A General Iterative Method for Constrained Convex Minimization Problems in Hilbert Spaces

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Abstract: It is well known that the gradient-projection algorithm plays an important role in solving constrained convex minimization problems. In this paper, based on Xu's method [Xu, H. K.: Averaged mappings and the gradient-projection algorithm, J. Optim. Theory Appl. 150, 360-378(2011)], we use the idea of regularization to establish implicit and explicit iterative methods for finding the approximate minimizer of a constrained convex minimization problem and prove that the sequences generated by our methods converge strongly to a solution of the constrained convex minimization problem. Such a solution is also a solution of a variational inequality defined over the set of the solutions of the constrained convex minimization problem. As application, we apply our main result to solve the split feasibility problem in Hilbert spaces.

Key–Words: Variational inequality; Regularization algorithm; Constrained convex minimization; Fixed point; Averaged mapping; Nonexpansive mappings.

1 Introduction

Strong convergence is convergent in norm. In finite dimensional space, strong convergence is equivalent to the weak convergence. In infinite dimension space, strong convergence must be weak convergence, but weak convergence is not necessarily strong convergence. Therefore, we hope to obtain a iterative algorithm which converges in norm to a best solution of constrained convex minimization problems.

Now we give an example that is weakly but not strongly convergent.

Assume that $l^2 = \{x = \{x_k\} | \sum_{k=1}^{\infty} x_k^2 < \infty\}$, $\{e_i\} \in l^2$, $e_i = (0, 0, \cdots\cdots, 1, 0, \cdots\cdots)$. For $\forall\, v \in l^2$, we can obtain $\langle e_i, v\rangle = v_i \to \langle \bar{0}, v\rangle$, which implies that $e_i \rightharpoonup \bar{0}$. However, $\|e_i\| = 1$ $(i = 1, 2, 3, \cdots\cdots)$ implies that e_i is not strongly convergent to the $\bar{0}$.

Consider the following constrained convex minimization problem:

$$\min_{x\in C} f(x), \tag{1}$$

where C is a nonempty closed and convex subset of a real Hilbert space H and $f : C \to \mathbb{R}$ is a real-valued convex and continuously Fréchet differentiable function. Assume that the minimization problem (1) is consistent and let S denote its solution set.

It is well known that the gradient-projection algorithm is very useful in dealing with constrained convex minimization problems and has extensively been studied (see [1-8]and the references therein). It has recently been applied to solve split feasibility problems(see [9-14]) which find applications in image reconstructions and the intensity modulated radiation therapy (see [12, 13, 15, 16, 17]) .

The gradient-projection algorithm (GPA) generates a sequence $\{x_n\}_{n=0}^{\infty}$ using the following recursive formula:

$$x_{n+1} := \mathbf{Proj}_C(x_n - \gamma\nabla f(x_n)), \quad n \geq 0, \tag{2}$$

or more generally,

$$x_{n+1} := \mathbf{Proj}_C(x_n - \gamma_n\nabla f(x_n)), \quad n \geq 0, \tag{3}$$

where, in both (2) and (3), the initial guess x_0 is taken from C arbitrarily, the parameters γ or γ_n are positive real numbers satisfying appropriate conditions. The convergence of the algorithms (2) and (3) depends on the behavior of the gradient ∇f; see Levitin and Polyak [1]. As a matter of fact, it is known [1] that if f has a Lipschitz continuous and strongly monotone gradient, then $\{x_n\}_{n=0}^{\infty}$ generated by (2) and (3) can be strongly convergent to a minimizer of f in C, respectively. If the gradient of f fails to be strongly monotone, then $\{x_n\}_{n=0}^{\infty}$ can only be weakly convergent if H is infinite-dimensional. Regularization, in particular, the traditional Tikhonov regularization, is usually used to solve ill-posed optimization problems.

