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+)$ *given by*

$$\mathcal{C}_{\Phi_{\mathcal{C}},\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi_{\mathcal{C}} \cdot \mathcal{F},$$

*and with* $\Phi_{\mathcal{C}}$ *being the Fourier symbol defined by*

$$\Phi_{\mathcal{C}}(\xi)=\left[\begin{array}{cc:cc} e^{-i\xi a} & 0 & 0 & 0 \\ 0 & e^{-i\xi a} & 0 & 0 \\ \hdashline \multicolumn{2}{c:}{\frac{1}{2}\mathcal{C}_{22}(\xi)} & \begin{matrix} e^{i\xi a} & 0 \\ 0 & e^{i\xi a}\end{matrix} \end{array}\right].$$

with

$$\mathcal{C}_{22}(\xi) = \begin{bmatrix} p^+ - q^+t^{-1}(\xi) & -p^+t(\xi)+q^+ \\ -p^- - q^-t^{-1}(\xi) & -p^-t(\xi)-q^- \end{bmatrix}.$$

**Proof:** We present only a sketch of the proof because it is a hard procedure well-known and discussed in [25].

In order to simplify the notation we will consider $\mathcal{H}^{\alpha}_{\beta}(\cdot)$ instead of $\mathcal{H}^{\alpha}(\cdot) \times \mathcal{H}^{\beta}(\cdot)$.

The equivalence is consequence of Kuijper's extension methods and, for more details, we advise to see [7] and [26]. In abridged form and without many details, the equivalence after extension relation can be directly obtained by computing the following operator composition

$$\begin{bmatrix} C_{\Phi_C,\Sigma} & 0 & 0 \\ 0 & I_{\mathcal{H}^{-1/2-n+\varepsilon}_{1/2-n+\varepsilon}(\mathbb{R}_+)} & 0 \\ 0 & 0 & I_{\left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2} \end{bmatrix} = ETF,$$

where $T$ is defined between $Ker\,A \times N \times \left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ and $\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times Im\,B \times M$ by

$$T = \begin{bmatrix} A_{|_{Ker\,A}} & A_{|N} & 0 \\ C_1 & C_2 & B_{Im\,B} \\ C_3 & C_4 & 0 \end{bmatrix},$$

$E$ and $F$ are invertible operators defined between the spaces $\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times Im\,B \times M$ and $\left[\mathcal{H}^{-1/2-n+\varepsilon}(\Sigma)\right]^2 \times \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times \left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ for $E$ and between $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\Sigma) \times \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times \left[\widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ and $Ker\,A \times N \times \left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ for $F$ and defined by

$$E = \begin{bmatrix} -q_M C_4\left(A_{|N}\right)^{-1} & 0 & q_M \\ I_{\mathcal{H}^{-1/2-n+\varepsilon}_{1/2-n+\varepsilon}(\mathbb{R}_+)} & 0 & 0 \\ -\left(B_{Im\,B}\right)^{-1} C_2 \left(A_{|N}\right)^{-1} & \left(B_{Im\,B}\right)^{-1} & 0 \end{bmatrix}$$

and

$$F = \begin{bmatrix} R^{-1} & 0 & 0 \\ 0 & \left(A_{|N}\right)^{-1} & 0 \\ -\left(B_{Im\,B}\right)^{-1} C_1 R & 0 & I_{\left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2} \end{bmatrix},$$

for some algebraic decompositions (see [22])

$$\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\mathbb{R}_+) = Ker\,A \times N$$

with $N = N_1^{-1/2-n+\varepsilon} \times N_2^{1/2-n+\varepsilon}$ and

$$\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) = Im\,B \times M$$

with $M = M_1^{-1/2-n+\varepsilon} \times M_2^{-1/2-n+\varepsilon}$ for convenient subspaces

$$N_1^{-1/2-n+\varepsilon} \subset \widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+),$$
$$N_2^{1/2-n+\varepsilon} \subset \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\mathbb{R}_+),$$
$$M_1^{-1/2-n+\varepsilon} \subset \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+),$$
$$M_2^{-1/2-n+\varepsilon} \subset \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+),$$

and where $q$ is the quotient map from $\left[\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ to $\left[\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2/Im\,B$, $A := r_+\mathcal{F}^{-1}e^{-i\xi a}I \cdot \mathcal{F}$ defined between $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\mathbb{R}_+)$ and $\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+)$, $B_{Im\,B}$ is the isomorphism defined between $\left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ and $Im\,B$ where $B := r_+\mathcal{F}^{-1}e^{i\xi a}I \cdot \mathcal{F}$ is defined between $\left[\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$ and $\left[\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\right]^2$, $C : Ker\,A \times N \longrightarrow Im\,B \times M$ is defined by $C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} := r_+\mathcal{F}^{-1}\Phi_C \cdot \mathcal{F}$ where $\Phi_C$ is the Fourier symbol of the operator $C_{\Phi_C,\Sigma}$ and $R$, taking into account the geometry of our problem, is the identity operator defined between $Ker\,A$ and $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\Sigma)$. □

Due to the use of the lifting procedure, and choosing convenient auxiliary bounded invertible operators, we now obtain a new operator relation for an operator acting between Lebesgue spaces – which is presented in the next result.

