Weak Greedy Algorithms for Nonlinear Approximation with Quasi-Greedy Bases

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Abstract: We study weak greedy approximation with respect to quasi-greedy bases. For a Hilbert space, we prove that the error of the m-th weak greedy approximation is bounded by the error of best m-term approximation multiplied by an extra factor of order $\ln m$. However we show weak greedy algorithm realizes the best expansional m-term approximation for individual element and best m-term approximation for some sparse classes. Furthermore we establish sharp Lebesgue-type inequality for expansional m-term approximation with quasi-greedy bases in Banach spaces.

Key–Words: Quasi-greedy basis, Weak greedy algorithm, m-term approximation, Sparse classes, Optimality, Hilbert spaces

1 Introduction

Nonlinear m-term approximation with respect to a basis or a dictionary is important in applications, such as signal and information processing, PDE solver and the design of the neural networks, cf. [1]-[7]. Furthermore, it also can be used in two hot topic areas in numerical mathematics: learning theory and compressed sensing, cf. [1]-[3],[7], [20]-[22]. The fundamental problem of nonlinear m-term approximation is how to construct good algorithms of this approximation scheme. It turns out that greedy type algorithms are suitable methods for nonlinear m-term approximation with respect to bases or redundant systems. In this paper we study the efficiency of greedy type algorithms for m-term approximation with respect to a basis. We formulate this problem in a Banach space setting.

Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_k\}_{k=1}^{\infty}$ be a normalized basis for X ($\|\psi_k\| = 1, k \in \mathbb{N}$). All bases considered in our paper are assumed to be normalized. For a given $x \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(x) := \sigma_m(x,\Psi)_X := \inf_{b_k,\Lambda} \|x - \sum_{k\in\Lambda} b_k\psi_k\|_X,$$

where the infimum is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $x \in X$ we consider the expansion

$$x = \sum_{k=1}^{\infty} c_k(x)\psi_k.$$

We call a permutation ρ, $\rho(j) = k_j, j = 1, 2, ...,$ of the positive integers $decreasing$ and write $\rho \in D(x)$ if

$$|c_{k_1}(x)| \geq |c_{k_2}(x)| \geq \dots .$$

In the case of strict inequalities $D(x)$ consists of only one permutation. We define the m-th greedy approximant of x with regard to the basis Ψ corresponding to a permutation $\rho \in D(x)$ by formula

$$G_m(x) := G_m(x,\Psi) := G_m(x,\Psi,\rho) := \sum_{j=1}^{m} c_{k_j}(x)\psi_{k_j}.$$

It is a simple algorithm which describes a theoretical scheme for m-term approximation of an element x. This algorithm is known in the theory of nonlinear approximation under the name of Thresholding Greedy Algorithm (TGA). The beset error we can achieve with the algorithm G_m is

$$\|x - G_m(x)\|_X = \sigma_m(x,\Psi)_X,$$

or a little weaker

$$\|x - G_m(x)\|_X \leq C\sigma_m(x,\Psi)_X$$

for all $x \in X$ with a constant C independent of x and m. The following concept of a greedy basis has been introduced in [14].

Definition 1 *We call a basis Ψ a greedy basis if for every $x \in X$ there exists a permutation $\rho \in D(x)$ such that*

$$\|x - G_m(x, \Psi, \rho)\|_X \leq C\sigma_m(x, \Psi)_X$$

with a constant C independent of x and m.

The examples of greedy basis can be found from [23],[32]. Then we recall the definitions of unconditional and democratic bases.

Definition 2 *A basis $\{\psi_n\}_{n=1}^{\infty}$ of a Banach space X is said to be unconditional if for every choice of sign $\theta = \{\theta_n\}_{n=1}^{\infty}$, $\theta_n = 1$ or -1, $n = 1, 2, \cdots$ the linear operator M_θ, defined by*

$$M_\theta\left(\sum_{n=1}^{\infty} a_n\psi_n\right) = \sum_{n=1}^{\infty} a_n\theta_n\psi_n,$$

is a bounded operator from X to X.

Definition 3 *We recall that a basis $\{\psi_n\}_{n=1}^{\infty}$ in a Banach space X is called democratic if for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$, we have*

$$\|\sum_{n\in P} \psi_n\| \leq D\|\sum_{n\in Q} \psi_n\|$$

with a constant D independent of P and Q.

The following characterization of greedy bases was proved in [14].

