Some comparison results with new effective preconditioners for *L*-matrices

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Abstract: Based on the work of Wang and Li [A new preconditioned AOR iterative method for L-matrices, J. Comput. Appl. Math. 229 (2009) 47-53], in this paper, a new preconditioner for the AOR method is proposed for solving linear systems whose coefficient matrix is an L-matrix. Several comparison theorems are shown for the proposed method with two preconditioners. It follows from the comparison results that the proposed new method can achieve faster convergence than the preconditioner introduced by Wang and Li.

Key–Words: Preconditioner; *L*-matrix; AOR mehtod; SOR mehtod

1 Introduction

Consider the linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$ are given and $x \in \mathbb{R}^{n\times 1}$ is unknown. For any matrix splitting $A = M-N$ with $\det(M) \neq 0$, the basic iterative method for solving (1) is

$$x^{k+1} = M^{-1}Nx^k + M^{-1}b,\ k = 0, 1, 2, \dots$$

For simplicity, without loss of generality, we assume that $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and upper triangular matrices obtained from A, respectively. Then, the iteration matrix of the classical AOR iterative method [2] is defined

$$T_{rw} = (I-rL)^{-1}((1-w)I+(w-r)L+wU), \tag{2}$$

where w and r are real parameters with $w \neq 0$.

The spectral radius of the iteration matrix is decisive for the convergence and stability, and the smaller it is, the faster the iterative method converges when the spectral radius is smaller than one. The effective method to decrease the spectral radius is to precondition the linear systems (1), namely,

$$PAx = Pb, \tag{3}$$

where P is a nonsingular matrix. Let

$$PA = D^\star - L^\star - U^\star.$$

Applying the AOR method, we get the corresponding preconditioned AOR iterative method, whose iteration matrix is

$$T^\star(w) = (D^\star-rL^\star)^{-1}((1-w)D^\star+(w-r)L^\star+wU^\star). \tag{4}$$

So far, various types of preconditioners have been considered [3–7,10,11]. Recently, in [3], Wang and Li presented a preconditioned AOR iterative method by using the preconditioners $\tilde{P} = I + \tilde{S}$ and $\bar{P} = I + \bar{S}$ with

$$\tilde{S} = \begin{bmatrix} 0 & \cdots & 0 & -\frac{a_{1n}}{\alpha} - \beta \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\bar{S} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{\alpha} - \beta & 0 & \cdots & 0 \end{bmatrix},$$

where α and β are real parameters.

Let

$$\tilde{A}x = \tilde{b}, \tag{5}$$

where $\tilde{A} = (I + \tilde{S})A$, $\tilde{b} = (I + \tilde{S})b$, and

$$\bar{A}x = \bar{b}, \tag{6}$$

where $\bar{A} = (I + \bar{S})A$ and $\bar{b} = (I + \bar{S})b$.

Let $\tilde{A} = \tilde{P}A$ and $\tilde{S}L = \tilde{D}_1 + \tilde{U}_1$, where $\tilde{D}_1$ is a diagonal matrix and $\tilde{U}_1$ is a strictly upper triangular matrix. Then, from $\tilde{S}U = 0$ one obtains

$$\begin{aligned}\tilde{A} &= (I + \tilde{S})(I - L - U) \\ &= I - L - U + \tilde{S} - \tilde{S}L \\ &= \tilde{D} - \tilde{L} - \tilde{U},\end{aligned}$$

where $\tilde{D} = I - \tilde{D}_1$, $\tilde{L} = L$ and $\tilde{U} = U + \tilde{U}_1 - \tilde{S}$.

Let $\bar{A} = \bar{P}A$ and $\bar{S}U = \bar{D}_1 + \bar{L}_1$, where $\bar{D}_1$ is a diagonal matrix and $\bar{L}_1$ is a strictly lower triangular matrix. Then, from $\bar{S}L = 0$ one obtains

$$\begin{aligned}\bar{A} &= (I + \bar{S})(I - L - U) \\ &= I - L - U + \bar{S} - \bar{S}U \\ &= \bar{D} - \bar{L} - \bar{U},\end{aligned}$$

where $\bar{D} = I - \bar{D}_1$, $\bar{L} = L - \bar{S} + \bar{L}_1$ and $\bar{U} = U$.

If we apply the AOR iterative method to the preconditioned linear system (3) with $\bar{A}$ and $\tilde{A}$, then two different forms of AOR iteration matrix can be denoted by

$$\begin{aligned}\tilde{T}_{rw} =&(\tilde{D} - r\tilde{L})^{-1}((1-w)\tilde{D} \\ &+ (w-r)\tilde{L} + w\tilde{U}),\end{aligned} \tag{7}$$

and

$$\begin{aligned}\bar{T}_{rw} =&(\bar{D} - r\bar{L})^{-1}((1-w)\bar{D} \\ &+ (w-r)\bar{L} + w\bar{U}).\end{aligned} \tag{8}$$

In this paper, we propose a new preconditioned AOR method for solving (1) and discuss its convergence property. From the comparison theorems, we can conclude that the new method is better than the preconditioner of Wang and Li in [3].