We will use the notation $L^2_+(\mathbb{R}) := \widetilde{\mathcal{H}}^0(\mathbb{R}_+)$.

**Theorem 3** *The Wiener-Hopf operator $\mathcal{C}_{\Phi_{\mathcal{C}},\mathbb{R}_+}$ defined above between Bessel potential spaces is equivalent to the Wiener-Hopf operator*

$$\widehat{\mathcal{C}}_{\Phi_{\widehat{\mathcal{C}}},\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi_{\widehat{\mathcal{C}}}\cdot\mathcal{F} : \left[L^2_+(\mathbb{R})\right]^4 \to \left[L^2(\mathbb{R}_+)\right]^4,$$

*where $\Phi_{\widehat{\mathcal{C}}}$ has the block matricial representation*

$$\Phi_{\widehat{\mathcal{C}}}(\xi) = \left[\begin{array}{c:c} \mathfrak{A}(\xi) & 0_2 \\ \hdashline \frac{1}{2}\mathfrak{C}(\xi) & \mathfrak{B}(\xi) \end{array}\right] \qquad (6)$$

*where*

$$\mathfrak{A}(\xi)=\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi)e^{-i\xi a} & 0 \\ 0 & \zeta^{1/2-n+\varepsilon}(\xi)e^{-i\xi a}\end{bmatrix},$$

$$\mathfrak{B}(\xi)=\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi)e^{i\xi a} & 0 \\ 0 & \zeta^{-1/2-n+\varepsilon}(\xi)e^{i\xi a}\end{bmatrix},$$

$$\mathfrak{C}(\xi)=\begin{bmatrix} \mathfrak{C}_{11}(\xi) & \mathfrak{C}_{12}(\xi) \\ \mathfrak{C}_{21}(\xi) & \mathfrak{C}_{22}(\xi)\end{bmatrix},$$

*with*

$$\mathfrak{C}_{11}(\xi) = p^+\zeta^{-1/2-n+\varepsilon}(\xi) - q^+\zeta^{-n+\varepsilon}(\xi)(\xi-k)^{-1},$$

$$\mathfrak{C}_{12}(\xi) = -p^+\zeta^{-n+\varepsilon}(\xi) + q^+\zeta^{-1/2-n+\varepsilon}(\xi)(\xi+k)^{-1},$$

$$\mathfrak{C}_{21}(\xi) = -p^-\zeta^{-1/2-n+\varepsilon}(\xi) - q^-\zeta^{-n+\varepsilon}(\xi)(\xi-k)^{-1},$$

$$\mathfrak{C}_{22}(\xi) = -p^-\zeta^{-n+\varepsilon}(\xi) - q^-\zeta^{-1/2-n+\varepsilon}(\xi)(\xi+k)^{-1},$$

$\zeta(\xi) = \frac{\xi-k}{\xi+k} = \frac{\lambda_-}{\lambda_+}$, $\xi\in\mathbb{R}$ *and* $0_2$ *denotes the* $2\times 2$ *zero matrix.*

**Proof:** The equivalence relation can be directly obtained by computing the following operator composition

$$\mathcal{C}_{\Phi_{\mathcal{C}},\mathbb{R}_+} = \mathrm{W}_{\Phi_E,\mathbb{R}_+}\, l_0\, \widehat{\mathcal{C}}_{\Phi_{\widehat{\mathcal{C}}},\mathbb{R}_+} l_0 \mathrm{W}_{\Phi_F,\mathbb{R}_+},$$

where

$$l_0 : \left[L^2(\mathbb{R}_+)\right]^4 \to \left[L^2_+(\mathbb{R})\right]^4$$

denotes de zero extension operator and where $\mathrm{W}_{\Phi_E,\mathbb{R}_+}l_0$ is defined between the spaces $\left[L^2(\mathbb{R}_+)\right]^4$ and $\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{1/2-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\times\mathcal{H}^{-1/2-n+\varepsilon}(\mathbb{R}_+)$ by

$$\mathrm{W}_{\Phi_E,\mathbb{R}_+}l_0 := r_+\mathcal{F}^{-1}\Phi_E\cdot\mathcal{F}l_0$$

