Power-law index and its application based on the degree of distribution and scale-free network evolution

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Abstract: In realistic networks, many scholars have found that the connection between nodes does not only depend on the size of the node degree probability. This paper sets up a kind of changing scaling exponent scale-free network evolution model(NABA) based on the BA model. In NABA model, the preferential probability depends on sites' degree and the growth rate of degree. Using mean field theory, we calculate the degree distribution function and give the range of scaling exponent. Furthermore we perform some simulations to testify the point of view. Simulation results verify that the distribution function is correct, and scaling exponent will vary in the range $(2, +\infty]$ with the variation of parameters. The NABA model can degenerate into BA model when $\alpha = 1$ and $r = 0$, and hence the BA model can be regarded as its special form.

Key–Words: scale-free, degree distribution, evolution model, computer simulation, mean field theory

1 Introduction

The complex network is a network which has a high degree of complexity. In the real world, there are many networks, most of them are the complex network, such as the social relationship among people, the cooperation network of scientists, science citation, aviation network, biological metabolism network, protein network, gene regulatory network, Internet, WWW network and so on. They constitute a complex system, and has some kinds of connection relationship. The complexity of these complex network mainly includes in the following several aspects:

1. Network structure complexity. The number of the nodes are huge, and the construction of the network shows a variety of different characteristics.

2. Connection diversity. The connections weight between nodes are different, and it exists the possibility of directional difference. For example, in the disease transmission network, the process which spreads from a patient to another has obvious directionality.

3. Nodes diversity. The node of network can represent everything, for example, the nodes in the complex network which are composed of interpersonal relationship represent individual, and the nodes in the complex network which are composed of WWW represent different webpages.

4. The evolution capacity of network. When the time passes by, the connection and disappearance of the nodes in the network. For example, in the WWW, the webpages or links may show or disconnect at any time. It will cause the structure of the network change constantly.

5. Dynamics complexity. The nodes in the network may belong to nonlinear dynamic system, such as the knowledge dissemination network. Due to the situation of the knowledge innovation, join or leave specific knowledge network of individual, the node station occurs complex change over time.

6. Multiple complexity fusion. The multiple complexity affect each other, it can lead to unexpected results. For example, for the long term plan of electric power system complex network, we should consider the evolution of network which includes the change of generating capacity, the change and distribution of the demand for power. According to the evolution of the network, we decide the topology of the network. The generating capacity is related to the factors of resource distribution, and the change and distribution of the demand for power is also related to the social factors, such as economics development.

Using graph theory, statistical physics, uncertainty theory, the complex network theory mainly study the network characteristics, evolution mechanism and regularity, dynamic characteristics of the complex system, which provides a good research tool for studying the structure, development function of the complex system. The research on the complex network is now in a stage of vigorous development due to its much more advantages in the study of complex system. A great number of scholars focus their attention on research to the complexity science.

Degree is a simple but important characters of a node in the graph theory. With the use of degree theory, we can abstract the actual problems in real life, and describe it through the way of establishing models. We can do some relative analysis and researches using the mathematical analysis method with the computer simulation, and get some relevant conclusions. Therefore, the degree theory has a wide range of applications in the field of management science, computer science, communications theory, automatic control, the system engineering and operational research.

Graph theory has been considered as the mathematic foundation of the complex network. We can use an abstract diagram to express the real world. One specific network of nodes N and links M can be abstracted as a diagram whose node set is V and link set is E. It can be shown as $G = (V, E)$, where $N = V, M = E$. Every link in the link set E associates with two nodes in the node set V.

The degree of node i which is often expressed as k_i is defined as the number of links at the node. For a directed network, the degree of a node is represented by out-degree and in-degree. Out-degree is the number of edges from node i to other nodes, and in-degree is the number of edges from other nodes to node i. The degree of pendant point is 1, and the degree of isolate point is 0. Look from the intuitive, the bigger a node's degree is, the more important the node in the network is in some sense. The mean of all nodes' degree in the network is called the network's mean degree expressed in $< k >$,

$$< k >= \frac{\sum_i k_i}{N} \tag{1}$$

where N is the scale of the network, is also the number of the nodes in the network, and $i = 1, ..., N$. The distribution of nodes' degree could be described by distribution function $P(k)$. $P(k)$ means the probability of a random node's degree in k.

For a random network, its degree distribution is approximate to Poisson distribution as in Figure 1. From the shape of the distribution we know the node whose degree is k is not exist actually when $k \gg \langle k \rangle$. Because of that, this kind of networks are also called homogeneous network.

Recent studies have pointed toward a number of findings. The degree distributions of many actual networks are obviously different from Poisson distribution. Especially the distributions of many actual networks' degree could be described by a power law form like $P(k) \propto k^{-\gamma}$ as Figure 2. The curve of the distribution of power law declines slower than the poisson distribution curve.