2 Preparatory knowledge

For an $n \times n$ matrix A, the directed graph $\Gamma(A)$ of A is defined to be the pair (V, E), where $V = \{1, 2, \ldots, n\}$ is a set vertices and $E = \{(i, j) : a_{ij} \neq 0,\ i, j = 1, 2, \ldots, n\}$ is a set of arcs. A path from i_1 to i_p is an ordered tuple of vertices $(i_1, i_1, \ldots, i_p)$ such that each k, $(i_k, i_{k+1}) \in E$. A path $(i_1, i_1, \ldots, i_p)$ is said to by a cycle provided that $i_1, i_1, \ldots, i_p$ are pairwise distinct and $i_1 = i_p$. A directed graph is strongly connected if for any two vertices i, j there is a path from i to j. $\rho(\cdot)$ denotes the spectral radius of a matrix.

Definition 1 *[8] A matrix A is irreducible if the directed graph associated with A is strongly connected.*

Definition 2 *[9] A matrix A is an L-matrix if $a_{ii} > 0$; $i = 1, \cdots, n$ and $a_{ij} \leq 0$, for all $i, j = 1, 2, \cdots, n$; $i \neq j$.*

Lemma 3 *[8] Let $A \geq 0$ be an irreducible matrix. Then*

(a) A has a positive eigenvalue equal to its spectral radius.

(b) A has an eigenvector $x > 0$ corresponding to $\rho(A)$.

(c) $\rho(A)$ is a simple eigenvalue of A.

Lemma 4 *[1] Let A be a nonnegative matrix. Then*

(i) If $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$.

(ii) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then

$$\alpha \leq \rho(A) \leq \beta$$

and x is a positive vector.

3 The new preconditioned AOR iteration

Now, we consider the preconditioned linear system

$$\hat{A}x = \hat{b}, \tag{9}$$

where $\hat{A} = (I + \hat{S})A$ and $\hat{b} = (I + \hat{S})b$ with

$$\hat{S} = \begin{bmatrix} 0 & 0 & \cdots & -\frac{a_{1n}}{\alpha} - \beta \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{\alpha} - \beta & 0 & \ldots & 0 \end{bmatrix}.$$

The coefficient matrix of (9) can be expressed as

$$\hat{A} = \hat{D} - \hat{L} - \hat{U},$$

where $\hat{D}$=diag($\hat{A}$), $\hat{L}$ and $\hat{U}$ are strictly lower and upper triangular matrices obtained from $\hat{A}$, respectively. By calculation, we obtain that

$$\hat{D} \;\; = \left[\begin{array}{ccc} 1 - (\frac{a_{1n}}{\alpha} + \beta)a_{n1} & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots \end{array} \right.$$

$$\begin{array}{c} 0 \\ 0 \\ \vdots \\ 1-(\frac{a_{n1}}{\alpha}+\beta)a_{1n} \end{array}\Bigg] \tag{10}$$

$$\hat{L} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -a_{n-1,1} & -a_{n-1,2} & & \\ (\frac{a_{n1}}{\alpha}+\beta)-a_{n1} & (\frac{a_{n1}}{\alpha}+\beta)a_{12}-a_{n2} & (\frac{a_{n1}}{\alpha}+\beta)a_{1,n-1}-a_{n,n-1} & 0 \end{bmatrix} \tag{11}$$

$$\hat{U} = \begin{bmatrix} 0 & (\frac{a_{1n}}{\alpha}+\beta)a_{n2}-a_{12} & \cdots & \cdots & (\frac{a_{1n}}{\alpha}+\beta)-a_{1n} \\ 0 & 0 & -a_{23} & \cdots & -a_{2,n-1} & -a_{2n} \\ 0 & & \ddots & & \ddots & \vdots \\ 0 & 0 & & \ddots & -a_{n-1,n} & \\ 0 & 0 & \cdots & & & 0 \end{bmatrix}. \tag{12}$$

Applying the AOR method to the preconditioned linear system (9), we have the corresponding iteration matrix of the preconditioned AOR iterative method,

$$\begin{aligned} \hat{T}_{rw} =& (\hat{D}-r\hat{L})^{-1}[(1-w)\hat{D} \\ &+ (w-r)\hat{L}+w\hat{U}]. \end{aligned} \tag{13}$$

4 Analysis of convergence

Similar to Lemma 3 and Theorem 1 in [3], we have the following Lemma 5 and Theorem 6.

Lemma 5 *Let A and $\tilde{A}$ be coefficient matrices of the linear systems* (1) *and* (5)*, respectively. If* $0 \le r \le w \le 1$ $(w \ne 1$ *and* $r \ne 1)$ *and A is an irreducible L-matrix with* $0 < a_{1n}a_{n1} < \alpha$ $(\alpha > 1)$, $\beta \in (-\frac{a_{1n}}{\alpha}+\frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha}) \cap ((1-\frac{1}{\alpha})a_{1n}, -\frac{a_{1n}}{\alpha})$*, then the iterative matrices T_{rw} and $\tilde{T}_{rw}$ are nonnegative and irreducible.*

Proof. The proof of the iterative matrix T_{rw} is similar to Lemma 3 in [5]. Here we only give the proof for $\tilde{T}_{rw}$.