with

$$\Phi_E(\xi)=\begin{bmatrix} \lambda_-^{1/2+n-\varepsilon} & 0 & 0 & 0\\ 0 & \lambda_-^{-1/2+n-\varepsilon} & 0 & 0 \\ 0 & 0 & \lambda_-^{1/2+n-\varepsilon} & 0 \\ 0 & 0 & 0 & \lambda_-^{1/2+n-\varepsilon}\end{bmatrix}$$

and $l_0\mathrm{W}_{\Phi_F,\mathbb{R}_+}$ is defined between $\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\times \widetilde{\mathcal{H}}^{1/2-n+\varepsilon}(\mathbb{R}_+)\times\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)\times\widetilde{\mathcal{H}}^{-1/2-n+\varepsilon}(\mathbb{R}_+)$ and $\left[L^2_+(\mathbb{R})\right]^4$ by

$$l_0\mathrm{W}_{\Phi_F,\mathbb{R}_+} := l_0 r_+\mathcal{F}^{-1}\Phi_F\cdot\mathcal{F}$$

with

$$\Phi_F(\xi)=\begin{bmatrix} \lambda_+^{-1/2-n+\varepsilon} & 0 & 0 & 0\\ 0 & \lambda_+^{1/2-n+\varepsilon} & 0 & 0 \\ 0 & 0 & \lambda_+^{-1/2-n+\varepsilon} & 0 \\ 0 & 0 & 0 & \lambda_+^{-1/2-n+\varepsilon}\end{bmatrix}.$$

Notice that the bounded operators $\mathrm{W}_{\Phi_E,\mathbb{R}_+}l_0$ and $l_0\mathrm{W}_{\Phi_F,\mathbb{R}_+}$ are invertible as pointed out in [37, §2.10.3]. □

## 5 Fredholm Analysis

Our main goal is to study and characterize the Fredholm property of the finite interval convolution type operator $C_{\Phi_C,\Sigma}$ for general $\varepsilon$. We will use different factorization procedures applied to the operators introduced in the last section. We start by recalling the definition of Fredholm operator.

**Definition 4** *Let $X$, $Y$ be two Banach spaces and $A : X\to Y$ a bounded linear operator with closed image. The operator $A$ is called a Fredholm operator if*

$$n(A) := \dim\, Ker\, A < \infty$$

*and*

$$d(A) := \dim\, Y/Im\, A < \infty.$$

*If $A$ is a Fredholm operator, then the Fredholm index of $A$ is the integer defined by*

$$Ind\, A = n(A) - d(A).$$

**Theorem 5** *Let $\Phi_{\widehat{\mathcal{C}}}$ be defined by (6) and*

$$\det\mathfrak{C}(\pm\infty)\neq 0.$$

*The operator $\widehat{\mathcal{C}}_{\Phi_{\widehat{\mathcal{C}}},\mathbb{R}_+}$ presented in the last theorem admits the factorization*

$$\widehat{\mathcal{C}}_{\Phi_{\widehat{\mathcal{C}},\mathbb{R}_+}} = \widehat{\mathrm{W}}_{\widehat{\Phi}_-,\mathbb{R}_+}\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}},\mathbb{R}_+}}\widehat{\mathrm{W}}_{\widehat{\Phi}_+,\mathbb{R}_+}$$

*where $\widehat{\mathrm{W}}_{\widehat{\Phi}_-,\mathbb{R}_+}$ and $\widehat{\mathrm{W}}_{\widehat{\Phi}_+,\mathbb{R}_+}$ are invertible operators having Fourier symbols*

$$\widehat{\Phi}_-(\xi) = \left[\begin{array}{cc:cc} -1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$-e^{-ia\xi}\tau^-(\xi)$}} \\ 0 & -1 & & \\ \hdashline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1\end{array}\right]$$

*and*

$$\widehat{\Phi}_{+}(\xi)=\left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$e^{ia\xi}\tau^{+}(\xi)$}} \\ 0 & 1 & & \end{array}\right],$$

*which admit bounded analytic extensions in $\Im m\,\xi<0$ and $\Im m\,\xi>0$, respectively, and with*

$$\tau^{-}(\xi)=S^{-}(\xi)\mathfrak{C}^{-1}(-\infty)+ S^{+}(\xi)\begin{bmatrix} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(1-2n+2\varepsilon)} \end{bmatrix}\mathfrak{C}^{-1}(+\infty)$$

*and*

$$\tau^{+}(\xi)=S^{-}(\xi)\mathfrak{C}^{-1}(-\infty)+ S^{+}(\xi)\begin{bmatrix} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(-1-2n+2\varepsilon)} \end{bmatrix}\mathfrak{C}^{-1}(+\infty)$$