Next consider the regularized minimization problem:

$$\min_{x\in C} f_\alpha(x) := f(x) + \frac{\alpha}{2}\|x\|^2, \tag{4}$$

here $\alpha > 0$ is the regularization parameter. f is convex and ∇f is inverse strongly monotone. Since now the gradient ∇f_α is α-strongly monotone and $(L+\alpha)$-Lipschitzian, (4) has a unique solution which is denoted as $x_\alpha \in C$.

Assume that the minimization problem (1) is consistent. If we appropriately select the regularized parameter α and the parameter γ in the projection-gradient algorithm, we can get a single iterative algorithm that generates a sequence $\{x_n\}_{n=0}^{\infty}$ in the following manner:

$$x_{n+1} = \mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_n})x_n. \tag{5}$$

It is proved that if the sequence $\{\alpha_n\}$ and the parameter γ satisfy appropriate conditions, the sequence $\{x_n\}$ generated by (5) converges weakly to a minimizer of (1) [18].

One question arises in the literature naturally: Is it possible to get strong convergence of (5) when we make some changes?

In 2001, Yamada [19] introduced the following hybrid iterative algorithm for solving the variational inequality

$$x_{n+1} = Tx_n - \mu\alpha_n FTx_n, \quad n \geq 0, \tag{6}$$

where F is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$, $\eta > 0$. Then he proved that if $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution of variational inequality

$$\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad x \in Fix(T). \tag{7}$$

Recently, Xu [20] provided a modification of GPA so that strong convergence is guaranteed. He considered the following hybrid gradient-projection algorithm

$$x_{n+1} = \theta_n h(x_n) + (1-\theta_n)\mathbf{Proj}_C(x_n - \lambda_n\nabla f(x_n)). \tag{8}$$

Assume that the minimization problem (1) is consistent, it is proved that if the sequences $\{\theta_n\}$ and $\{\lambda_n\}$ satisfy appropriate conditions, the sequence $\{x_n\}$ generated by (8) converges in norm to a minimizer of (1) which solves the variational inequality

$$x^* \in S, \quad \langle (I-h)x^*, x - x^*\rangle \geq 0, \qquad x \in S. \tag{9}$$

In this article, motivated and inspired by the research work of [20], we will combine the iterative method (6) with the iterative method (5) and consider the following hybrid algorithm for the idea of regularization approach:

$$\begin{cases} y_n = (I - \mu\theta_n F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_n})x_n, \\ x_{n+1} = \mathbf{Proj}_C y_n, \qquad n \geq 0. \end{cases} \tag{10}$$

Assume that the minimization problem (1) is consistent, we will prove that if the sequence $\{\theta_n\}$ of parameters and the sequence $\{\alpha_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ generated by (10) converges in norm to a minimizer of (1) which solves the variational inequality (VI)

$$x^* \in S, \quad \langle Fx^*, x - x^*\rangle \geq 0, \quad \forall\ x \in S,$$

where S is the solution set of the minimization problem (1).

Finally, in Sect.4, we apply this algorithm to the split feasibility problem, and give the numerical result in Sect. 5

2 Preliminaries

Throughout this paper, we assume that H is a Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, and C is a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

Definition 1 *A mapping $T : H \to H$ is said to be*

(a) *nonexpansive, if and only if*

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in H.$$

(b) *firmly nonexpansive, if and only if $2T - I$ is nonexpansive, or equivalently*

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2, \qquad \forall x, y \in H.$$

Alternatively, T is firmly nonexpansive, if and only if T can be expressed as

$$T = \frac{1}{2}(I + W),$$

where $W : H \to H$ is nonexpansive. Projections are firmly nonexpansive.

(c) *an averaged mapping, if and only if it can be written as the average of the identity I and a nonexpansive mapping; that is,*

$$T = (1-\varepsilon)I + \varepsilon W,$$

where ε is a number in $(0,1)$ and $W : H \to H$ is nonexpansive. More precisely, when the above expression holds, we say that T is ε-averaged. Thus firmly nonexpansive mappings (in particular, projections) are (1/2)-averaged maps.