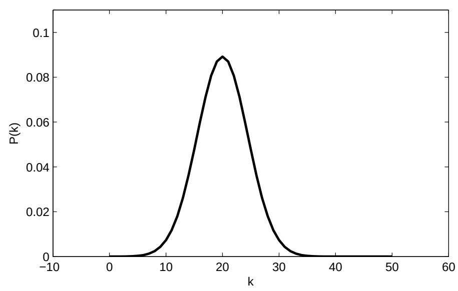

Figure 1: Poisson Distribution

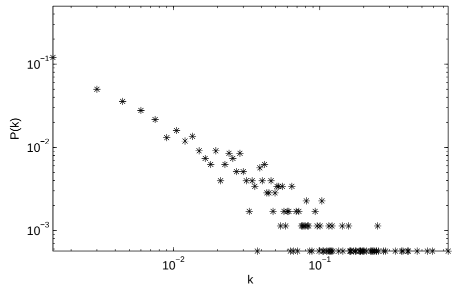

Figure 2: Power Law Distribution

Power law distribution is also called scale-free distribution. The network whose degree distribution follows a power law distribution is called scale-free network. The scale-free characters of a paw law function are as follows.

For a probability distribution function $f(x)$, if for any given constant a, there exists a constant b such that $f(x)$ satisfies the following condition

$$f(ax) = bf(x). \tag{2}$$

Then it will be

$$f(x) = f(1)x^{-\gamma} \tag{3}$$

and

$$\gamma = -f(1)/f'(1). \tag{4}$$

That means power law function is only the probability distribution function to meet the condition of "scale-free".

The meaning of scale-free character is that evolution and structure of a network is indivisible. It means the real network is not static but dynamic. The network is developing and changing in constant.

In general, we assume that the degree of node i is k_i. It means that there are k_i nodes connected to it, hence the k_i nodes are the neighbour of node i. It is obvious that there are no more than $\frac{k_i(k_i-1)}{2}$ links among k_i neighbour nodes. If the actual existence links among k_i nodes are E_i, then using C_i represents the ratio of E_i and $\frac{k_i(k_i-1)}{2}$ is

$$C_i = \frac{2E_i}{k_i(k_i-1)} \tag{5}$$

The clustering coefficient C of the whole network is the average of clustering coefficient of node i,

$$C = \frac{\sum_i C_i}{N} \tag{6}$$

where N is the scale of the network, it also is the number of the nodes in the network.

In a completely random network whose scale is N, $C = O(N^{-1})$ when N is large enough. Lots of large scale real network have obviously clustering effect, the clustering coefficient of them is smaller than 1, but much larger than $O(N^{-1})$. In many kinds of network, $C \to O(1)$ when $N \to \infty$.

Barabási and Albert found that there is a common property of many large networks that they are characterized by a power-law degree distribution. In the same year, in order to understand the feature of power-law distribution, they proposed a famous scale-free network model-BA Model. They attributed the happening mechanism to two generic mechanisms: one is that networks expand continuously by adding new vertices, the other one is the new site attaching preferentially with old sites. This is the advantage of the BA model, but it's inevitable to have some obvious limitations compared with the real network. For example, the real model often has some non power-law characteristics, such as exponential cutoff, saturation for small variables and so on. And the networks generated by BA model have a fixed exponent $\gamma = 3$. However, many networks existing in nature have exponent $1 \leq \gamma \leq 3$ [2]. That means BA model cannot explain these real networks even if their degrees follow a power-law distribution.

Watts and his professor Stronatz, in the Cornell University, and Albert and his professor, in Barabasi of the Madonna University, published two papers which are concerned about complex network model. Since then, the research on the complex network has entered to a new era. The theory of complex network has earn scholars' attentions of many fields, such as physics, biology, engineering technology and social science, which is not limited to the mathematics.

Siying Zhang published a paper in 2005 which emphasizes the study of complex network need to focus on the whole rules, and need to pay attention to the qualitative research, since the qualitative research focuses on the overall generality summarization instead of the local precise determination [3]. In the paper of 'The evolution of complex system process, $N(N-1)$ law, self aggregation', the author considered the complex network composed of vertex and the edge between them. The vertex represents the object of research, edge represents the interaction and relationship between objects. He also thought that complex system and complex network are both for studying the whole rule, they are linked closely, and complex network is one of the most important way to study complex system [4]. In 2005, Siying Zhang made a prophecy that the combination of complex system and complex network may have broad development prospects in the self aggregation, attract nuclear and accumulation [5].