*with*

$$S^{\pm}(\xi)=\frac{1\pm S(\xi)}{2}$$

*where $S:\mathbb{C}\to\mathbb{C}$ is the normalized sine-integral function given by*

$$S(\xi)=\frac{2}{\pi}\int_0^{\xi}\frac{\sin x}{x}dx$$

*and where $\mathfrak{C}^{-1}(-\infty)$ and $\mathfrak{C}^{-1}(+\infty)$ are defined by*

$$\mathfrak{C}^{-1}(-\infty)=\begin{bmatrix} \frac{1}{p^{+}} & -\frac{1}{p^{-}} \\ -\frac{1}{p^{+}} & -\frac{1}{p^{-}} \end{bmatrix}$$

*and*

$$\mathfrak{C}^{-1}(+\infty)=\begin{bmatrix} \frac{1}{p^{+}}e^{i\pi(1+2n-2\varepsilon)} & -\frac{1}{p^{-}}e^{i\pi(1+2n-2\varepsilon)} \\ -\frac{1}{p^{+}}e^{i\pi(2n-2\varepsilon)} & -\frac{1}{p^{-}}e^{i\pi(2n-2\varepsilon)} \end{bmatrix}$$

*if*

$$\det\mathfrak{C}(-\infty)=-\frac{p^{+}p^{-}}{2}\neq 0$$

*and*

$$\det\mathfrak{C}(+\infty)=-\frac{p^{+}p^{-}}{2}e^{i\pi(-1-4n+4\varepsilon)}\neq 0,$$

*respectively.*

*The Fourier symbol $\Phi_{\widetilde{C}}$ belongs to $PC^{4\times4}(\dot{\mathbb{R}})$, the space of four by four matrix-valued functions with piecewise continuous entries on $\dot{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$, and is given by*

$$\Phi_{\widetilde{C}}(\xi)=\left[\begin{array}{c:c} \mathfrak{A}(\xi) & \mathfrak{B}(\xi) \\ \hdashline \mathfrak{D}(\xi) & -\mathfrak{C}(\xi) \end{array}\right] \tag{7}$$

*where*

$$\mathfrak{A}(\xi)=\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{1/2-n+\varepsilon}(\xi) \end{bmatrix}\tau^{+}(\xi)- \tau^{-}(\xi)\mathfrak{C}(\xi)\tau^{+}(\xi)+ \tau^{-}(\xi)\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{-1/2-n+\varepsilon}(\xi) \end{bmatrix},$$

$$\mathfrak{B}(\xi)=e^{-ia\xi}\tau^{-}(\xi)\mathfrak{C}(\xi)- e^{-ia\xi}\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{1/2-n+\varepsilon}(\xi) \end{bmatrix},$$

$$\mathfrak{D}(\xi)=e^{ia\xi}\mathfrak{C}(\xi)\tau^{+}(\xi)- e^{ia\xi}\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{-1/2-n+\varepsilon}(\xi) \end{bmatrix}.$$

The proof of the last result can be done by direct computation and therefore is here omitted. Anyway, we have,

$$\lim_{\xi\to\pm\infty}\left(\tau^{-}(\xi)\mathfrak{C}(\xi)-\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{1/2-n+\varepsilon}(\xi) \end{bmatrix}\right)=0, \tag{8}$$

$$\lim_{\xi\to\pm\infty}\left(\mathfrak{C}(\xi)\tau^{+}(\xi)-\begin{bmatrix} \zeta^{-1/2-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{-1/2-n+\varepsilon}(\xi) \end{bmatrix}\right)=0. \tag{9}$$

These last two results are a consequence of the fact that we agree that

$$\lim_{\xi\to-\infty}\zeta^{\sigma}(\xi)=1$$

and

$$\lim_{\xi\to+\infty}\zeta^{\sigma}(\xi)=e^{i2\pi\sigma},$$

for $\sigma\in\mathbb{R}$.

In order to continue, let us consider, for $\Phi\in PC^{n\times n}(\dot{\mathbb{R}})$, the function

$$\overline{\Phi}:\dot{\mathbb{R}}\times[0,1]\to\mathbb{C}^{n\times n}$$

defined by

$$\overline{\Phi}(\xi,\mu):=(1-\mu)\Phi(\xi-0)+\mu\Phi(\xi+0),$$

$(\xi,\mu)\in\dot{\mathbb{R}}\times[0,1]$, where

$$\Phi(\infty-0):=\Phi(+\infty)$$

and

$$\Phi(\infty+0):=\Phi(-\infty).$$

The following result [4, Theorem 5.9] helps us to study the Fredholm property for the operator $C_{\Phi_C,\Sigma}$.