Theorem 4 *A basis $\{\psi_n\}_{n=1}^{\infty}$ is greedy if and only if it is unconditional and democratic.*

We refer the reader to [26] and [14] for further discussion of greedy type bases. By Definition 1, greedy bases are those for which we have ideal (up to a multiplicative constant) Lebesgue-type inequalities for greedy approximation. By the Lebesgue-type inequalities we mean an inequality that provides an upper estimate for the error of a particular method of approximation of x by elements of a special form, say form $\mathcal{A}$ in terms of the best possible approximation of x by elements of form $\mathcal{A}$.

In this paper we concentrate on a wider class of bases than greedy bases – quasi-greedy bases. The concept of quasi-greedy basis was introduced in [14].

Definition 5 *The basis Ψ is called quasi-greedy if for every $x \in X$ and every permutation $\rho \in D(x)$ there exists some constant C such that*

$$\sup_m \|G_m(x, \Psi)\| \leq C\|x\|.$$

Subsequently, Wojtaszczyk [11] proved that these are precisely the bases for which the TGA merely converges, i.e.,

$$\lim_{n\to\infty} G_n(x) = x.$$

Some examples of quasi-greedy bases in Banach spaces can be found from [8].

The greedy approximant $G_m(x)$ considered above was defined as the sum

$$\sum_{j=1}^{m} c_{k_j}(x)\psi_{k_j}$$

of the expansion terms with the m biggest (in absolute value) coefficients.

Now we consider a more flexible way of construction of a greedy approximant. The rule of choosing the expansion terms for approximation will be weaker than in the greedy algorithm $G_m(\cdot)$. Instead of taking m terms with the biggest coefficients we now take m terms with near biggest coefficients. We proceed to a formal definition of the Weak Greedy Algorithm with regard to a basis Ψ (see [15]).

Let $t \in (0, 1]$ be a fixed parameter. For a given basis Ψ and a given $x \in X$ denote $\Lambda_m(t)$ any set of m indices such that

$$\min_{k\in\Lambda_m(t)} |c_k(x)| \geq t \max_{k\notin\Lambda_m(t)} |c_k(x)| \tag{1}$$

and define

$$G_m^t(x) := G_m^t(x, \Psi) := \sum_{k\in\Lambda_m(t)} c_k(x)\psi_k.$$

We call it the Weak Greedy Algorithm with the weakness parameter t. It is clear $G_m^1(x, \Psi) = G_m(x, \Psi)$.

The motivation of our paper is the following Lebesgue-type inequality for greedy approximation with respect to a quasi-greedy basis in the L_p space, $1 < p < \infty$, cf. [30]. Here L_p is the space of all p-th power Lebesgue-integrable functions, endowed with the usual norm.

Theorem 6 *Let $1 < p < \infty$, $p \neq 2$, and let Ψ be a quasi-greedy basis of the L_p space. Then for each $f \in L_p$ we have*

$$\begin{aligned}&\|f - G_m^t(f)\|_p \\ &\leq C(\Psi, p, t)m^{|1/2-1/p|}\sigma_m(f)_p.\end{aligned} \tag{2}$$

Theorem 6 does not cover the case $p = 2$. Note that L_2 is a Hilbert space. On the other hand, in [31], the author announced without detailed proof the following result. For any quasi-greedy basis in a Hilbert space,

$$\|x - G_m(x)\| \leq C(\ln m)\sigma_m(x). \quad (3)$$

In this paper, we first show in section 2 that (3) still holds for the weak greedy approximant $G_m^t(x)$. Note that it can be inferred from Theorem 4 the factor $\ln m$ cannot be replaced by a constant, otherwise it implies Ψ is an unconditional basis and this is in contrast with the existence of the conditional quasi-greedy basis. It is known that the above inequality can be improved in the sense that an extra factor $\ln m$ can be replaced by a slower growing factor. In fact for uniformly bounded quasi-greedy bases in a Hilbert space the factor can be improved to $(\ln m)^{1/2}$. On the other hand, we discover the optimality of $G_m^t(\cdot, \Psi)$ in the following two cases. One is $G_m^t(\cdot, \Psi)$ realizes the expansional best m-term approximation up to a constant. We will prove this fact in section 3. The other is $G_m^t(\cdot, \Psi)$ provide the optimal convergence rate of best m-term approximation on some sparse classes. We will prove this result in section 4.

2 Efficiency of Weak Greedy Algorithms for m-term approximation

To compare the error of weak greedy algorithm with best possible m-term approximation, we define the following quantity

$$r_m^t(\Psi) = \sup_{x \in X} \frac{\|x - G_m^t(x, \Psi)\|}{\sigma_m(\Psi)}.$$

The upper estimates of r_m is called Lebesgue-type inequality. The main result of this section is the following Lebesgue-type inequality. In what follows, for two nonnegative sequences a_n and b_n the order inequality $a_n \ll b_n$ means that there is a number C such that $a_n \leq C \cdot b_n$.