By simple computations, when $a_{1n}a_{n1} < \alpha$ and $\beta \in (-\frac{a_{1n}}{\alpha}+\frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha}) \cap ((1-\frac{1}{\alpha})a_{1n}, -\frac{a_{1n}}{\alpha})$, we can obtain $1-(\frac{a_{1n}}{\alpha}+\beta)a_{n1} > 0$, $(\frac{a_{1n}}{\alpha}+\beta)-a_{1n} > 0$ and $\frac{a_{1n}}{\alpha}+\beta < 0$. It is easy to get that $\tilde{D} > 0$, $\tilde{L} \ge 0$ and $\tilde{U} \ge 0$. Then,

$$\begin{aligned} \tilde{T}_{rw} =& (\tilde{D}-r\tilde{L})^{-1}((1-w)\tilde{D}+(w-r)\tilde{L}+w\tilde{U}) \\ =& (I-r\tilde{D}^{-1}\tilde{L})^{-1}[(1-w)I+(w-r)\tilde{D}^{-1}\tilde{L} \\ &+ w\tilde{D}^{-1}\tilde{U}] \\ =& (1-w)I+w(1-r)\tilde{D}^{-1}\tilde{L}+w\tilde{D}^{-1}\tilde{U}+T, \end{aligned}$$

where

$$\begin{aligned} T =& r\tilde{D}^{-1}\tilde{L}[(w-r)\tilde{D}^{-1}\tilde{L}+w\tilde{D}^{-1}\tilde{U}] \\ &+ [r^2(\tilde{D}^{-1}\tilde{L})^2+\cdots+r^{n-1}(\tilde{D}^{-1}\tilde{L})^{n-1}] \\ &\times [(1-w)I+(w-r)\tilde{D}^{-1}\tilde{L}+w\tilde{D}^{-1}\tilde{U}] \\ \ge& 0. \end{aligned}$$

So $\tilde{T}_{rw}$ is nonnegative. We can also get that $(1-w)I+(w-r)\tilde{D}^{-1}\tilde{L}+w\tilde{D}^{-1}\tilde{U}$ is irreducible for A is irreducible. So $\tilde{T}_{rw}$ is irreducible, too. □

Theorem 6 *If A is an irreducible L-matrix with* $0 < a_{1n}a_{n1} < \alpha$ $(\alpha > 1)$, $\beta \in (-\frac{a_{1n}}{\alpha}+\frac{1}{a_{n1}}, -\frac{a_{1n}}{\alpha}) \cap ((1-\frac{1}{\alpha})a_{1n}, -\frac{a_{1n}}{\alpha})$ *and* $0 \le r \le w \le 1$ $(w \ne 1$ *and* $r \ne 1)$*, then the following holds.*

(1) $\rho(\tilde{T}_{rw}) < \rho(T_{rw})$*, if* $\rho(T_{rw}) < 1$*;*

(2) $\rho(\tilde{T}_{rw}) = \rho(T_{rw})$*, if* $\rho(T_{rw}) = 1$*;*

(3) $\rho(\tilde{T}_{rw}) > \rho(T_{rw})$*, if* $\rho(T_{rw}) > 1$.

Proof. From Lemma 5, T_{rw} and $\tilde{T}_{rw}$ are nonnegative and irreducible matrices. Thus, from Lemma 3 there is a positive vector x such that $T_{rw}x = \lambda x$, where $\lambda = \rho(T_{rw})$. From $T_{rw}x = \lambda x$, one easily obtains

$$((1-w)I+(w-r)L+wU)x = \lambda(I-rL)x \tag{14}$$

and

$$(w-r+\lambda r)\tilde{S}Lx = (w+\lambda-1)\tilde{S}x. \tag{15}$$

Using (14) and (15), one easily obtains

$$\begin{aligned} &\tilde{T}_{rw}x-\lambda x \\ =& (\tilde{D}-r\tilde{L})^{-1}[(1-w)\tilde{D}+(w-r)\tilde{L}+w\tilde{U} \\ &- \lambda(\tilde{D}-r\tilde{L})]x \\ =& (\tilde{D}-r\tilde{L})^{-1}[(w+\lambda-1)\tilde{D}_1+w(\tilde{U}_1-\tilde{S})]x \\ =& (\tilde{D}-r\tilde{L})^{-1}[(w+\lambda-1)\tilde{D}_1+w((\tilde{S}L-\tilde{D}_1) \\ &- \tilde{S})]x \\ =& (\tilde{D}-r\tilde{L})^{-1}[(\lambda-1)\tilde{D}_1+w(\tilde{S}L-\tilde{S})]x \\ =& (\lambda-1)(\tilde{D}-r\tilde{L})^{-1}\Big[\tilde{D}_1+\frac{w(1-r)}{w-r+\lambda r}\tilde{S}\Big]x. \end{aligned}$$

We are now ready to deduce (1)-(3).

(1) If $\lambda < 1$, then $\tilde{T}_{rw}x - \lambda x \leq 0$ but is not equal to the null vector. Therefore $\tilde{T}_{rw}x \leq \lambda x$. By Lemma 4, we get $\rho(\tilde{T}_{rw}) < \lambda = \rho(T_{rw})$.

(2) If $\lambda = 1$, then $\tilde{T}_{rw}x - \lambda x = 0$. Therefore $\tilde{T}_{rw}x = \lambda x$. By Lemma 4, we get $\rho(\tilde{T}_{rw}) = \lambda = \rho(T_{rw})$.