**Theorem 6** *For $\Phi \in PC^{n\times n}(\dot{\mathbb{R}})$, it follows that*

$$\det \overline{\Phi}(\xi, \mu) \neq 0$$

*for all $(\xi,\mu) \in \dot{\mathbb{R}} \times [0,1]$ if and only if*

$$\mathcal{W}_{\Phi,\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F} : \left[L^2_+(\mathbb{R})\right]^n \to \left[L^2(\mathbb{R}_+)\right]^n$$

*is a Fredholm operator.*

*In case of having the Fredholm property, the Fredholm index of $\mathcal{W}_{\Phi,\mathbb{R}_+}$ is given by*

$$Ind\,\mathcal{W}_{\Phi,\mathbb{R}_+} = -wind(\det \overline{\Phi}),$$

*where $wind$ denotes the winding number.*

Finally, we are able to present the Fredholm characterization to our operator $C_{\Phi_C,\Sigma}$ and, consequently, to our initial problem.

**Theorem 7** *The finite interval convolution type operator $C_{\Phi_C,\Sigma}$ is a Fredholm operator with zero Fredholm index if and only if*

$$\varepsilon \neq \frac{q}{2} \quad for \quad q \in \mathbb{Z}. \tag{10}$$

**Proof:** First of all, we notice that from Theorems 2–5 we conclude that the operator $C_{\Phi_C,\Sigma}$ is algebraically equivalent after extension to the operator

$$\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi_{\widetilde{\mathcal{C}}} \cdot \mathcal{F} : \left[L^2_+(\mathbb{R})\right]^4 \to \left[L^2(\mathbb{R}_+)\right]^4$$

where $\Phi_{\widetilde{\mathcal{C}}}$ is given by (7). Therefore, in view to obtain the desired conclusion, that $C_{\Phi_C,\Sigma}$ is a Fredholm operator, we start by deducing the conditions which characterize the Fredholm property of $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+}$.

Letting

$$\overline{\Phi}_{\widetilde{\mathcal{C}}}(\xi,\mu) = (1-\mu)\Phi_{\widetilde{\mathcal{C}}}(\xi - 0) + \mu\Phi_{\widetilde{\mathcal{C}}}(\xi + 0)$$

and

$$\Phi_{\widetilde{\mathcal{C}}}(\infty \pm 0) := \Phi_{\widetilde{\mathcal{C}}}(\mp\infty),$$

by Theorem 6, we have that

$$\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\xi,\mu) \neq 0$$

for $(\xi,\mu) \in \dot{\mathbb{R}} \times [0,1]$ if and only if the operator $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+}$ has the Fredholm property. Additionally, from Theorem 5, we already know that the Fourier symbol $\Phi_{\widetilde{\mathcal{C}}}$ can be written as

$$\Phi_{\widetilde{\mathcal{C}}}(\xi) = \widehat{\Phi}_-^{-1}(\xi)\Phi_{\widehat{\mathcal{C}}}(\xi)\widehat{\Phi}_+^{-1}(\xi).$$

Thus, for any $\xi \in \mathbb{R}$ we have

$$\det \Phi_{\widetilde{\mathcal{C}}}(\xi \pm 0) = \det \Phi_{\widehat{\mathcal{C}}}(\xi)$$

because $\Phi_{\widehat{\mathcal{C}}}(\xi)$ has no discontinuities on the real line, $\det \widehat{\Phi}_\pm^{-1}$ also have no discontinuities on the real line and, moreover, $\det \widehat{\Phi}_\pm^{-1} \equiv 1$. Therefore,

$$\begin{aligned}
\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\xi,\mu) &= \det\left[(1-\mu)\Phi_{\widehat{\mathcal{C}}}(\xi) + \mu\Phi_{\widehat{\mathcal{C}}}(\xi)\right] \\
&= \det \Phi_{\widehat{\mathcal{C}}}(\xi) \\
&= \zeta^{-1-4n+4\varepsilon}(\xi) \\
&\neq 0,
\end{aligned}$$

in the case of $\xi \in \mathbb{R}$.

For $\xi = \infty$, we have,

$$\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\infty,\mu) = \det\left[(1-\mu)\Phi_{\widetilde{\mathcal{C}}}(+\infty) + \mu\Phi_{\widetilde{\mathcal{C}}}(-\infty)\right].$$