Theorem 7 *Let Ψ be a quasi-greedy basis of a Hilbert space H. Then*

$$r_m^t(\Psi) \ll \ln m.$$

In the proof of Theorem 7 we will use the notation

$$a_n(x) := |c_{k_n}(x)|$$

for the decreasing rearrangement of the coefficients of x. For a set of indices Λ we define the corresponding projection operator as follows

$$P_\Lambda(x) := \sum_{k \in \Lambda} c_k(x)\psi_k.$$

The next theorem was proved in [11].

Theorem 8 *Let $\Psi = \{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of a Hilbert space H. Then for each $x \in H$ we have*

$$C_1 \sup_n n^{1/2} a_n(x) \leq \|x\| \leq C_2 \sum_{n=1}^\infty n^{-1/2} a_n(x).$$

The following theorem is a corollary of the above Theorem 8.

Theorem 9 *Let Ψ be a quasi-greedy basis of a Hilbert space H. Then for any set of indices Λ, one has the following inequality*

$$\|P_\Lambda(x)\| \leq C \ln |\Lambda| \|x\|.$$

Proof of Theorem 9: Let $m := |\Lambda|$. Using Theorem 8 we get

$$\begin{aligned}
\|P_\Lambda(x)\| &\leq C_4 \sum_{n=1}^m n^{-1/2} a_n(P_\Lambda(x)) \\
&= C_4 \sum_{n=1}^m n^{-1}(n^{1/2} a_n(P_\Lambda(x))) \\
&\leq C_4 \sum_{n=1}^m n^{-1}(n^{1/2} a_n(x)) \\
&\leq C_5 \sup_n n^{1/2} a_n(x) \sum_{n=1}^m n^{-1} \\
&\leq (C_5 C_3^{-1} \ln m)\|x\|
\end{aligned}$$

where in the second inequality we use $a_n(P_\Lambda(x)) \leq a_n(x)$. So the proof of Theorem 9 is complete.

We know from the above definitions that a quasi-greedy basis is not necessarily an unconditional basis. However, quasi-greedy bases have some properties that are close to those of unconditional bases. We formulate one of them which will be used in our study of quasi-greedy bases. We cite the following known Lemma 10 from [9]. It will be convenient to define the quasi-greedy constant K to be the least constant such that

$$\|G_m(x)\| \leq K\|x\|$$

and

$$\|x - G_m(x)\| \leq K\|x\|, \quad x \in X.$$

Lemma 10 *Suppose Ψ is a quasi-greedy basis with a quasi-greedy constant K. Then, for any numbers a_j and any finite set of indices P, we have*

$$(2K)^{-2}\min_{j\in P}|a_j|\|\sum_{j\in P}\psi_j\| \le \|\sum_{j\in P}a_j\psi_j\|$$
$$\le 2K\max_{j\in P}|a_j|\|\sum_{j\in P}\psi_j\|.$$

Proof of Theorem 7: It is known from the definition of σ_m that for a given $\epsilon > 0$, there is a polynomial

$$p_m(x) = \sum_{k\in P} b_k\psi_k, \quad |P| = m,$$

satisfying the inequality

$$\|x - p_m(x)\| \le \sigma_m(x) + \epsilon. \tag{4}$$

Denote by Q the set of indices picked by the greedy algorithm after m iterations

$$G^t_m(x) = \sum_{k\in Q} c_k(x)\psi_k.$$

We use the representation

$$\begin{aligned} x - G^t_m(x) &= x - S_Q(x) \\ &= x - S_P(x) + S_P(x) - S_Q(x). \end{aligned} \tag{5}$$

First, we bound

$$\begin{aligned} &\|x - S_P(x)\| \\ &= \|x - p_m(x) - S_P(x - p_m(x))\| \\ &\le (1 + \ln(m))\|x - p_m(x)\|. \end{aligned} \tag{6}$$

Second, we write

$$\begin{aligned} &\|S_P(x) - S_Q(x)\| = \|S_{P\setminus Q}(x) - S_{Q\setminus P}(x)\| \\ &\le \|S_{P\setminus Q}(x)\| + \|S_{Q\setminus P}(x)\|. \end{aligned} \tag{7}$$

We begin with estimating the second term in the right side of (7)

$$\begin{aligned} &\|S_{Q\setminus P}(x)\| = \|S_{Q\setminus P}(x - p_m(x))\| \\ &\le \ln(m)\|x - p_m(x)\|. \end{aligned} \tag{8}$$