Proposition 2 *([15, 21]) Let $T : H \to H$ be given. We have:*

(i) *if T is υ-ism, then for $\gamma > 0$, γT is (υ/γ)-ism;*

(ii) *T is averaged, iff the complement $I - T$ is υ-ism for some $\upsilon > 1/2$; indeed, for $\varepsilon \in (0,1)$, T is ε-averaged, iff $I - T$ is $(1/2\varepsilon)$-ism;*

(iii) *the composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$(see [22]).*

In particular, an averaged mapping is a nonexpansive mapping.

Definition 3 *(See[23]) for comprehensive theory of monotone operators.)*

(i) *A is monotone if and only if,*

$$\langle x - y, Ax - Ay\rangle \geq 0, \qquad \forall x, y \in H.$$

(ii) *Given is a number $\upsilon > 0$. $A : H \to H$ is said to be υ-inverse strongly monotone, if and only if*

$$\langle x-y, Ax-Ay\rangle \geq \upsilon\|Ax-Ay\|^2, \qquad \forall x, y \in H.$$

(iii) *Given is a number $\zeta > 0$. A is said to be ζ-strongly monotone, if and only if*

$$\langle x - y, Ax - Ay\rangle \geq \zeta\|x - y\|^2, \qquad \forall x, y \in H.$$

It is easily seen that, if T is nonexpansive, then $I - T$ is monotone. It is also easily seen that a projection is a one-ism.

Inverse strongly monotone operators have widely been applied to solve practical problems in various fields; for instance, in traffic assignment problems(see[24, 25]).

Definition 4 *Let the operators $S, T, V : H \to H$ be given.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive, if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *T is nonexopansive, if and only if the complement $I - T$ is (1/2)-ism.*

Lemma 5 [26] *Assume that $\{a_n\}_{n=0}^\infty$ is a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in (0,1) and $\{\delta_n\}_{n=0}^\infty$ is a sequence in $\mathbb{R}$ such that

(i) $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum\limits_{n=0}^{\infty} \gamma_n|\delta_n| < \infty$;

(iii) $\sum\limits_{n=0}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 6 [27] *Let C be a closed and convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping with $FixT \neq \emptyset$. If $\{x_n\}_{n=1}^\infty$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}_{n=1}^\infty$ converges strongly to y, then $(I-T)x = y$.*

Lemma 7 *Let C be a closed subset of a real Hilbert space H, given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall\ z \in C.$$

3 Main results

Assume that the minimization problem (1) is consistent and let S denote its solution set. Assume that the gradient ∇f is $\frac{1}{L}$-inverse strongly monotone ($\frac{1}{L}$-ism) (see [23]) with a constant $L > 0$. Throughout the rest of this paper, we always let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Since S is a closed convex subset, the nearest point projection from H onto S is well defined. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with κ, $\eta > 0$.

Let $0 < \mu < 2\eta/\kappa^2$, $\tau = \mu(\eta - \frac{\mu\kappa^2}{2})$. Assume that α_t is continuous with respect to t and in addition, $\lim_{t\to 0} \frac{\alpha_t}{t} = 0$; that is, there is a constant $B > 0$ so as to $|\frac{\alpha_t}{t}| < B$. For $t \in (0,1)$, we consider a mapping X_t on C defined by

$$X_t(x) = \mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}(x), \quad x \in C, \quad (11)$$

where

$$V_{\alpha_t} := \mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t}).$$

It is obvious that V_{α_t} is a nonexpansive mapping. It is also easy to see that X_t is a contraction.