(3) If $\lambda > 1$, then $\tilde{T}_{rw}x - \lambda x \geq 0$ but is not equal to the null vector. Therefore $\tilde{T}_{rw}x \geq \lambda x$. By Lemma 4, we get $\rho(\tilde{T}_{rw}) > \lambda = \rho(T_{rw})$. □

In the sequel, we will present some theorems to compare the convergence rates of the preconditioned AOR (SOR and Jacobi) method proposed in this paper with the methods in [3].

From (10), (11) and (12), it is easy to get that $\hat{D} > 0$, $\hat{L} \geq 0$ and $\hat{U} \geq 0$, when $0 < a_{1n}a_{n1} < \alpha$ $(\alpha > 1)$ and $\beta \in (\eta, \delta) \cap (\kappa, \delta)$ with

$$\begin{aligned}\eta &= \max\Big\{\frac{1}{a_{n1}} - \frac{a_{1n}}{\alpha}, \frac{1}{a_{1n}} - \frac{a_{n1}}{\alpha}\Big\},\\ \kappa &= \max\Big\{(1-\frac{1}{\alpha})a_{n1}, (1-\frac{1}{\alpha})a_{1n}\Big\},\\ \delta &= \min\Big\{\frac{-a_{n1}}{\alpha}, \frac{-a_{1n}}{\alpha}\Big\}.\end{aligned}$$

Further, we have the following results.

Theorem 7 *Let $\hat{T}_{rw}$ and $\tilde{T}_{rw}$ be the iteration matrices of the AOR method given in (13) and (7), respectively. If $0 \leq r \leq w \leq 1$ ($w \neq 1$ and $r \neq 1$) and A is an irreducible L-matrix with $0 < a_{1n}a_{n1} < \alpha$ ($\alpha > 1$) and $\beta \in (\eta, \delta) \cap (\kappa, \delta)$ where $\eta = \max\{\frac{1}{a_{n1}} - \frac{a_{1n}}{\alpha}, \frac{1}{a_{1n}} - \frac{a_{n1}}{\alpha}\}$, $\kappa = \max\{(1-\frac{1}{\alpha})a_{n1}, (1-\frac{1}{\alpha})a_{1n}\}$ and $\delta = \min\{\frac{-a_{n1}}{\alpha}, \frac{-a_{1n}}{\alpha}\}$, then*

$$\rho(\hat{T}_{rw}) < \rho(\tilde{T}_{rw}), \text{ if } \rho(\tilde{T}_{rw}) < 1.$$

Proof. In fact, it is easy to get that $\hat{T}_{rw}$ is nonnegative and irreducible matrix, when A is an irreducible L-matrix with $0 < a_{1n}a_{n1} < \alpha$ $(\alpha > 1)$ and $\beta \in (\eta, \delta) \cap (\kappa, \delta)$ where

$$\eta = \max\{\frac{1}{a_{n1}} - \frac{a_{1n}}{\alpha}, \frac{1}{a_{1n}} - \frac{a_{n1}}{\alpha}\},$$

$$\kappa = \max\{(1-\frac{1}{\alpha})a_{n1}, (1-\frac{1}{\alpha})a_{1n}\}$$

and $\delta = \min\{\frac{-a_{n1}}{\alpha}, \frac{-a_{1n}}{\alpha}\}$.

Since $\tilde{T}_{rw}$ is nonnegative and irreducible matrix, there exist a positive vector x such that $\tilde{T}_{rw}x = \lambda x$, where $\lambda = \rho(\tilde{T}_{rw})$. From $\tilde{T}_{rw}x = \lambda x$, one easily obtains

$$((1-w)\tilde{D} + (w-r)\tilde{L} + w\tilde{U})x = \lambda(\tilde{D} - r\tilde{L})x \quad (16)$$

and

$$\begin{aligned} w\hat{U}x =& w\tilde{U}x\\ =&(w+\lambda-1)\tilde{D}x + (r-w-\lambda r)\tilde{L}x.\end{aligned} \quad (17)$$

Since $(I + \tilde{S} + \bar{S})(I - L - U) = \hat{D} - \hat{L} - \hat{U}$, by simple computations, we obtain

$$\hat{D} - \hat{L} = \tilde{D} - \tilde{L} + \bar{S} - \bar{S}U$$

and

$$\hat{L} - \tilde{L} = \hat{D} - \tilde{D} - \bar{S} + \bar{S}U.$$

Further,

$$\begin{aligned}\hat{D} - r\hat{L} =& (1-r)\hat{D} + r(\hat{D} - \hat{L})\\ =& (1-r)\hat{D} + r(\tilde{D} - \tilde{L} + \bar{S} - \bar{S}U).\end{aligned} \quad (18)$$

By (16), (17) and (18), we have

$$\begin{aligned}&\hat{T}_{rw}x - \lambda x\\ =&(\hat{D} - r\hat{L})^{-1}[(1-w)\hat{D} + (w-r)\hat{L}\\ &+ w\hat{U} - \lambda(\hat{D} - r\hat{L})]x\\ =&(\hat{D} - r\hat{L})^{-1}[(1-w)\hat{D} + (w-r)\hat{L}\\ &+ (w+\lambda-1)\tilde{D} + (r-w-\lambda r)\tilde{L}\\ &- \lambda(1-r)\hat{D} - r\lambda(\tilde{D} - \tilde{L} + \bar{S}\\ &- \bar{S}U)]x\\ =&(\hat{D} - r\hat{L})^{-1}[(1-w-\lambda+r\lambda)\hat{D}\\ &+ \lambda(\tilde{D} - r\tilde{L}) - (1-w)\tilde{D}\\ &+ (w-r)(\hat{L} - \tilde{L}) - r\lambda(\tilde{D} - \tilde{L}\\ &+ \bar{S} - \bar{S}U)]x\\ =&(\hat{D} - r\hat{L})^{-1}[(1-\lambda)(1-r)(\hat{D} - \tilde{D})\\ &+ (w-r+\lambda r)(\bar{S}U - \bar{S})]x.\end{aligned}$$