For the first term we have by Lemma 10

$$\begin{aligned} &\|S_{P\setminus Q}(x)\| \le 2K\max_{k\in P\setminus Q}|c_k(x)|\|\sum_{k\in P\setminus Q}\psi_k\| \\ &\le 2Kt^{-1}\min_{k\in Q\setminus P}|c_k(x)|\|\sum_{k\in Q\setminus P}\psi_k\| \\ &\le t^{-1}(2K)^3\|S_{Q\setminus P}(x)\|, \end{aligned} \tag{9}$$

where in the first inequality we use the fact any quasi-greedy basis in a Hilbert space is democratic, cf. [31].

Combining (4) – (9) we obtain

$$\|x - G^t_m(x)\| \le (1 + 2\ln(m) + t^{-1}(2K)^3\ln(m))\sigma_m(x)$$

which completes the proof of Theorem 7.

3 Efficiency of Weak Greedy Algorithms for expansional m-term approximation

The conclusion of Theorem 7 seems a little discouraging. However if one replace the best m-term approximation by expansional best m-term approximation then the factor $\ln m$ can be removed. To prove this, we first recall the definition of $\tilde{\sigma}_m$ expansional best m-term approximation error of x with regard to Ψ from [26].

$$\tilde{\sigma}_m(x) := \tilde{\sigma}_m(x,\Psi) := \inf_{|\Lambda|=m}\|x - \sum_{k\in\Lambda}c_k(x)\psi_k\|.$$

In this section we show for any quasi-greedy basis in a Hilbert space H the inequality

$$\|x - G^t_m x\| \le C(t)\tilde{\sigma}_m(x,\Psi) \tag{10}$$

holds for every $x \in H$.

In fact the inequality (10) can be derived from a more general result for a Banach space X. Similarly, to compare the error of weak greedy algorithm with the best possible expansional m-term approximation, we define the following quantity

$$\tilde{r}^t_m(\Psi) = \sup_{x\in X}\frac{\|x - G^t_m x\|}{\tilde{\sigma}_m(\Psi)}.$$

We will show that this quantity is equivalent to the quantity

$$\mu_m(\Psi) := \sup_{k\le m}(\sup_{|\Lambda|=k}\|\sum_{i\in\Lambda}\psi_i\|/\inf_{|\Lambda|=k}\|\sum_{i\in\Lambda}\psi_i\|)$$

which measures the democratic properties of the basis Ψ. To state our results we introduce the notation $\asymp$. For two nonnegative sequences a_n and b_n the relation $a_n \asymp b_n$ means $a_n \ll b_n$ and $b_n \ll a_n$.

Theorem 11 *Let Ψ be a quasi-greedy basis of a Banach space X with a quasi-greedy constant K. Then for a fixed $t \in (0,1]$, we have*

$$\tilde{r}^t_m(\Psi) \asymp \mu_m(\Psi).$$

Theorem 12 *Let Ψ be an unconditional basis of a Banach space X. Then for a fixed $t \in (0,1]$, we have*

$$r^t_m(\Psi) \asymp \mu_m(\Psi).$$

Proof of Theorem 11: We first prove the upper bound. Note that $\mu_m(\Psi)$ is an increasing function of

m and for any two sets of indices A and B, $|A| = |B| = m$ we have

$$\|\sum_{k\in A}\psi_k\| \le \mu_m(\Psi)\|\sum_{k\in B}\psi_k\|.$$

Therefore the upper bound follows from Theorem 8 in [30]. Then we turn to the proof of the lower bound. To this end we need the following lemma.

Lemma 13 *If Ψ is a quasi-greedy basis with a quasi-greedy constant K of a Banach space X, then for each m there exist disjoint sets A and B with $|A| = |B| \le m$ such that*

$$\|\sum_{i\in A}\psi_i\|\|\sum_{i\in B}\psi_i\|^{-1} \ge (2K)^{-1}\mu_m.$$

Proof of Lemma 13: If $\mu_m \le 2K$ then the conclusion is obvious. Otherwise for a given $\varepsilon > 0$ take sets A_1 and A_2 with $|A_1| = |A_2| \le m$ such that

$$\|\sum_{i\in A_1}\psi_i\|\|\sum_{i\in A_2}\psi_i\|^{-1} > \max\{2K, \mu_m - \varepsilon\}.$$