For $t \in (0,1)$, we can get

$$
\begin{aligned}
& \|(I - t\mu F)V_{\alpha_t}(x) - (I - t\mu F)V_{\alpha_t}(y)\|^2 \\
= \; & \|V_{\alpha_t}(x) - V_{\alpha_t}(y) \\
& -t\mu(FV_{\alpha_t}(x) - FV_{\alpha_t}(y))\|^2 \\
= \; & \|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
& -2t\mu\langle V_{\alpha_t}(x) - V_{\alpha_t}(y), FV_{\alpha_t}(x) - FV_{\alpha_t}(y)\rangle \\
& +t^2\mu^2\|FV_{\alpha_t}(x) - FV_{\alpha_t}(y)\|^2 \\
\leq \; & \|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
& -2t\mu\eta\|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
& +t^2\mu^2\kappa^2\|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
= \; & (1 - t\mu(2\eta - t\mu\kappa^2))\|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
\leq \; & (1 - \frac{s\mu(2\eta - s\mu\kappa^2)}{2})^2\|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
\leq \; & (1 - t\mu(\eta - \frac{\mu\kappa^2}{2}))^2\|V_{\alpha_t}(x) - V_{\alpha_t}(y)\|^2 \\
= \; & (1 - t\tau)^2\|x - y\|^2.
\end{aligned}
$$

Indeed, we have

$$
\begin{aligned}
& \|X_t(x) - X_t(y)\| \\
= \; & \|\mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}(x) \\
& -\mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}(y)\| \\
\leq \; & \|(I - t\mu F)V_{\alpha_t}(x) - (I - t\mu F)V_{\alpha_t}(y)\| \\
\leq \; & (1 - t\tau)\|x - y\|.
\end{aligned}
$$

Hence X_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation

$$x_t = \mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}(x_t). \tag{12}$$

The next proposition summarizes the properties of $\{x_t\}$.

Proposition 8 *Let x_t be defined by* (12)

(i) $\{x_t\}$ *is bounded for* $t \in (0, \frac{1}{\tau})$.

(ii) $\lim_{t\to 0}\|x_t - \mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t)\| = 0$.

(iii) x_t *defines a continuous curve from* $(0, 1/\tau)$ *into* H.

Proof: (i) For a $p \in S$, then we have

$$
\begin{aligned}
& \|x_t - p\| \\
= \; & \|\mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}(x_t) - p\| \\
\leq \; & \|(I - t\mu F)V_{\alpha_t}(x_t) - p\| \\
= \; & \|(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t) \\
& -(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f)p \\
& -t\mu F\mathbf{Proj}_C(I - \gamma\nabla f)p\| \\
\leq \; & (1 - t\tau)\|\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t) \\
& -\mathbf{Proj}_C(I - \gamma\nabla f)p\| + t\|\mu Fp\| \\
\leq \; & (1 - t\tau)(\|x_t - p\| + \gamma\alpha_t\|p\|) + t\|\mu Fp\| \\
\leq \; & (1 - t\tau)\|x_t - p\| + t\|\mu Fp\| + \gamma\alpha_t\|p\| \\
= \; & (1 - t\tau)\|x_t - p\| + t(\|\mu Fp\| + \gamma\frac{\alpha_t}{t}\|p\|) \\
\leq \; & (1 - t\tau)\|x_t - p\| + t(\|\mu F(p)\| + \gamma B\|p\|).
\end{aligned}
$$

It follows that

$$\|x_t - p\| \leq \frac{\|\mu F(p)\| + \gamma B\|p\|}{\tau}.$$

Hence, $\{x_t\}$ is bounded.

(ii) By the definition of $\{x_t\}$, we have

$$
\begin{aligned}
& \|x_t - \mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t)\| \\
= \; & \|\mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}x_t \\
& -\mathbf{Proj}_C\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t)\| \\
\leq \; & \|(I - t\mu F)V_{\alpha_t}x_t - \mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t)\| \\
= \; & t\mu\|FV_{\alpha_t}x_t\| \to 0.
\end{aligned}
$$

$\{x_t\}$ is bounded, so is $\{FV_{\alpha_t}x_t\}$.