Appealing to the limits (8) and (9), we obtain

$$\Phi_{\widetilde{\mathcal{C}}}(-\infty) = \left[\begin{array}{c:c} \mathfrak{C}^{-1}(-\infty) & 0_2 \\ \hdashline 0_2 & -\mathfrak{C}(-\infty) \end{array}\right]$$

and

$$\Phi_{\widetilde{\mathcal{C}}}(+\infty) = \left[\begin{array}{c:c} \mathfrak{A}\mathfrak{C}^{-1}(+\infty)\mathfrak{B} & 0_2 \\ \hdashline 0_2 & -\mathfrak{C}(+\infty) \end{array}\right],$$

with

$$\mathfrak{A} = \begin{bmatrix} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(1-2n+2\varepsilon)} \end{bmatrix},$$

and

$$\mathfrak{B} = \begin{bmatrix} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(-1-2n+2\varepsilon)} \end{bmatrix}.$$

Thus, by direct computation, we have

$$\Phi_{\widetilde{\mathcal{C}}}(-\infty) = \begin{bmatrix} \frac{1}{p^+} & -\frac{1}{p^-} & 0 & 0 \\ -\frac{1}{p^+} & -\frac{1}{p^-} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}p^+ & \frac{1}{2}p^+ \\ 0 & 0 & \frac{1}{2}p^- & \frac{1}{2}p^- \end{bmatrix}$$

and

$$\Phi_{\widetilde{\mathcal{C}}}(+\infty) = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix},$$

with

$$\begin{aligned}
a_{11} &= \frac{1}{p^+} e^{i\pi(-1-2n+2\varepsilon)}, \\
a_{12} &= -\frac{1}{p^-} e^{i\pi(-1-2n+2\varepsilon)}, \\
a_{21} &= -\frac{1}{p^+} e^{i\pi(-2n+2\varepsilon)}, \\
a_{22} &= -\frac{1}{p^-} e^{i\pi(-2n+2\varepsilon)}, \\
a_{33} &= -\frac{p^+}{2} e^{i\pi(-1-2n+2\varepsilon)}, \\
a_{34} &= \frac{p^+}{2} e^{i\pi(-2n+2\varepsilon)}, \\
a_{43} &= \frac{p^-}{2} e^{i\pi(-1-2n+2\varepsilon)}, \\
a_{44} &= \frac{p^-}{2} e^{i\pi(-2n+2\varepsilon)}.
\end{aligned}$$

Finally, the last results, tell us that

$$\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\infty,\mu) = \begin{vmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{vmatrix},$$

where

$$\begin{aligned}
b_{11} &= \frac{1-\mu}{p^+} e^{i\pi(-1-2n+2\varepsilon)} + \frac{\mu}{p^+}, \\
b_{12} &= -\frac{1-\mu}{p^-} e^{i\pi(-1-2n+2\varepsilon)} - \frac{\mu}{p^-}, \\
b_{21} &= -\frac{1-\mu}{p^+} e^{i\pi(-2n+2\varepsilon)} - \frac{\mu}{p^+}, \\
b_{22} &= -\frac{1-\mu}{p^-} e^{i\pi(-2n+2\varepsilon)} - \frac{\mu}{p^-}, \\
b_{33} &= -\frac{(1-\mu)p^+}{2} e^{i\pi(-1-2n+2\varepsilon)} - \frac{\mu p^+}{2}, \\
b_{34} &= \frac{(1-\mu)p^+}{2} e^{i\pi(-2n+2\varepsilon)} + \frac{\mu p^+}{2}, \\
b_{43} &= \frac{(1-\mu)p^-}{2} e^{i\pi(-1-2n+2\varepsilon)} + \frac{\mu p^-}{2}, \\
b_{44} &= \frac{(1-\mu)p^-}{2} e^{i\pi(-2n+2\varepsilon)} + \frac{\mu p^-}{2}.
\end{aligned}$$

So,

$$\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\infty,\mu) = \Big[(1-\mu)e^{i\pi(-1-2n+2\varepsilon)} + \mu\Big]^2 \times \Big[(1-\mu)e^{i\pi(-2n+2\varepsilon)} + \mu\Big]^2.$$

As a consequence, $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+}$ is a Fredholm operator if and only if

$$(1-\mu)e^{i\pi(-1-2n+2\varepsilon)} + \mu \neq 0 \tag{11}$$

and

$$(1-\mu)e^{i\pi(-2n+2\varepsilon)} + \mu \neq 0, \tag{12}$$

$\mu \in [0,1]$.