We write

$$a = \|\sum_{i\in A_1}\psi_i\| \quad b = \|\sum_{i\in A_2}\psi_i\|$$

and

$$a_1 = \|\sum_{i\in A_1\cap A_2}\psi_i\| \quad a_2 = \|\sum_{i\in A_1\setminus A_2}\psi_i\|.$$

Note that since Ψ is a quasi-greedy basis, we have $a_1 \le Kb$, thus

$$2K < a/b \le Ka/a_1$$

and hence $a_1 < a/2$. This implies

$$a/b \le (a_1 + a_2)/b < a/2b + a_2/b$$

so $a_2/b > (1/2)(a/b)$. Let A_3 be any set of cardinality m which contains $A_1 \setminus A_2$ and is disjoint with A_2. Define $a_3 = \|\sum_{i\in A_3}\psi_i\|$. Again we have $a_2 \le Ka_3$, and hence

$$a_3/b > (1/2K)(a/b) > (1/2K)(\mu_m - \varepsilon).$$

Since ε was arbitrary we get $a_3/b \ge (1/2K)\mu_m$. Set $A = A_3, B = A_2$, we complete the proof.

Now we return to the proof of the lower bound of Theorem 11. Let us take sets as in Lemma 13 and denote $|A| = |B| = k \le m$. Let $C \supset A$ be a set of cardinality m disjoint with B. Consider

$$x = (1+\varepsilon)(\sum_{i\in B}\psi_i + \sum_{i\in C\setminus A}\psi_i) + t\sum_{i\in A}\psi_i,$$

where $\varepsilon > 0$. It is clear that

$$x - G_m^t x = \sum_{i\in A} t\psi_i$$

and

$$\tilde{\sigma}_m(x) \le \|P_B x\| = (1+\varepsilon)\|\sum_{i\in B}\psi_i\|.$$

This inequality and Lemma 11 give

$$\begin{aligned} \tilde{r}_m^t &\ge \frac{\|x - G_m^t x\|}{\tilde{\sigma}_m(x)} \ge \frac{\|\sum_{i\in A} t\psi_i\|}{(1+\varepsilon)\|\sum_{i\in B}\psi_i\|} \\ &\ge \frac{t}{(1+\varepsilon)2K}\mu_m. \end{aligned}$$

Since ε was arbitrary we get the desired result.

Proof of Theorem 12: It is well known that an unconditional basis is a quasi-greedy basis. It also clear that for an unconditional basis

$$\tilde{\sigma}_m(x) \asymp \sigma_m(x).$$

Therefore the conclusion of Theorem 12 follows directly from Theorem 11.

Note that the case of $t = 1$ the result of Theorem 12 was obtained in [31]. As mentioned before Ψ is democratic. Therefore $\mu_m(\Psi) \asymp 1$. Thus the inequality (10) follows from Theorem 11.

4 Optimality of weak greedy approximation on some sparse classes

In this section we prove that for some sparse classes the weak greedy algorithm G_m^t realizes the best m-term approximation. In what follows, we recall the definition of the class.

For $r > 0$, we define

$$\begin{aligned} F^r(\Psi) &:= F^r_\infty(\Psi) := \{x : |c_{n_k}(x,\Psi)| \\ &\le k^{-r},\ k = 1, 2, \ldots\} \end{aligned}$$

where the number r describes the sparsity of the class. This class plays an important role in the study of the nonlinear m-term approximation with respect to bases, cf. [29].

To study the worst case of the best m-term approximation error on F^r, we define the following quantities

$$\sigma_m(F^r) := \sup_{x\in F^r}\sigma_m(x)$$

and

$$\tilde{\sigma}_m(F^r) := \sup_{x\in F^r}\tilde{\sigma}_m(x).$$

We will also introduce the m-th greedy remainder

$$H^t_m(x) := x - G^t_m(x)$$

and define

$$H^t_m(F^r) := \sup_{x\in F^r} H^t_m(x).$$

For simplicity we write $H_m(x)$ for $H^1_m(x)$.

The main result of this section is the following theorem.

Theorem 14 *If Ψ is a quasi-greedy basis of a Hilbert space H, then for $r > 1/2$*

$$\sigma_m(F^r) \asymp H^t_m(F^r) \asymp m^{1/2-r}.$$

In fact the upper bound of Theorem 14 can be established for a wider class of bases which are defined as follows.