Because of the limitations of the BA model which has a fixed scaling exponent, it appears lots of researches on the extended model based on the original BA model. Many scholars set up some important extended models which modify the connection mechanism of BA model. Bianconi and Barabási introduced the concept of fitness and established Fitness Model in [6]. In this model, the preferential probability was proportional to the product of degree and fitness. Li and Chen in [7] established Local-world Evolving Network Model in 2003. This model differed from BA model in which the new adding site did not link with all old sites, but linked with the sites belong a fixed local-world according preferential probability. On the basis of the previous work, Tan Jinsong and He Zheng in [8] built five cluster self-organization models with their different connection mechanisms to explain clusters' evolution. Considering the interval of operating time is not equal and the number of operating within a period of time is a random variable of the variation as time, Xiangmin Geng in [9]established a model and proved that this system self-organizes into scale-free structures with scaling exponent $\gamma = 3 + \alpha/m$.

Complex networks can properly describe many collective dynamics in social, biological and communication systems. Many academics did research about realistic network, such as Internet [10] which had a research on the power-law relationships of Topology, traffic network [11,12], world trade network [13,14] and so on. Hongwu Wang Junhai Ma in [15] studied the chaos control and synchronization of a fractional-order autonomous system, and Guanhui Wang Junhai Ma in [16] studied the complexity of multi-enterprise output game in supply chain. A lot of real network demonstrate that the connection mechanism of node is not only related to the value of the degree. For example, in the WWW network, several new site can obtain a great number of the hypertext link in the short time according to the good content and market promoting. Some valuable papers can get a huge mass of references in a relatively short period of time. However, to real networks, BA model is too simple to describe their evolution progresses. So it is necessary to improve BA model for further research. In other words, more factors should be considered in building a new model.

This paper introduces the concept of the attraction degree. Considering the real network, connection probability is not just determined by degree. In this paper, we consider that the preferential probability which new nodes connect with the original old nodes is determined not only by degree of old nodes but also by degree's growth rate of them. In other words, to the same degree sites, the faster the degree increases, the bigger the preferential probability is. Considering the function of attraction degree in the connection mechanism, we set up scale-free networks evolution model based on the attraction degree of node, which is named as NABA model. We also analyze and calculate the distribution function and the power law index of the model, and discuss the value range of the power law index. From the result, BA model can be regarded as a special one of this model.

This paper is organized as follows. In Section 2, the intrinsic growth mechanics is briefly introduced. In Section 3, we calculate the degree distribution function and perform simulations. Finally, in Section 4, we will draw our conclusion.

2 The basic model of complex network

In the middle of the twentieth century, Paul Erdos and Alfred Rnyi who are Hungarian mathematician develop the random graph theory based on the previous studies. It was recognized as a milestone of the development from the traditional graph theory to the modern network. They encourage lots of mathematician to do research on the random graph theory.

2.1 ER random network model

ER random network model can be expressed as:

Step 1, setting the scale of the network as N.

Step 2, selecting random two nodes at one time step, and adding link between them with the proportion of p.

$$p = \frac{2n}{N(N-1)} \tag{7}$$

where n is the sum of the links which are fixed ($n < \frac{N(N-1)}{2}$), and $\frac{N(N-1)}{2}$ is the maximum possible links.

Step 3, repeating Step 2, stop evolution until the links reach n.

Step 4, developing $\begin{pmatrix} \frac{N(N-1)}{2} \\ n \end{pmatrix}$ network with this model, and the proportion of occurrence is the same, the average link is $\frac{N(N-1)}{2}$.

2.2 The 'Small World' network model

There are two famous 'Small World' network models, one is the WS (Watts Strogtz)'Small World' network model which is developed by Watts and Strogtz, the other is NW (Newman Watts) 'Small World' network model which is developed by Newman and Watts. Then we introduce the two famous models in the next part.

Watts and Strogtz introduce WS 'Small World' network model in 1998, which is a phase transition model from completely rules network to completely random networks. The rules are as follows:

Step 1, start from the rule diagram. Considering a nearest-neighbor coupled network which consist of N nodes. They are making a circle, each node is connected with his $\frac{k}{2}$ neighbour nodes, where k is even.

Step 2, randomization reconnection. Reconnect every links in the network with the probability of p randomly. In other words, one site of the links is unchanged, and the other site chooses a optional node. It is assumed that there is no multiple edges, and don't exist ring.

When $p = 0$, this model is the completely rule network. And this model is completely random network as $p = 1$. We can achieve the transitioin from the regular network to completely random network, if we change the value of p.

The construction algorithm of WS 'Small World' model may destroy the connection of the network, so Newman and Watts develop NW 'Small World' model in 1999. NW model uses "randomization adding links" instead of "randomization reconnection" of the WS 'Small World' model. The rules are as follows:

Step 1, making a rule diagram. In a network which consist of N nodes, making them as a circle. Each node is connected with his $\frac{k}{2}$ neighbour nodes, where k is even.