Since the sets

$$\mathcal{S}_1 = \Big\{(1-\mu)e^{i\pi(-1-2n+2\varepsilon)} + \mu \ : \ \mu \in [0,1]\Big\}$$

and

$$\mathcal{S}_2 = \Big\{(1-\mu)e^{i\pi(-2n+2\varepsilon)} + \mu \ : \ \mu \in [0,1]\Big\}$$

define the line segments joining 1 to $e^{i\pi(-1-2n+2\varepsilon)}$ and 1 to $e^{i\pi(-2n+2\varepsilon)}$, respectively, for holding the inequalities in (11) and (12), we need that

$$e^{i\pi(-1-2n+2\varepsilon)} \notin \mathbb{R}_-$$

and

$$e^{i\pi(-2n+2\varepsilon)} \notin \mathbb{R}_-.$$

Thus

$$\pi(-1-2n+2\varepsilon) \neq \pi + 2\pi q$$

and

$$\pi(-2n+2\varepsilon) \neq \pi + 2\pi q,$$

$q \in \mathbb{Z}$, i.e.,

$$\varepsilon \neq 1 + n + q$$

and

$$\varepsilon \neq \frac{1}{2} + n + q,$$

$q \in \mathbb{Z}$. So, we have $\varepsilon \neq \frac{q}{2}$, $q \in \mathbb{Z}$.

Therefore, from the operator identities provided by both the above mentioned algebraic and topological equivalence relations, given in Theorems 2–5, we conclude that $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+}$ and $C_{\Phi_C,\Sigma}$ are Fredholm operators if and only if condition (10) holds, and that the corresponding defect spaces of these operators have the same dimensions. From this, and since by [2, Theorem 3] Fredholm operators in Banach spaces are equivalent after extension if and only if their corresponding defect spaces have equal dimensions, we even arrive at the conclusion that $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+}$ and $C_{\Phi_C,\Sigma}$ are not only algebraically equivalent after extension but also topologically equivalent after extension.

Finally, jointing the last conclusion with Theorem 6, we obtain the following result for the Fredholm index of $C_{\Phi_C,\Sigma}$,

$$\begin{aligned} Ind\, C_{\Phi_C,\Sigma} &= Ind\, \widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+} \\ &= -wind\left(\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\xi,\mu)\right) \\ &= -\frac{1}{2\pi}\Big(\left[\arg\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\xi,\mu)\right]_{\mathbb{R}} + \\ &\qquad \left[\arg\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\infty,\mu)\right]_{[0,1]}\Big) \\ &= -\frac{1}{2\pi}\Big(\left[\arg\det \Phi_{\widetilde{\mathcal{C}}}(\xi)\right]_{\mathbb{R}} + \\ &\qquad \left[\arg\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\infty,\mu)\right]_{[0,1]}\Big), \end{aligned}$$

where $[f(\xi)]_{\mathbb{R}}$ denotes the increment of $f(\xi)$ when $\xi$ varies through $\mathbb{R}$ from $-\infty$ to $+\infty$ and $[f(\infty,\mu)]_{[0,1]}$ is the increment of $f(\infty,\mu)$ when $\mu$ varies through $\mathbb{R}$ from 0 to 1. Directly, we obtain

$$\left[\arg\det \Phi_{\widetilde{\mathcal{C}}}(\xi)\right]_{\mathbb{R}} = \pi(-2-8n+8\varepsilon)$$

and

$$\left[\arg\det \overline{\Phi}_{\widetilde{\mathcal{C}}}(\infty,\mu)\right]_{[0,1]} = \pi(2+8n-8\varepsilon).$$

So, we have the desired result $Ind\, C_{\Phi_C,\Sigma} = 0$. □

## 6 Image Normalization

In this last section, we will study the normalization of our problem in the particular case where the lifted operator Fourier symbol $\Phi_{\widetilde{\mathcal{C}}}$ belongs to

$$\mathcal{GC}^{\nu}([-\infty,+\infty])^{n\times n} \quad \text{with } \nu \in ]0,1[,$$

the space of Hölder continuous matricial functions with order $n$, invertible and with exponent $\nu$ and with $\det \Phi_{\widetilde{\mathcal{C}}}(\xi) \neq 0$, $\xi \in [-\infty,+\infty]$. The normalization relies on the fact that if an operator is not Fredholm, which is equivalent to say that is not normally solvable, [34], it is possible to make an extension of the image space and/or a restriction of the domain such that the resulting operator becomes Fredholm. By physical reasons, we will choose to do the normalization without changing simultaneously both spaces. We will do the image normalization. For more details see [8, 11, 31, 32, 34].

**Theorem 8** *The operator*

$$\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi_{\widetilde{\mathcal{C}}}\cdot\mathcal{F} : \left[L^2_+(\mathbb{R})\right]^4 \to \left[L^2(\mathbb{R}_+)\right]^4$$

*with Fourier symbol $\Phi_{\widetilde{\mathcal{C}}}$ given by (7) is not normally solvable if and only if $\varepsilon = \frac{q}{2}$, $q \in \mathbb{Z}$.*

*For $\varepsilon = \frac{q}{2}$, $q \in \mathbb{Z}$ we have*

*(i) for $|q|$ even or 0, the image normalizer operator*

$$\overset{<}{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+,even} = \widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+} : \left[L^2_+(\mathbb{R})\right]^4 \to Y_1\times Y_2\times\left[L^2(\mathbb{R}_+)\right]^2$$