Definition 15 *A basis Ψ is called unconditional for constant coefficients (UCC) if there exist constants C_1 and C_2 such that for each finite subset $A \subset \mathbb{N}$ and for each choice of signs $\varepsilon_i = \pm 1$ we have*

$$C_1 \left\|\sum_{i\in A}\psi_i\right\| \le \left\|\sum_{i\in A}\varepsilon_i\psi_i\right\| \le C_2\left\|\sum_{i\in A}\psi_i\right\|.$$

To prove Theorem 14 we need some known results which were proved in [9],[29]. To formulate these results we need some of the basic concepts of the Banach space theory from [16]. First, let us recall the definition of type and co-type of a Banach space. Let $(\varepsilon_i = r_i(\omega))$ be a sequence of independent Rademacher variables, we write

$$\left(Ave_{\varepsilon_k=\pm1}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|^p\right) := \int_0^1 \sum_{k=1}^n \|r_k(\omega)f_k\|^p\, d\omega.$$

We say that a Banach space X has type p if there exists a universal constant C_3 such that for $x_k \in X$

$$\left(Ave_{\varepsilon_k=\pm1}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|^p\right)^{1/p} \le C_3\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p},$$

and X is of co-type q if there exists a universal constant C_4 such that for $x_k \in X$

$$\left(Ave_{\varepsilon_k=\pm1}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|^q\right)^{1/q} \ge C_4\left(\sum_{k=1}^n \|x_k\|^q\right)^{1/q}.$$

The following two theorems were proved in [29].

Theorem 16 *Let X be a Banach space with type p, $1 < p \le 2$. If a basis Ψ of X is UCC, then for $r > 1/p$*

$$H_m(F^r) \ll m^{1/p-r}.$$

Theorem 17 *Let Ψ be a quasi-greedy basis of a Banach space X. Then for $a > 0$ the following two statements are equivalent.*

i) For any finite set Λ of indices

$$\left\|\sum_{k\in\Lambda}\psi_k\right\| \ge c|\Lambda|^a.$$

ii) For any two positive integers $N < M$ we have

$$a_M(f) \ll \|H_N(f)\|(M-N)^{-a}.$$

The next theorem was obtained in [9].

Theorem 18 *Let Ψ be a normalized quasi-greedy basis of a Hilbert space H. Then, for any $x \in H$ and $\lambda > 1$*

$$\|x - G_{\lambda m}(x,\Psi)\| \le C(\lambda)\sigma_m(x,\Psi).$$

Proof of Theorem 14: Upper bound: It is well-known that any Hilbert space H is of type 2, see [16], [31]. It is also known that a quasi-greedy base is unconditional for constant, cf. [16],[31]. Thus by Theorem 16

$$H_m(F^r) \ll m^{1/2-r}. \tag{11}$$

By (10) we have

$$H^t_m(F^r) \ll \tilde{\sigma}_m(F^r). \tag{12}$$

Note that

$$\tilde{\sigma}_m(F^r) \ll H_m(F^r).$$

Thus combining (11) and (12) we complete the proof of the upper bound. Lower bound: Consider the element

$$x_m = m^{-r}\sum_{k=1}^{2m}\psi_k.$$

It is known from [29] for a UCC basis Ψ of a Hilbert space H

$$\left\|\sum_{k\in\Lambda}\psi_k\right\| \asymp |\Lambda|^{1/2}. \tag{13}$$

Therefore applying Theorem 17 by setting $M = 2N = 2m$, we have

$$H_m(\,x_m) \ge c\cdot m^{1/2-r}.$$

Further by Theorem 18 where we take $\lambda = 2$, we have

$$\sigma_m(x_{2m}) \geq C \cdot H_{2m}(x_{2m}) \geq C \cdot m^{1/2-r}.$$

It is clear $x_{2m} \in F^r$. Thus we complete the proof of the lower bound.

For UCC bases in Hilbert spaces we have the following theorem.

Theorem 19 *If Ψ is a basis of UCC of a Hilbert space H, then for $r > 1/2$*

$$\tilde{\sigma}_m(F^r) \asymp H_m(F^r) \asymp m^{1/2-r}.$$

Proof of Theorem 19: *The upper bound is known from the proof of Theorem 14. The lower bound can be derived from the following theorem.*

Theorem 20 *Let X be a Banach space with co-type q, $q \geq 2$. If a basis Ψ of X is UCC, then there exists a $f_m \in F^r$ such that*

$$\tilde{\sigma}_m(f_m) \geq c \cdot m^{1/q-r}.$$

Proof of Theorem 20: Let Λ be a set of indices satisfying $\Lambda \subset \{1, 2, ..., 2m\}$ and $|\Lambda| = m$. Since X has co-type q, we have

$$\begin{aligned} &\left(Ave_{\varepsilon=\pm1}\left\|\sum_{k\in\Lambda}\varepsilon_k\psi_k\right\|^q\right)^{1/q} \\ &\geq C\left(\sum_{k\in\Lambda}\|\psi_k\|^q\right)^{1/q} \\ &\geq c\cdot m^{1/q}. \end{aligned}$$