Step 2, adding links with probability p. Choosing a couple of nodes with the probability of p, and adding a link between them. There is no multiple edges, and each node can not connect with its own.

In the NW network, this model is a nearest-neighbor coupled network when $p = 0$. And when $p = 1$, it is the global coupled network. From the view of theory, NW model is more simple than WS model. NW 'Small World' model and WS 'Small

World' model are essentially the same when p is small enough and N is big enough.

2.3 Local-world evolving network model

Xiang Li and Guanrong Chen make researches on the example of World Trade Web in 2003. They view the country as a node in this network, and the trade relationship between countries is seen as the links between nodes. They found that in some groups, such as ASEAN (ASEAN), European Union (EU) and the North American Free Trade area (NAFTA), new countries always strength the cooperation of economics and trade relationship inside of the reginal economic group. So in the world trade network, the global priority mechanism does not apply to those with only a few (less than 20) nodes connected [17]. According to the local-world evolving network model of Xiang Li and Guanrong Chen established, the specific algorithm is as follows:

Step 1, increasing the length. The initial network has initial node m_0 and links e_0.

Step 2, selecting the local world. Choosing nodes from the network randomly, the number is M. The new nodes are added as the 'local world'.

Step 3, local world preferential attachment. The new nodes are connected with chosen nodes which are in the local world with the probability of Pi_{local}, just like the BA model. The preferential probability Pi_{local} is:

$$Pi_{local} = Pi' \frac{k_i}{\sum\limits_j k} = \frac{M}{m_0 + t} \cdot \frac{k_i}{\sum\limits_j k}, i \in LW \quad (8)$$

2.4 The BA Model

In recent years, there is a major discovery in the research of complex network. It is that the degree distribution functions of many complex networks, such as Internet, WWW and economic networks, following a power law form. Because the nodes of the network connectivity have no obvious characteristic length, they are called scale-free network.

Barabási and Albert[1] considered the feature of many large networks following scale-free power-law distribution, scale-free distribution was found to be a consequence of two generic mechanisms:

(1) Networks expand continuously by the addition of new unceasing increase nodes in the research of the network. For example, a large number of new web pages are produced in a large number of research articles published every day and every year.

(2) New vertices attach preferentially to sites that are already well connected. The new nodes prefer to make connection with the nodes having bigger degree. This phenomenon is also referred to as "rich get richer" or "Matthew effect". BA model can explain many of the phenomenon in the real life, such as job seekers' choices on the companies choosing. In this network, the quantity of job seekers and company are increasing, and job seekers tend to choose the company which has a large scale and more relevant employees.

Medina, A., I. Matta and J. Byers [18] conclude four factors on that basis:

(1) preferential connectivity of a new node to existing nodes;

(2) incremental growth of the network;

(3) distribution of nodes in space;

(4) locality of edge connections.

The generation algorithm of BA(Barabase-Ablert) Model is as follows:

Step 1-Growth: Starting with a small number m_0 of connected nodes, at every step a new node is added with $m \leq m_0$ new edges linking to the structure. The parameter m is the minimum connectivity that must have each node in the network.

Step 2-Preferential attachment: An old site was connected according to the preferential probability "P_i".

$$\pi_i = \frac{k_i}{\sum\limits_j k_j} n \quad (9)$$

where k_i is the degree of site i.

After t steps, this algorithm produces a network with $N = t + m_0$ nodes and mt edges.We could use the master equation method to get the BA model's degree distribution function.

The numerical simulations of Barabsi and Albert demonstrate that when t is big enough, the degree distribution of model must obey the power law distribution. And we get $p_k \propto e^{\frac{-k}{m}}$ according to the meanfield theory, the degree distribution follows an exponential function. Fig.3 presents the degree distribution p_k of BA scale-free network model.

Firstly, we define $p(k, t_i, t)$ as the probability of the degree of node i adding network at $t = t_i$ is k. In BA model,when a new node added into the network, the probability of added value of node i's degree is

$$m\Pi_i = k/2t \quad (10)$$

we obtain

$$p(k, t_i, t+1) = \frac{k-1}{2t} p(k-1, t_i, t) + (1 - \frac{k}{2t}) p(k, t_i, t) \quad (11)$$

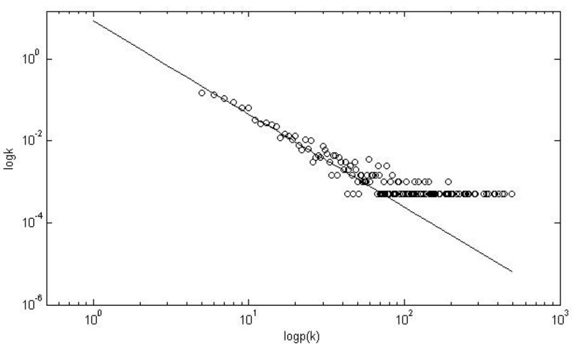

Figure 3: The degree distribution p_k of BA scale-free network model

The degree distribution function of BA model is

$$P(k) = \lim_{t\to\infty}\left(\frac{1}{t}\sum_{t_i} p(k, t_i, t)\right) \tag{12}$$