Since

$$\begin{aligned}&(I + \tilde{S})(I - L - U)\\ =&I - L - U + \tilde{S} - \tilde{S}L - \tilde{S}U\\ =&\tilde{D} - \tilde{L} - \tilde{U},\end{aligned}$$

we obtain

$$U = I + \tilde{S} - \tilde{S}L - \tilde{D} + \tilde{U}.$$

Therefore,

$$\begin{aligned}
&(w-r+r\lambda)(\bar{S}U-\bar{S})\\
=&(w-r+r\lambda)\bar{S}(I+\tilde{S}-\tilde{S}L-\tilde{D}+\tilde{U}-I)\\
=&(w-r+r\lambda)\bar{S}(\tilde{S}+\tilde{U}-\tilde{S}L-\tilde{D})\\
=&\frac{1}{w}(w-r+r\lambda)\bar{S}[(w+\lambda-1)\tilde{D}+(r-w-r\lambda)\tilde{L}]\\
&+(w-r+r\lambda)\bar{S}(\tilde{S}-\tilde{S}L-\tilde{D})\\
=&\frac{1}{w}(w-r+r\lambda)(\lambda-1+w)\bar{S}\tilde{D}\\
&+(w-r+r\lambda)\bar{S}(\tilde{S}-\tilde{S}L-\tilde{D})\\
=&\frac{1}{w}[w^2+w(\lambda-1)(1+r)+r(\lambda-1)^2]\bar{S}\tilde{D}\\
&+(w-r+r\lambda)\bar{S}(\tilde{S}-\tilde{S}L-\tilde{D}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\hat{T}_{rw}x-\lambda x\\
=&(\hat{D}-r\hat{L})^{-1}\{(1-\lambda)(1-r)(\hat{D}-\tilde{D})\\
&+[w+(\lambda-1)(1+r)+\frac{r(\lambda-1)^2}{w}]\bar{S}\tilde{D}\\
&+(w-r+r\lambda)\bar{S}(\tilde{S}-\tilde{S}L-\tilde{D})\}x.
\end{aligned}$$

Let

$$\begin{aligned}
C=&(1-\lambda)(1-r)(\hat{D}-\tilde{D})\\
&+[w+(\lambda-1)(1+r)+\frac{r(\lambda-1)^2}{w}]\bar{S}\tilde{D}\\
&+(w-r+r\lambda)\bar{S}(\tilde{S}-\tilde{S}L-\tilde{D}).
\end{aligned}$$

Then

$$\hat{T}_{rw}x-\lambda x=(\hat{D}-r\hat{L})^{-1}Cx,$$

where $\bar{S}\tilde{D}=\gamma e_n e_1^T$, where e_1 and e_n are the first and the last column of the identity matrix and $\gamma=-(\frac{a_{n1}}{\alpha}+\beta)[1-(\frac{a_{1n}}{\alpha}+\beta)a_{n1}]$, $\hat{D}-\tilde{D}$=diag$(0,\cdots,0,-(\frac{a_{n1}}{\alpha}+\beta)a_{1n})$,

$$-\tilde{S}L=\begin{bmatrix}
-(\frac{a_{1n}}{\alpha}+\beta)a_{n1} & -(\frac{a_{1n}}{\alpha}+\beta)a_{n2} & \cdots & -(\frac{a_{1n}}{\alpha}+\beta)a_{n,n-1} & 0\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}$$

Note that

$$\tilde{D}=\begin{bmatrix}
1-(\frac{a_{1n}}{\alpha}+\beta)a_{n1} & & & \\
 & 1 & & \\
 & & \ddots & \\
 & & & 1
\end{bmatrix}.$$

Combing $\tilde{S}$, $\tilde{S}L$ with $\tilde{D}$, by direct computations, we have

$$\tilde{S}-\tilde{S}L-\tilde{D}=\begin{bmatrix}
-1 & -(\frac{a_{1n}}{\alpha}+\beta)a_{n2} & \cdots & -(\frac{a_{1n}}{\alpha}+\beta)a_{n,n-1} & -(\frac{a_{1n}}{\alpha}+\beta)\\
 & -1 & \cdots & 0 & 0\\
 & & \ddots & \ddots & \vdots\\
 & & & \ddots & 0\\
 & & & & -1
\end{bmatrix},$$

$$\bar{S}(\tilde{S}-\tilde{S}L-\tilde{D})=(\frac{a_{n1}}{\alpha}+\beta)\begin{bmatrix}
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
1 & (\frac{a_{1n}}{\alpha}+\beta)a_{n2} & \cdots & (\frac{a_{1n}}{\alpha}+\beta)a_{n,n-1} & (\frac{a_{1n}}{\alpha}+\beta)
\end{bmatrix}.$$

By simple calculation, we get that $c_{ij}=0, i=1,2,\ldots,n-1; j=1,2,\ldots,n-1$.