*with*

$$Y_1 = r_+\Lambda_-^{\frac{1}{2}+n-\varepsilon} Sl_0\{\overset{<}{\mathcal{H}}{}^0(\mathbb{R}_+)\},$$

*and*

$$Y_2 = r_+\Lambda_-^{-\frac{1}{2}+n-\varepsilon} Sl_0\{\overset{<}{\mathcal{H}}{}^0(\mathbb{R}_+)\},$$

*(ii) for $|q|$ odd, the image normalizer operator*

$$\overset{<}{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+,odd} = \widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+} : \left[L^2_+(\mathbb{R})\right]^4 \to \left[L^2(\mathbb{R}_+)\right]^2\times[Y_3]^2$$

*with*

$$Y_3 = r_+\Lambda_-^{\frac{1}{2}+n-\varepsilon} Sl_0\{\overset{<}{\mathcal{H}}{}^0(\mathbb{R}_+)\},$$

*where*

$$\overset{<}{\mathcal{H}}{}^0(\mathbb{R}_+) = r_+\Lambda_-^{-\frac{1}{2}}\Lambda_+^{\frac{1}{2}}L^2_+(\mathbb{R}),$$

*$S$ is a matriz from the normal Jordan form*

$$\widetilde{\Phi}^{-1}(+\infty)\widetilde{\Phi}(-\infty) = S^{-1}JS$$

*and $\Lambda_\pm^\alpha = \mathcal{F}^{-1}(\xi\pm k)^\alpha\cdot\mathcal{F}$.*

**Proof:** We have the jump at infinity defined by

$$\widetilde{\Phi}^{-1}(+\infty)\widetilde{\Phi}(-\infty) = \frac{1}{2}\begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{bmatrix}$$

with

$$\begin{aligned} c_{11} &= e^{i\pi(1+2n-2\varepsilon)} + e^{i\pi(2n-2\varepsilon)}, \\ c_{12} &= -\frac{p_+}{p_-}e^{i\pi(1+2n-2\varepsilon)} + \frac{p_+}{p_-}e^{i\pi(2n-2\varepsilon)}, \\ c_{21} &= -\frac{p_-}{p_+}e^{i\pi(1+2n-2\varepsilon)} + \frac{p_-}{p_+}e^{i\pi(2n-2\varepsilon)}, \\ c_{22} &= e^{i\pi(1+2n-2\varepsilon)} + e^{i\pi(2n-2\varepsilon)}, \\ c_{33} &= 2e^{i\pi(1+2n-2\varepsilon)}, \\ c_{44} &= 2e^{i\pi(2n-2\varepsilon)}. \end{aligned}$$

The jump at infinity in the normal Jordan form is

$$\widetilde{\Phi}^{-1}(+\infty)\widetilde{\Phi}(-\infty) = S^{-1}JS$$

with

$$S=\begin{bmatrix} 0 & 0 & 1 & 0 \\ -\frac{p_-}{2p_+} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{p_-}{2p_+} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

and

$$J=\begin{bmatrix} e^{i\pi(1+2n-2\varepsilon)} & 0 & 0 & 0 \\ 0 & e^{i\pi(1+2n-2\varepsilon)} & 0 & 0 \\ 0 & 0 & e^{i\pi(2n-2\varepsilon)} & 0 \\ 0 & 0 & 0 & e^{i\pi(2n-2\varepsilon)} \end{bmatrix}.$$

For the critical parameters $\varepsilon = \frac{q}{2}$ we have

$$J=\pm\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

according to $|q|$ being even (or 0) or odd, respectively. For the even case, the eigenvalues of $J$ are $\lambda_1 = -1$ with multiplicity 2 and $\lambda_2 = 1$ with multiplicity 2. Introducing the notation $\lambda_j = e^{2\pi i\omega_j}$, we know that the operator $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{\mathcal{C}}},\mathbb{R}_+}$ is normally solvable if and only if $\Re e(\omega_j) \neq \frac{1}{2}$, $j = 1, 2$. So, the operator is not normally solvable due to the eigenvalue $\lambda_1$. The image normalizer operator is a direct consequence of the results presented in [34, §6] and [11]. For the even case we have a similar procedure. $\square$

# 7 Conclusion

In the present paper we were able to characterize the Fredholm property of particular operators associated with an impedance boundary problem which are a generalization of the results presented in [15] and present the image normalization for a particular case. For practical and theoretical reasons, with the Fredholm property we are able to answer further questions about this kind of diffraction problems in particular the invertibility and the image normalization in general of the operators related with the problem. We plan to do this in future works.

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