The function meets the following recursive formula

$$P(k) = \begin{cases} \frac{k-1}{k+2}P(k-1), & k \geq m+1 \\ \frac{2}{m+2}, & k = m \end{cases} \tag{13}$$

Then we get the degree distribution of BA model as

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \propto 2m^2 k^{-3} \tag{14}$$

The BA model have a fixed exponent $\gamma \approx 3$. But most real networks follows a scale-free paw-law distribution and $2 \leq \gamma \leq 3$ [19-26](Table 1). It means that the BA model describes the linking way of real networks to some extent, but it is necessary to improve it more practicable, $P(k)$ follows a power law. We indicate their size and average degree $< k >$. We list the indegree (γ_{in}) and out-degree ($gmma_{out}$) exponents separately for directed networks , while for the undirected networks, marked with an asterisk (*).

2.5 The NABA Model

Considering the real network, sites connect with others not just depend on degree. Actually there are many other reasons. In many cases, the increasing rate of sites' degree is an important factor. The first mechanism of BA model: growth mechanism, which describes the scale of a network is growing or the network nodes is growing. But at a certain period, the situation of the increase rate and scale of the nodes, and what influence of the change will do to network overall, this problem is one of the important issue in the future research. Based on above, we set up a model calling NABA model. The evolving rules are as follows.

Step1-Giving an initial network: Starting with a small number m_0 of connected nodes.

Step2-Growth and preferential attachment: at every step a new node is added with $m \leq m_0$ new edges linking to the structure. The parameter m is the minimum connectivity that must have each node in the network. An old site was connected according to the preferential probability "Π_i".

$$\Pi_i = \alpha \times \frac{k_i}{\sum_j k_j} + (1-\alpha)\frac{k_i(t) - k_i(t-1)}{\sum_j [k_j(t) - k_j(t-1)]}$$
$$0 < \alpha \leq 1 \tag{15}$$

where k_i and A_i are the degree and the growth rate of degree of site i.

Step 3-Adding edges between old sites: r new edges are added between old sites each time-gap. Both two points is chosen according to the probability "Pi_i".

Repeating step2 and step3 until the network contains N nodes. After t time-gaps, there is a network which has $m_0 + t$ sites and $2(m+r)t$ edges. The connection probability "Π_i" depends on its degree and the growth rate of degree.

Figure 4-7 shows the detail evolutions when $m_0 = 5$, $m = 3$, $r = 2$. Each instant a new sit and $m+r$ new links are added. These links are distributed between the sites according to the rule introduced in the text. Figure 1 shows the initial network with five isolated sites($t = 0$).When $t = 1$, a new site 5 add this net and link with three old sites, No.1,2 and 4. In the same instant, two new edges are added between No.1 and No.4, No.0 and No.3. It means that at every time step five new links appear. When $t = 2$, we add a new site 6, and 3 new links are emerged between the new site 6 and old sites. The site 5 which is added in the $t = 1$ is seen as a old one. And 2 new links are emerged among old sites. The smaller links represent the added connections between the new site and old sites, while the thicker links represent the added connections between old sites in the time step marked under figures.

3 The Distribution Function and Simulations

Many concepts and methods were presented in order to characterize the statistic features of complex networks, which include degree distribution, shortest path and clustering coefficient. Degree distribution is one of the best important characters[23]. Therefore, we calculate the degree distribution function of NABA and perform simulation in this section.

Table 1: The scale-free character of several real networks

Networks	γ_{out}	γ_{in}	Size	$< k >$	Reference
Movie actors[12]	2.3	2.3	4×10^7	28.76	Barabasi A, Albert R, Jeong H, 1999
Bus Networks[13]	2.12	2.12	1281		Chu Y. et al., 2010
WorldWideWeb*[14]	2.45	2.1	325729	4.51	Albert R., H. Jeong, et al,1999
Co-authors, neuro.[15]	2.4	2.4	209293	11.54	Barabasi et al., 2002
Co-authors, math.[15]	2.1	2.1	70975	3.9	Barabasi et al., 2002
Metabolic, E. coli[16]	2.2	2.2	778	7.4	Jeong J. , A. P. McMahon, 2011
Protein, S. cerev.[17]	2.4	2.4	1870	2.39	Jeong, Mason, et al., 2001
Phone call[18]	2.1	2.1	$53 \times (10^6)$	3.16	Aiello et al., 2000
Words, synonyms[19]	2.8	2.8	22311	13.48	Yook S. H., H. Jeong, et al.,2001

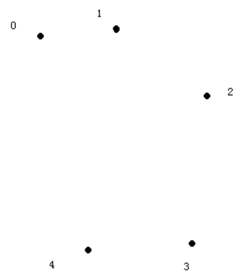

Figure 4: Illustration of the growing network under consideration($m_0 = 5, m = 3, r = 2, t = 0$) .