(iii) For t, $t_0 \in (0, 1/\tau)$, we have

$$
\begin{aligned}
& \|x_t - x_{t_0}\| \\
= \; & \|\mathbf{Proj}_C(I - t\mu F)V_{\alpha_t}(x_t) \\
& -\mathbf{Proj}_C(I - t_0\mu F)V_{\alpha_{t_0}}(x_{t_0})\| \\
\leq \; & \|(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t) \\
& -(I - t_0\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_{t_0})\| \\
= \; & \|(I - \mu tF)V_{\alpha_t}x_t - (I - \mu t_0F)V_{\alpha_{t_0}}x_t \\
& +(I - \mu t_0F)V_{\alpha_{t_0}}x_t - (I - \mu t_0F)V_{\alpha_{t_0}}x_{t_0}\| \\
\leq \; & \|(I - t_0\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t) \\
& -(I - t_0\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_{t_0})\| \\
& +\|(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t) \\
& -(I - t_0\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t)\| \\
\leq \; & (1 - t_0\tau)\|x_t - x_{t_0}\| \\
& +\|(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})(x_t) \\
& -(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t)\| \\
& +\|(I - t\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t) \\
& -(I - t_0\mu F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t)\| \\
\leq \; & (1 - t_0\tau)\|x_t - x_{t_0}\| \\
& +\|\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})x_t \\
& -\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})x_t\| \\
& +\|t_0\mu F\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t) \\
& -t\mu F\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t)\| \\
\leq \; & (1 - t_0\tau)\|x_t - x_{t_0}\| \\
& +\|(I - \gamma\nabla f_{\alpha_t})x_t - (I - \gamma\nabla f_{\alpha_{t_0}})x_t\| \\
& +\mu|t - t_0|\|F\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t)\|
\end{aligned}
$$

$$\begin{aligned} = \;& (1 - t_0\tau)\|x_t - x_{t_0}\| \\ & +\gamma|\alpha_t - \alpha_{t_0}|\|x_t\| \\ & +\mu|t - t_0|\|F\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_{t_0}})(x_t)\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_t - x_{t_0}\| \;\leq\; & \tfrac{\mu\|F\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{t_0}})x_t\|}{t_0\tau}|t - t_0| \\ & +\tfrac{\gamma\|x_t\|}{t_0\tau}|\alpha_t - \alpha_{t_0}|. \end{aligned}$$

Therefore $x_t \to x_{t_0}$ as $t \to t_0$. This means x_t is continuous.

Our main result below shows that $\{x_t\}$ converges in norm to a minimizer of (1) which solves a variational inequality.

Theorem 9 *Assume that the minimization problem* (1) *is consistent and let S denote its solution set. Assume that the gradient ∇f is $\frac{1}{L}$-ism. Let $F : C \to H$ is η-strongly monotone and κ-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/\kappa^2$ and a constant γ satisfying $0 < \gamma < 2/L$. Assume also that $t \in (0, 1)$ satisfies the condition $\alpha_t = o(t)$. Let $\{x_t\}$ be defined by* (12). *Then x_t converges in norm, as $t \to 0$, to a minimizer of* (1) *which solves the variational inequality*

$$\langle Fx^*, \tilde{x} - x^*\rangle \geq 0, \;\; \forall\; \tilde{x} \in S. \tag{13}$$

Equivalently, we have

$$\mathbf{Proj}_S(I - F)x^* = x^*.$$

Proof: It is easy to see that the uniqueness of a solution of the variational inequality (13). Let $x^* \in S$ denote the unique solution of (13).