Since Ψ is UCC, we get

$$\begin{aligned} \left\|\sum_{k\in\Lambda}\psi_k\right\| &\asymp \left(Ave_{\varepsilon=\pm1}\left\|\sum_{k\in\Lambda}\varepsilon_k\psi_k\right\|^q\right)^{1/q} \\ &\geq c\cdot m^{1/q}. \end{aligned} \tag{14}$$

Consider the element

$$f_m = m^{-r}\sum_{k=1}^{2m}\psi_k.$$

It follows from (14) that

$$\tilde{\sigma}_m(f_m) \geq c \cdot m^{1/q-r}.$$

It is clear $f_m \in F^r$, thus we complete the proof of Theorem 20.

Now we return to the proof of Theorem 19. It is well-known that a Hilbert space H is of co-type 2, cf. [16]. Thus the lower bound of Theorem 19 follows from Theorem 20. So we complete the proof of Theorem 19.

5 Scheme of constructing quasi-greedy bases

In this section we describe a scheme of constructing quasi-greedy bases in Banach spaces from [10]. This would help the reader have a better understand of the structure of this type of bases.

Let X be a separable Banach space and Φ be a Besselian basis of X. Assume that Φ can be split into two systems $F = \{f_s\}_{s=1}^{\infty}$, $f_s = \phi_{m(s)}$ and $E = \{e_j\}_{j=1}^{\infty}$, $e_j = \phi_{n(j)}$ with increasing sequences $m(s)$ and $n(j)$ in such a way that E has the following property. For any sequence $\{c_j\}$ we have

$$\|\sum_{j=1}^{\infty}\|c_je_j\| \leq C(\sum_{j=1}^{\infty}|c_j|^2)^{1/2}.$$

In our construction of quasi-greedy bases we will use special matrices. Let $\mathcal{A} = \{A(n)\}_{n=1}^{\infty}$ be a collection of matrices which satisfies the following properties.

S1. Singular numbers of matrix $A(n)$ and their inverse are uniformly bounded.

S2. For the elements of the first column of matrix $A(n) = [a_{ij}(n)]$ we have

$$|a_{i1}(n)| \leq Cn^{-1/2}.$$

Let n_k be an increasing sequence of integers such that

$$n_{k+1} \geq n_k^2.$$

For a fixed natural number k we pick the basis elements

$$g_1^k = f_k, g_i^k := e_{S_{k-1}+i-1},$$

$i = 2, ..., n_k$, where S_j is defined recursively as

$$S_j = S_{j-1} + n_j - 1,\ j = 1, 2, ..., S_0 = 0.$$

We build a new system of elements $\{\psi_i^k\}_{i=1}^{n_k}$ using a matrix $A(n_k)$ in the following way:

$$(\psi_1^k, ..., \psi_{n_k}^k)^T = A(n_k)(g_1^k, ..., g_{n_k}^k)^T.$$

That is, for $i \in [1, n_k]$ we have

$$\psi_i^k = \sum_{j=1}^{n_k} a_{ij}(n_k)g_j^k.$$

We define the system $\Psi = \{\psi_i^k\}_{i=1,k=1}^{n_k,\infty}$ ordered in the lexicographical way: $j(k', i') > j(k, i)$ if either $k' > k$ or $k' = k$ and $i' > i$. The following theorem was proved in [10].

Theorem 21 *The basis Ψ is a quasi-greedy basis of X.*

6 Some remarks

In this section we first survey some recent results related to greedy approximation with respect to quasi-greedy bases. Then we present some open problems which seem to be good candidates for future research on this topic.

The first one is the optimal bound of $r_m^t(\Psi)$ for quasi-greedy bases in a general Banach space (real or complex), see [12],[13].

To state this result we consider the sequence

$$k_m := \sup_{|\Lambda| \leq m} \|P_\Lambda\|$$

where P_Λ is the project operator defined as before. Note that the sequence k_m quantifies the conditionality of the basis Ψ.

Theorem 22 *Let Ψ be a quasi-greedy basis of a Banach space (real or complex) X. Then*

$$r_m^t(\Psi) \asymp \max\{k_m, \mu_m\}.$$

This theorem plays an important role in the study of the efficiency of TGA with quasi-greedy bases. Note that Theorem 22 is a generalization of Theorem 4. The case of $t = 1$ was proved in [12]. It is not difficult to generalize it to the case of Weak Greedy Algorithm. Here we omit the details. Then we cite a result from [11] which provides a lower bound of $k_m(\Psi)$ for quasi-greedy bases in Hilbert spaces.