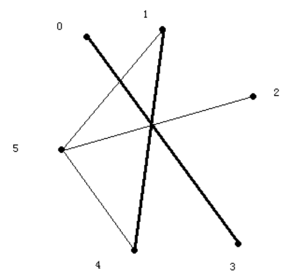

Figure 5: Illustration of the growing network under consideration($m_0 = 5, m = 3, r = 2, t = 1$) .

3.1 The Degree Distribution Function

There are four methods to study degree distribution, master-equation theory[27] and Markov chain[28] based probability theory, mean field theory[29] and rate equations theory[30] based continuity theory. Compared with other methods, mean field method has certain advantages to calculate degree distribution combining with BA degree distribution model. Seen from the calculation process, the calculation has a good effect. In order to calculate the degree distribution function of NABA model, the mean field method

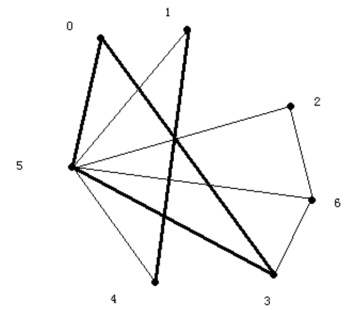

Figure 6: Illustration of the growing network under consideration($m_0 = 5, m = 3, r = 2, t = 2$).

is applied.

According to the evolving roles, there are $m_0 + t$ sites and $2(m + r)t$ edges in the system at t time. The degree of site i is $k_i(t)$. The delta of degree is $A_i(t)$.

$$A_i(t) = k_i(t) - k_i(t-1) \tag{16}$$

The connected probability in step 2 of Site i is both in step 3. According Equation(1), "Pi_i" depends on i's degree and the growth rate of degree. According to the network evolution rules, the selected probability of node i in step 2 and step 3 is Pi_i, and Pi_i is related to the node connectivity and attraction degree.

$$Pi_i = \frac{k_i}{\sum\limits_j k_j}, 0 < \alpha \leq 1 \tag{17}$$

Supposing $k_i(t)$ is contiguous to old site i, that is

$$\frac{\partial k_i(t)}{\partial t} = \Delta k_i(t) \cdot \Pi_i, \quad i = 1, 2, \cdots, m_0 + t - 1 \tag{18}$$

There are two reasons of the delta of k_i at the t time-gap. One is the adding of new site. The other is the increasing edges of old sites. That is

$$\Delta k_i(t) = m + 2r \tag{19}$$

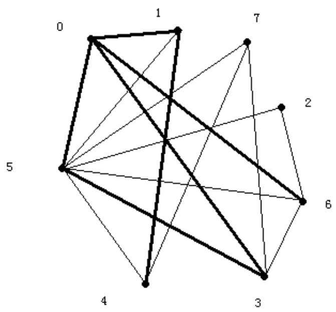

Figure 7: Illustration of the growing network under consideration($m_0 = 5$, $m = 3$, $r = 2$, $t = 3$).

Taking (17), (19) into (18), we can obtain

$$\frac{\partial k_i(t)}{\partial t} = (m+2r)(\alpha\frac{k_i(t)}{\sum\limits_j k_j}+(1-\alpha)\frac{A_i(t)}{\sum\limits_j A_j}) \quad (20)$$

Supposing site i added the net at t_i, the initial condition of function (20) is

$$k_i(t_i) = m \quad (21)$$

and

$$S(t) = \sum_j k_j = 2(m+r)t \quad (22)$$

$$S_A(t) = \sum_j A_j = 2(m+r) \quad (23)$$

Substituting (22) and (23) to (20), we obtain the degree function:

$$k_i(t) = m(\frac{t}{t_i})^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}} \quad (24)$$

Through (24), the probability of $k_i(t) < k$ is

$$P(k_i(t) < k) = P(t_i > t(\frac{m}{k})^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}}) \quad (25)$$

Supposing t follows uniform distribution, that is

$$\mathrm{P}(\mathrm{t_i} < T) = \mathrm{T}/(\mathrm{m_0}+\mathrm{t}) \quad (26)$$