Let us prove that $x_t \to x^*(t \to 0)$. Set

$$y_t := (I - t\mu F)V_{\alpha_t}(x_t)$$

and

$$V := \mathbf{Proj}_C(I - \gamma\nabla f).$$

Then we have $x_t = \mathbf{Proj}_C y_t$. For a given $\tilde{x} \in S$, we write

$$\begin{aligned} & x_t - \tilde{x} \\ = \;& \mathbf{Proj}_C y_t - \tilde{x} \\ = \;& \mathbf{Proj}_C y_t - y_t + y_t - \tilde{x} \\ = \;& \mathbf{Proj}_C y_t - y_t + (I - t\mu F)V_{\alpha_t}(x_t) \\ & -(I - t\mu F)\tilde{x} - t\mu F(\tilde{x}). \end{aligned}$$

Since $\mathbf{Proj}_C$ is the metric projection from H onto C, we have

$$\langle y_t - x_t, \tilde{x} - x_t\rangle \leq 0.$$

It follows that

$$\begin{aligned} & \|x_t - \tilde{x}\|^2 \\ = \;& \langle \mathbf{Proj}_C y_t - y_t, \mathbf{Proj}_C y_t - \tilde{x}\rangle \\ & +\langle (I - t\mu F)V_{\alpha_t}(x_t) - (I - t\mu F)\tilde{x}, x_t - \tilde{x}\rangle \\ & -t\mu\langle F(\tilde{x}), x_t - \tilde{x}\rangle \\ \leq \;& \langle (I - t\mu F)V_{\alpha_t}(x_t) - (I - t\mu F)\tilde{x}, x_t - \tilde{x}\rangle \\ & -t\mu\langle F(\tilde{x}), x_t - \tilde{x}\rangle \\ \leq \;& \|(I - t\mu F)V_{\alpha_t}(x_t) - (I - t\mu F)V\tilde{x}\|\|x_t - \tilde{x}\| \\ & -t\mu\langle F(\tilde{x}), x_t - \tilde{x}\rangle \\ \leq \;& (1 - t\tau)\|V_{\alpha_t}(x_t) - V\tilde{x}\|\|x_t - \tilde{x}\| \\ & -t\mu\langle F(\tilde{x}), x_t - \tilde{x}\rangle \\ = \;& (1 - t\tau)\|V_{\alpha_t}(x_t) - V_{\alpha_t}\tilde{x} \\ & +V_{\alpha_t}\tilde{x} - V\tilde{x}\|\|x_t - \tilde{x}\| \\ & -t\mu\langle F(\tilde{x}), x_t - \tilde{x}\rangle \\ \leq \;& (1 - t\tau)\|x_t - \tilde{x}\|^2 + \gamma\alpha_t\|\tilde{x}\|\|x_t - \tilde{x}\| \\ & -t\mu\langle F(\tilde{x}), x_t - \tilde{x}\rangle. \end{aligned}$$

To derive that

$$\begin{aligned} & \|x_t - \tilde{x}\|^2 \\ \leq \;& -\frac{1}{\tau}\langle \mu F(\tilde{x}), x_t - \tilde{x}\rangle \\ & +\frac{\alpha_t}{t}\frac{\gamma}{\tau}\|\tilde{x}\|\|x_t - \tilde{x}\|. \end{aligned} \tag{14}$$

Since $\{x_t\}$ is bounded as $t \to 0$, we see that if $\{t_n\}$ is a sequence in (0,1) such that $t_n \to 0$ and $x_{t_n} \rightharpoonup \bar{x}$, then by (14), $x_{t_n} \to \bar{x}$. We may further assume that $\alpha_{t_n} \to 0$. Notice that $\mathbf{Proj}_C(I - \gamma\nabla f)$ is nonexpansive. It turns out that

$$\begin{aligned} & \|x_{t_n} - \mathbf{Proj}_C(I - \gamma\nabla f)x_{t_n}\| \\ \leq \;& \|x_{t_n} - \mathbf{Proj}_C(I - \gamma\nabla f\alpha_{t_n})x_{t_n}\| \\ & +\|\mathbf{Proj}_C(I - \gamma\nabla f\alpha_{t_n})x_{t_n} \\ & -\mathbf{Proj}_C(I - \gamma\nabla f)x_{t_n}\| \\ \leq \;& \|x_{t_n} - \mathbf{Proj}_C(I - \gamma\nabla f\alpha_{t_n})x_{t_n}\| + \gamma\alpha_{t_n}\|x_{t_n}\|. \end{aligned}$$