$$\begin{aligned}
c_{n1}=&\Big[w+(r+1)(\lambda-1)+\frac{r(\lambda-1)^2}{w}\Big]\\
&\times(\frac{a_{n1}}{\alpha}+\beta)[(\frac{a_{1n}}{\alpha}+\beta)a_{n1}-1]\\
&+(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)\\
=&(\frac{a_{n1}}{\alpha}+\beta)\Big\{\Big[w+(r+1)(\lambda-1)\\
&+\frac{r(\lambda-1)^2}{w}\Big]\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}-\alpha}{\alpha}\\
&+w-r+r\lambda\Big\}
\end{aligned}$$

$$
\begin{aligned}
=&(\frac{a_{n1}}{\alpha}+\beta)\Big\{\Big[w-r+r\lambda-w-(r+1)(\lambda-1)\\
&-\frac{r(\lambda-1)^2}{w}\Big]\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}\\
&+(w-r+r\lambda)\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}}{\alpha}\Big\}\\
=&(\frac{a_{n1}}{\alpha}+\beta)\Big[\frac{(1-\lambda)(w-r+r\lambda)}{w}\\
&\times\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}\\
&+(w-r+r\lambda)\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}}{\alpha}\Big]\\
=&(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)\Big[\frac{(1-\lambda)}{w}\\
&\times\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}\\
&+\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}}{\alpha}\Big],
\end{aligned}
$$

$c_{nj}=(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)a_{nj}, j=2,\ldots,n-1$, and

$$
\begin{aligned}
c_{nn}=&-(1-\lambda)(1-r)(\frac{a_{n1}}{\alpha}+\beta)a_{1n}\\
&+(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)\\
\le&-(1-\lambda)(1-r)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)\\
&+(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)\\
=&(\lambda+w-1)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta).
\end{aligned}
$$

Therefore, $(Cx)_j=0, j=1,2,\ldots,n-1$. From the n-th entry of (16), we have

$$
\begin{aligned}
&(1-w)x_n-(w-r)\sum_{j=1}^{n-1}a_{nj}x_j\\
&=\lambda x_n+r\lambda\sum_{j=1}^{n-1}a_{nj}x_j,
\end{aligned}
$$

that is,

$$(w+\lambda-1)x_n+(w+r\lambda-r)\sum_{j=1}^{n-1}a_{nj}x_j=0.$$

Then

$$
\begin{aligned}
&(Cx)_n=c_{n1}x_1+\sum_{j=2}^{n-1}c_{nj}x_j+c_{nn}x_n\\
\le&(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)\Big[\frac{(1-\lambda)}{w}\\
&\times\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}\\
&+\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}}{\alpha}\Big]x_1\\
&+(w+r\lambda-r)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)\sum_{j=2}^{n-1}a_{nj}x_j\\
&+(\lambda+w-1)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)x_n\\
=&(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)\Big[\frac{(1-\lambda)}{w}\\
&\times\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}+\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}}{\alpha}\Big]x_1\\
&+(w+r\lambda-r)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)\sum_{j=1}^{n-1}a_{nj}x_j\\
&+(\lambda+w-1)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)x_n\\
&-(w+r\lambda-r)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)a_{n1}x_1\\
=&(w-r+r\lambda)(\frac{a_{n1}}{\alpha}+\beta)\Big[\frac{(1-\lambda)}{w}\\
&\times\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}+\frac{a_{1n}a_{n1}+\beta\alpha a_{n1}}{\alpha}\Big]x_1\\
&-(w+r\lambda-r)(\frac{a_{n1}}{\alpha}+\beta)(\frac{a_{1n}}{\alpha}+\beta)a_{n1}x_1\\
=&(w+r\lambda-r)(\frac{a_{n1}}{\alpha}+\beta)\cdot\frac{1-\lambda}{w}\\
&\times\frac{\alpha-a_{1n}a_{n1}-\beta\alpha a_{n1}}{\alpha}x_1.
\end{aligned}
$$

From the above discussion and $(\hat{D}-r\hat{L})^{-1}\ge 0$, it is easy to get that if $\lambda<1$, then $\hat{T}_{rw}x-\lambda x\le 0$ but is not equal to the null vector. Therefore $\hat{T}_{rw}x\le\lambda x$. By Lemma 4, we get $\rho(\hat{T}_{rw})<\lambda=\rho(\tilde{T}_{rw})$. □

Corollary 8 *Let T_{rw}, $\tilde{T}_{rw}$ and $\hat{T}_{rw}$ be defined by (2), (7) and (13), respectively. Under the assumptions in Theorem 7, we have*

$$\rho(\hat{T}_{rw})<\rho(\tilde{T}_{rw})<\rho(T_{rw}),\ \text{if}\ \ \rho(T_{rw})<1.$$

It is well known that, when $w=r$, AOR iteration is reduced to SOR iteration. So, we can easily obtain the following corollary.

Corollary 9 *Let T_w, $\tilde{T}_w$ and $\hat{T}_w$ be defined by (2), (7) and (13), respectively. Under the assumptions in Theorem 7, we have*

$$\rho(\hat{T}_w)<\rho(\tilde{T}_w)<\rho(T_w),\ \text{if}\ \rho(T_w)<1.$$

Similarly, let $w=1$ and $r=0$ in (2), (7) and (13), we can obtain the corresponding iterative matrices of Jacobi method. Therefore, we also have the following result.