Then

$$\begin{aligned} &P(t_i > t(\tfrac{m}{k})^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}}) \\ &= 1 - P(t_i < t(\tfrac{m}{k})^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}}) \\ &= 1 - \tfrac{1}{m_0+1}t(\tfrac{m}{k})^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}} \end{aligned} \quad (27)$$

Simplifying the function as

$$\begin{aligned} &P(k_i(t) < k) \\ &= 1 - \tfrac{t}{m_0+1}(\tfrac{m}{k})^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}} \end{aligned} \quad (28)$$

Finally we obtain the degree distribution function:

$$\begin{aligned} P(k) &= \tfrac{\partial P(k_i(t)<k)}{\partial k} \\ &= \tfrac{t}{m_0+1}\cdot\tfrac{1+2(m+r)-\alpha}{\alpha(m+2r)}\cdot m^{\frac{\alpha(m+2r)}{1+2(m+r)-\alpha}}\cdot k^{1+\frac{1+2(m+r)-\alpha}{\alpha(m+2r)}} \end{aligned} \quad (29)$$

When $t \to \infty$, we have

$$P(k) \sim \frac{1+2(m+r)-\alpha}{\alpha(m+2r)}\cdot m^{\frac{1+2(m+r)-\alpha}{\alpha(m+2r)}}\cdot k^{1+\frac{1+2(m+r)-\alpha}{\alpha(m+2r)}} \quad (30)$$

By (27), the network that we consider here belongs to the class of scale-free networks and the scaling exponent γ of distribution function is

$$\gamma = 1 + \frac{1+2(m+r)-\alpha}{(m+2r)\alpha} \quad (31)$$

3.2 Discussions of the Range of the Exponent

(1) Let $\alpha = 1$,$r = 0$, the preferential probability is

$$\Pi_i = \frac{k_i}{\sum\limits_j k_j} \quad (32)$$

The model degenerates to be a BA model, it means that BA model is a special form of NABA model.
(2) Let $m \gg r$, we obtain the scaling exponent γ:

$$\gamma = 1 + \frac{1+2(m+r)-\alpha}{\alpha(m+2r)} \approx \frac{2}{\alpha}+1 \quad (33)$$

While α adjusted, the scaling exponent is varying in the range of $[3, +\infty)$.
(3) Let $r \gg m$, we obtain the scaling exponent γ:

$$\gamma = 1 + \frac{1+2(m+r)-\alpha}{(m+2r)\alpha} \approx \frac{1}{\alpha}+1 \quad (34)$$

While α adjusted, the scaling exponent is varying in the range of $[2, +\infty)$.

3.3 Simulation

According to NABA model's growth mechanics, we simulate the evolution progress by using Matlab and present the simulation diagrams.

Fig 8 is the log-log plot of degree distribution P(k) at $m = r = 5$, $\alpha = 0.95$, for NABA network of size

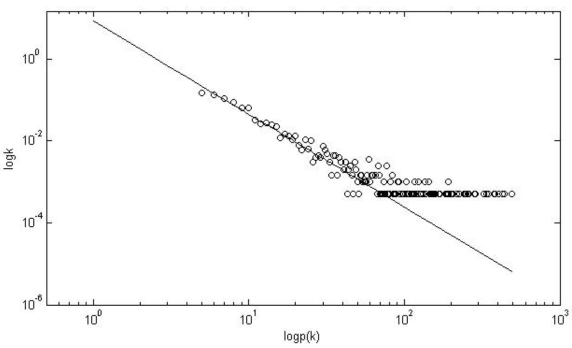

Figure 8: The Degree Distribution of NABA Model.

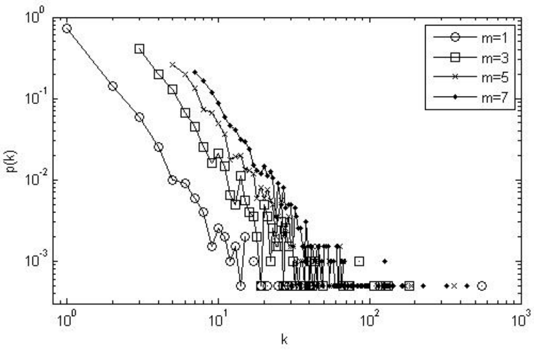

Figure 9: The Degree Distribution of NABA Model ($m = 1, 3, 5, 7$).

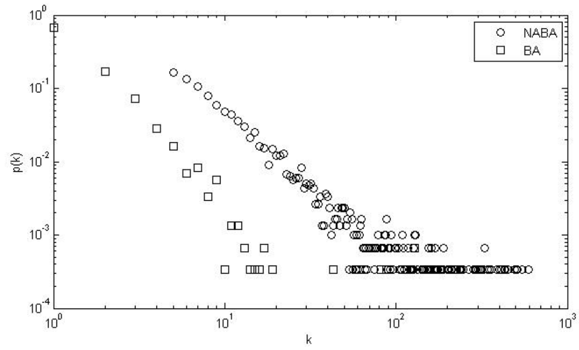

Figure 10: The Degree Distribution of BA Model and NABA Model.