From the boundedness of $\{x_t\}$ and

$$\lim_{t\to 0}\|\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_t})x_t - x_t\| = 0,$$

we conclude that

$$\lim_{n\to\infty}\|x_{t_n} - \mathbf{Proj}_C(I - \gamma\nabla f)x_{t_n}\| = 0.$$

Since $x_{t_n} \rightharpoonup \bar{x}$, by lemma 6, we obtain

$$\bar{x} = \mathbf{Proj}_C(I - \gamma\nabla f)\bar{x}.$$

This shows that $\bar{x} \in S$. We next prove that $\bar{x}$ is a solution of the variational inequality (13). Since

$$x_t = \mathbf{Proj}_C y_t - y_t + (I - t\mu F)V_{\alpha_t}(x_t),$$

we can derive that

$$\begin{aligned} & F(x_t) \\ = \ & \frac{1}{t\mu}(\mathbf{Proj}_C y_t - y_t) \\ & +\frac{1}{t\mu}((I-t\mu F)V_{\alpha_t}(x_t) - (I-t\mu F)(x_t)). \end{aligned}$$

Note that, for $\tilde{x} \in S$, $\langle \mathbf{Proj}_C y_t - y_t, \mathbf{Proj}_C y_t - \tilde{x}\rangle \leq 0$. Therefore, for $\tilde{x} \in S$,

$$\begin{aligned} & \langle F(x_t), x_t - \tilde{x}\rangle \\ = \ & \frac{1}{t\mu}\langle \mathbf{Proj}_C y_t - y_t, x_t - \tilde{x}\rangle + \frac{1}{t\mu}\langle (I-t\mu F) \\ & V_{\alpha_t}(x_t) - (I-t\mu F)(x_t), x_t - \tilde{x}\rangle \\ \leq \ & \frac{1}{t\mu}\langle (I-t\mu F)V_{\alpha_t}(x_t) - (I-t\mu F)(x_t), \\ & x_t - \tilde{x}\rangle \\ = \ & -\frac{1}{t\mu}\langle (I-t\mu F)(x_t) - (I-t\mu F)V_{\alpha_t}(x_t), \\ & x_t - \tilde{x}\rangle \\ = \ & -\frac{1}{t\mu}\langle (I-V_{\alpha_t})(x_t), x_t - \tilde{x}\rangle \\ & +\langle F(x_t) - FV_{\alpha_t}(x_t), x_t - \tilde{x}\rangle \\ = \ & -\frac{1}{t\mu}\langle (I-V_{\alpha_t})(x_t) - (I-V_{\alpha_t})\tilde{x}, x_t - \tilde{x}\rangle \\ & -\frac{1}{t\mu}\langle (I-V_{\alpha_t})\tilde{x}, x_t - \tilde{x}\rangle \\ & +\langle F(x_t) - FV_{\alpha_t}(x_t), x_t - \tilde{x}\rangle \\ \leq \ & -\frac{1}{t\mu}\langle (I-V_{\alpha_t})\tilde{x}, x_t - \tilde{x}\rangle \\ & +\langle F(x_t) - FV_{\alpha_t}(x_t), x_t - \tilde{x}\rangle \\ \leq \ & \frac{1}{t\mu}\|V_{\alpha_t}\tilde{x} - V\tilde{x}\|\|x_t - \tilde{x}\| \\ & +\langle F(x_t) - FV_{\alpha_t}(x_t), x_t - \tilde{x}\rangle \\ \leq \ & \frac{1}{t\mu}\gamma\alpha_t\|\tilde{x}\|\|x_t - \tilde{x}\| \\ & +\langle F(x_t) - FV_{\alpha_t}(x_t), x_t - \tilde{x}\rangle. \end{aligned}$$