Corollary 10 *Let B, $\tilde{B}$ and $\hat{B}$ the corresponding iterative matrices of Jacobi method. Under the assumptions in Theorem 7, we have*

$$\rho(\hat{B}) < \rho(\tilde{B}) < \rho(B), \; if \rho(B) < 1.$$

Remark 11 *From the above results, it is not difficult to find that the convergence rate of AOR (SOR and Jacobi) iterative methods can be indeed accelerated when the preconditioned methods are applied to solve the linear system (1). Especially, the methods proposed in [3] are indeed improved.*

5 Numerical examples

Now let us consider the following example to illustrate the results.

Suppose that the coefficient matrix A of (1) is given by

$$A = \begin{bmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ q_1 & s & 1 & q & \ddots & s \\ r & q_1 & s & 1 & \ddots & r \\ s & \ddots & \ddots & \ddots & \ddots & q \\ \cdots & s & r & q_1 & s & 1 \end{bmatrix}$$

where

$$q = -\frac{1}{n}, r = -\frac{1}{n+1}, s = -\frac{1}{n+2} \text{ and } q_1 = -\frac{2}{n}.$$

All experiments are done with Matlab 7.0. Tables 1-2 display the spectral radius of the corresponding iterative matrix with different parameters w and r by using the above methods.

Remark 12 *From Tables 1 and 2, it is easy to know that the numerical results are consistent with the theorems in Section 4. In the meantime, it can be concluded that our new preconditioner is far more effective to accelerate convergence of the AOR (SOR and Jacobi) method than the preconditioned method [3].*

In the sequel, we study the AOR iteration (2), preconditioned AOR iteration (7) and preconditioned AOR iteration (13). We try to solve the corresponding systems of linear equations $Ax = b$ by the mentioned (2), (7) and (13) methods, where A is described above,$b = Ae$ ($e = (1, 1, \ldots, 1)^T$). All tests are started from the zero vector, performed in MATLAB with

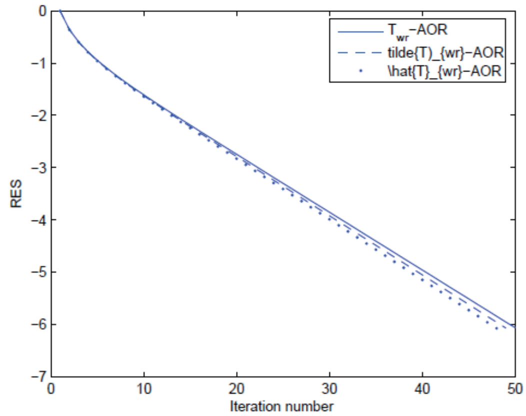

Figure 1: Iteration number for AOR with $n = 6$, $w = 0.9$, $r = 0.5$, $\alpha = 1.5$ and $\beta = -0.04$.

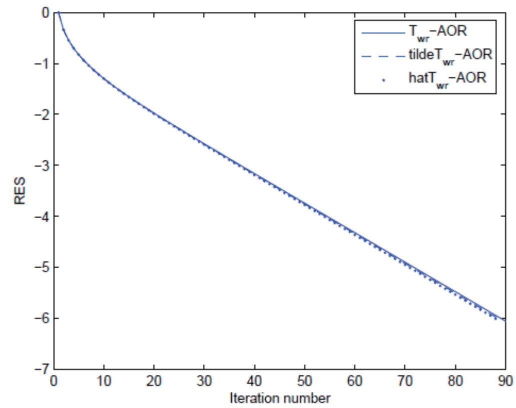

Figure 2: Iteration number for AOR with $n = 8$, $w = 0.8$, $r = 0.6$, $\alpha = 2$ and $\beta = 0$.

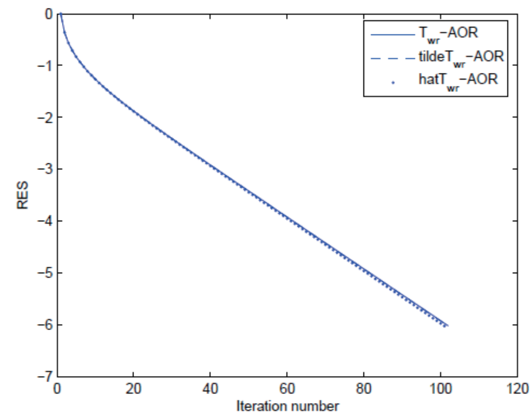

Figure 3: Iteration number for AOR with $n = 10$, $w = 0.95$, $r = 0.9$, $\alpha = 2.5$ and $\beta = 0.1$.

n	w	r	α	β	$\rho(\hat{T}_{rw})$	$\rho(\tilde{T}_{rw})$	$\rho(T_{rw})$
6	0.9	0.5	1.5	-0.04	0.7659	0.7707	0.7753
8	0.8	0.6	2	0	0.8736	0.8744	0.8753
10	0.95	0.9	2.5	0.01	0.8901	0.8912	0.8913

Table 1: The AOR method

n	w	r	α	β	$\rho(\hat{T}_{rw})$	$\rho(\tilde{T}_{rw})$	$\rho(T_{rw})$
6	0.9	0.9	6	0.005	0.7105	0.7118	0.7122
8	0.8	0.8	4	-0.1	0.8532	0.8557	0.8570
10	0.9	0.9	3	0	0.8569	0.8959	0.8970