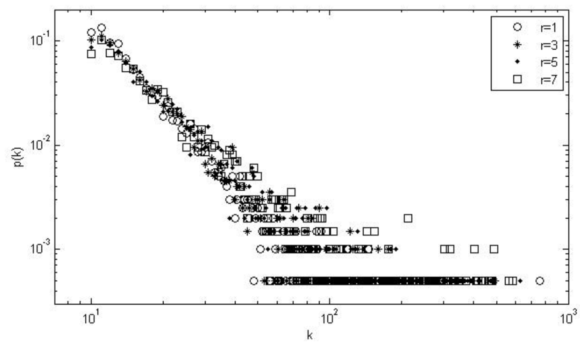

Figure 11: The Degree Distribution of NABA Model ($r = 1, 3, 5, 7$).

$N = 2000$ and with the sites' number of initial network is 10. The solid line of which the slope is -2.3 ($\gamma = 2.3$) presents the theoretical results according to Eq.(12). The scatter plots are the result of simulation. By fitting them, we obtain the exponent is 2.27. The result of model analysis is verified by the result of simulation. It proves that the distribution function is correct.

Fig 9 is the log-log plot of degree distribution $P(k)$ at $\alpha = 0.95$, $r = 10$, for NABA network of size $N = 2000$ and with the sites' number of initial network is 10 for different parameter m. It reflects the influence of different values of m on the degree distribution of the network. It shows that the network will maintain its power-law character for different parameter "m", but whole moves right. When the value of parameter m sets 1,3,5,7, the power-law index is 2.11, 2.14, 2.22, 2.29 respectively. So if other parameters remain the same, the power-law index is proportion to the value of the parameter "m". It means the power-law character is not related to parameter "m". This is also consistent with analytical result.

Fig 10 is the log-log plot of degree distribution $P(k)$ of BA model and NABA model. Fig 10 shows the log-log plot of degree distribution $P(k)$ when $m = 10$, $\alpha = 0.95$, $r = 1, 3, 5, 7$. From Fig 10, we can find that the degree distribution of NABA model is more uniform than BA model. The figures indicate that the degree distribution in double logarithmic coordinate still maintain the power law character with different parameter "r".

Fig 11 is the log-log plot of degree distribution p_k at $\alpha = 0.95$, $m = 5$, for NABA network of size $N = 2000$ and with the sites' number of initial network is 10 for different parameter r. It illustrates the effect of different values of r on the degree distribution of the network. When $r = 1, 3, 5, 7$, the degree distribution in double logarithmic coordinate still maintain the power law character.

4 Conclusion

With the development of the complex network's theory, it attracts lots of researchers who are in many fields to join in. It also makes the theory of complex network to relate with other different fields. Especially the papers which research on the complex network occur rapid growth in recently years. From Internet to Transportation network, from biological network to society network, it can be said that complex network has been applied into various subjects. Because people live in a world which is filled with all kinds of complex network. Economists and Management scientists think that we can view the whole interacting relationship in the economic system as a complex network. So we can do research on it simply, and study the construction of the network and the mechanism which influences the running and evolution of the whole network system. In the end, we can give the explanations of the results in economics view.

Based on the instruction of BA model, a kind of changing scaling exponent scale-free network evolution model(NABA) is set up in this paper which based on the review of a large number of domestic and foreign studies in this field. In NABA model, the preferential probability depends on sites' degree and the growth rate of degree. BA model is a special form of NABA. There are three differences between NABA model and BA mode. Firstly is the difference in preferential mechanism. The preferential probability is up to the combination of degree and the growth rate of degree in NABA model, while degree is the only factor in BA model. And the growth rate of degree has different influences by adjusting γ. Secondly, in BA model the scaling exponent $\gamma = 3$, and in NABA model the exponent $\gamma \in [3, +\infty)$. The last one is the difference in growth mechanism. BA model just considers to add links between new sites and old sites. NABA model also considered to add links among old sites based on the BA model. Compared with BA model, NABA model is more realistic. From the counting results of degree distribution of the NABA model, the power-law exponent is variable, it is changing with the parameters α, m, γ. The NABA network has changed scaling exponent in the range of $\gamma \in [2, +\infty)$ instead of fixed exponent $\gamma = 3$ in BA model. Therefore, we can tell that NABA model is a more realistic scale-free network in contrast with the BA model. Especially when $\alpha = 1, \gamma = 0$, NABA model will degenerate to be BA model. In other words, BA model is a special form of the NABA model. The study of NABA model has good theoretic value in the process of future research and usability in actual life.

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