Theorem 23 *For any $0 < \alpha < 1$ there exists a conditional quasi-greedy basis Ψ of a Hilbert space H such that*

$$k_m(\Psi) \geq c(\ln m)^\alpha.$$

Note that a quasi-greedy of a Hilbert space is democratic. Then $\mu_m \asymp 1$. Thus combining Theorem 23 with Theorem 22 we immediately yield the following theorem.

Theorem 24 *For any $0 < \alpha < 1$ there exists a conditional quasi-greedy basis Ψ of a Hilbert space H such that*

$$r_m^t(\Psi) \geq c(\ln m)^\alpha.$$

Using Theorem 22 one can reprove the Lebesgue-type inequality (1.2). Now a problem arise: is the factor $m^{|1/2-1/p|}$ sharp? That is, can we find a conditional quasi-greedy basis Ψ which satisfies the inequality

$$r_m^t(\Psi) \asymp m^{|1/2-1/p|}.$$

Note that for some unconditional quasi-greedy basis Ψ in L_p, P. Wojtaszczyk proved

$$r_m^1(\Psi) \asymp m^{|1/2-1/p|},$$

see [31]. By the way we point out using the method one can study the sparse class induced by democratic quasi-greedy bases in Banach spaces.

One more important problem in the study of greedy approximation with respect to non-greedy bases (including quasi-greedy bases) is how to construct greedy-type algorithm which realizes the best m-term approximation. Recent results of some authors show the Weak Chebyshev Greedy Algorithm (WCGA) seems to be a good candidate, see [17],[24],[28]. This algorithm is a generalization of the Weak Orthogonal Matching Pursuit which is widely used in compressed sensing, cf. [18],[19],[33]. Lebesgue-type inequalities for WCGA have been established in [17],[28]. In particular, applying the general results in [28] to the case of some quasi-greedy bases one can see that WCGA significantly improved the error bound. Now we recall this result.

Let X be a Banach space. A set of elements D from X is a dictionary if each $g \in D$ has norm one, and the closure of span D is X. For a nonzero element $g \in X$ we let F_g denote a norming functional for g

$$\|F_g\|_{X^*} = 1, \;\; F_g(g) = \|g\|_X$$

The existence of such a functional is guaranteed by the Hahn-Banach theorem.

Let f_0 be given. Then for each $m \geq 1$ we have the following inductive definition.

(1) $\psi_m \in D$ is any element satisfying

$$|F_{f_{m-1}}(\psi_m)| \geq t \sup_{g \in D} |F_{f_{m-1}}(g)|.$$

(2) Define

$$\Psi_m = span\,\{\psi_j\}_{j=1}^m$$

and define G_m to be the best approximant to f_0 from Ψ_m.

(3) Let

$$f_m = f_0 - G_m.$$

Theorem 25 *If Ψ is a uniformly bounded quasi-greedy basis of L_p, $2 \leq p < \infty$, then for $f_0 \in L_p$*

$$\|f_{m \ln\ln(m+3)}\| \leq \sigma_m(f_0).$$

The existence of uniformly bounded quasi-greedy bases was proved in [10]. In [28] the author also proved WCGA provides almost optimal sparse approximation for the trigonometric system which is not a quasi-greedy basis. However in many other important cases whether WCGA can be better than TGA is still unknown. It is interesting to study the Lebesgue-type inequalities for WCGA with quasi-greedy bases in a optimal way.

7 Conclusion

Our results show the weak greedy algorithm (WGA) is suitable for m-term approximation with quasi-greedy bases. In the case of Hilbert spaces the error of the m-th weak greedy approximation is bounded by the error of best m-term approximation multiplied by an extra factor of order $\ln m$. For some quasi-greedy bases $\ln m$ can be replaced by a slow growing factor. Moreover WGA realizes the best expansional m-term approximation for individual element. If we make some mild assumptions on the sparsity of the element, then WGA realizes best m-term approximation for some sparse classes. It is clear that in the case of $t < 1$, we have more flexibility in building a weak greedy approximant $G_m^t(x)$ than in building $G_m(x)$. Moreover the effect of the efficiency of the algorithm is minimal: it is only reflected in a multiplicative constant. This is the main advantage of the weak greedy approximant.

Finally we would like to say that although the theory of greedy approximation is developing rapidly, and results are spread over hundreds of papers by different authors, the field is still very active and many important problems remain open, see [25]-[27].

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