Table 2: The SOR method

n	w	r	α	β		$\hat{T}_{rw}$	$\tilde{T}_{rw}$	T_{rw}
6	0.9	0.5	1.5	-0.04	ρ	0.7659	0.7707	0.7753
					IT	50	49	48
					RES	2.6479×10^{-6}	2.5876×10^{-6}	2.5620×10^{-6}
8	0.8	0.6	2	0	ρ	0.8736	0.8744	0.8753
					IT	90	89	88
					RES	2.9257×10^{-6}	3.0840×10^{-6}	3.2684×10^{-6}
10	0.95	0.9	2.5	0.01	ρ	0.8901	0.8912	0.8913
					IT	102	102	101
					RES	3.222×10^{-6}	3.1640×10^{-6}	3.2032×10^{-6}

Table 3: Spectral radius, IT and RES of AOR iteration

n	w	r	α	β		$\hat{T}_{rw}$	$\tilde{T}_{rw}$	T_{rw}
6	0.9	0.9	6	0.005	ρ	0.7105	0.7118	0.7122
					IT	39	39	38
					RES	1.8029×10^{-6}	1.7659×10^{-6}	2.3384×10^{-6}
8	0.8	0.8	4	-0.1	ρ	0.8532	0.8557	0.8570
					IT	78	78	76
					RES	2.7666×10^{-6}	2.4681×10^{-6}	2.7701×10^{-6}
10	0.9	0.9	3	0	ρ	0.8569	0.8959	0.8970
					IT	108	107	106
					RES	2.8395×10^{-6}	3.1010×10^{-6}	3.1430×10^{-6}

Table 4: Spectral radius, IT and RES of SOR iteration

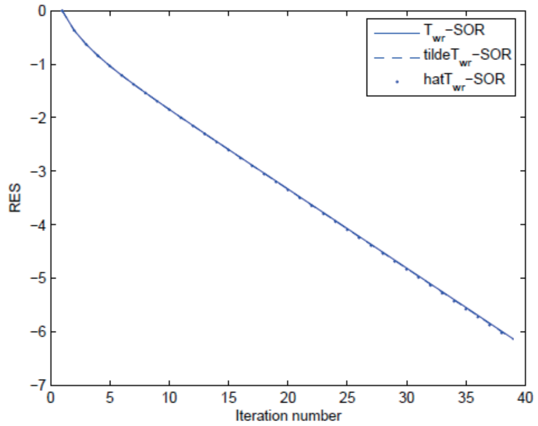

Figure 4: Iteration number for SOR with $n = 6$, $w = r = 0.9$, $\alpha = 6$ and $\beta = 0.005$.

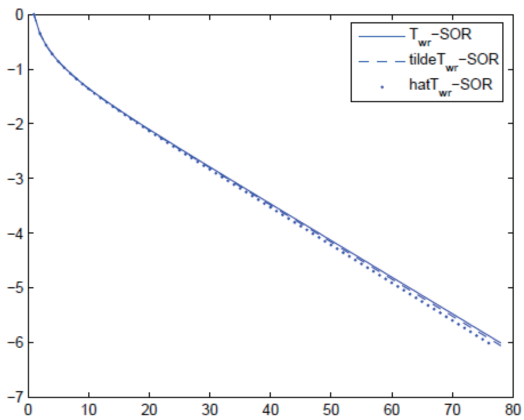

Figure 5: Iteration number for SOR with $n = 8$, $w = r = 0.8$, $\alpha = 4$ and $\beta = -0.1$.

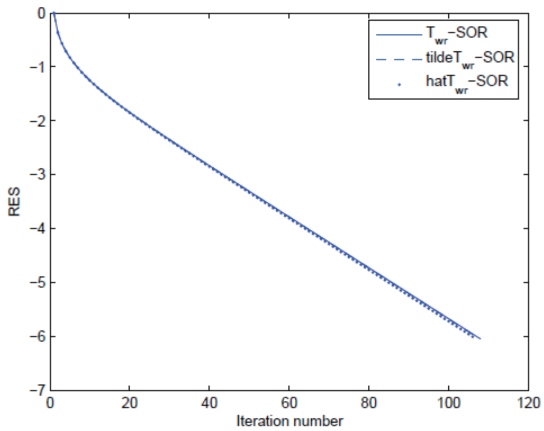

Figure 6: Iteration number for SOR with $n = 10$, $w = r = 0.9$, $\alpha = 3$ and $\beta = 0$.

machine precision 10^{-16}. The (2), (7) and (13) methods terminates if the relative residual error satisfies

$$\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} < 10^{-6}.$$

In Tables 3-4, we list the value of the spectral radius ρ of iterative matrix, the iteration number (IT), relative residual error (RES) with the different value of w and r when the AOR (SOR) iteration and the corresponding preconditioned AOR (SOR) are used to solve the linear systems (1). Figures 1-3 corresponds to Table 3 and Figures 4-6 corresponds to Table 4.

From Tables 3-4, it is easy to find that our preconditioner $I + \hat{S}$ is more better than the preconditioner $I + \tilde{S}$ in [3]. The AOR method with preconditioners $I + \hat{S}$ and $I + \tilde{S}$ are more efficient than the AOR method without preconditioner for the linear systems (1).

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