Since $\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_t})$ is nonexpansive, we obtain that $I-\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_t})$ is monotone, i.e.

$$\begin{aligned} & \langle (I-\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_t}))(x_t) \\ & -(I-\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_t}))(\tilde{x}), x_t - \tilde{x}\rangle \geq 0. \end{aligned}$$

Taking the limit through $t = t_n \to 0$ ensures that $\bar{x}$ is a solution to (13). That is to say

$$\langle F(\bar{x}), \bar{x} - \tilde{x}\rangle \leq 0.$$

Hence $\bar{x} = x^*$ by uniqueness. Therefore $x_t \to x^*$ as $t \to 0$. The variational inequality (13) can be written as

$$\langle (I-F)x^* - x^*, \tilde{x} - x^*\rangle \leq 0, \ \forall \ \tilde{x} \in S.$$

So, by lemma 7, it is equivalent to the fixed point equation

$$P_S(I-F)x^* = x^*.$$

Finally, we consider the following hybrid algorithm for the idea of regularization approach:

$$\begin{cases} y_n = (I-\mu\theta_n F)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_n, \\ x_{n+1} = \mathbf{Proj}_C y_n, \qquad n \geq 0, \end{cases} \tag{15}$$

where the initial guess x_0 is selected in C arbitrarily.

Assume that the parameter γ satisfies the condition $0 < \gamma < 2/L$ and in addition, that the following conditions are satisfied for $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty} \subset (0,1)$:

(i) $\theta_n \to 0$;

(ii) $\lim\limits_{n\to\infty} \frac{\alpha_n}{\theta_n} = 0$;

(iii) $\sum\limits_{n=0}^{\infty} \theta_n = \infty$;

(iv) $\sum\limits_{n=0}^{\infty} |\theta_{n+1} - \theta_n| < \infty$;

(v) $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Theorem 10 *Assume that the minimization problem* (1) *is consistent and the gradient* ∇f *is* $\frac{1}{L}$*-ism. Let* $F : C \to H$ *is* η*-strongly monotone and* κ*-Lipschitzian. Fix a constant* μ *satisfying* $0 < \mu < 2\eta/\kappa^2$ *and a constant* γ *satisfying* $0 < \gamma < 2/L$*. Let* $\{x_n\}$ *be generated by algorithm* (15) *with the sequences* $\{\theta_n\}$ *and* $\{\alpha_n\}$ *satisfying the above conditions. Then the sequence* $\{x_n\}$ *converges in norm to* x^* *that is obtained in Theorem 9.*

Proof: (1) The sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Set

$$V_{\alpha_n} := \mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n}).$$

First we see that

$$\begin{aligned} & \|(I-\mu\theta_n F)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_n \\ & -(I-\mu\theta_n F)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_{n-1}\|^2 \\ = \ & \|(I-\mu\theta_n F)V_{\alpha_n}x_n - (I-\mu\theta_n F)V_{\alpha_n}x_{n-1}\|^2 \\ = \ & \|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1} \\ & -\mu\theta_n(FV_{\alpha_n}x_n - FV_{\alpha_n}x_{n-1})\|^2 \\ = \ & \|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}\|^2 \\ & +\mu^2\theta_n^2\|FV_{\alpha_n}x_n - FV_{\alpha_n}x_{n-1}\|^2 \\ & -2\mu\theta_n\langle V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}, \\ & FV_{\alpha_n}x_n - FV_{\alpha_n}x_{n-1}\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}\|^2 \\
&\quad +\mu^2\theta_n^2k^2\|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}\|^2 \\
&\quad -2\mu\theta_n\eta\|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}\|^2 \\
&= (1-2\mu\theta_n\eta+\mu^2\theta_n^2k^2)\|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}\|^2 \\
&\leq (1-\frac{\mu\theta_n(2\eta-\mu\theta_n)k^2}{2})^2\|V_{\alpha_n}x_n - V_{\alpha_n}x_{n-1}\|^2 \\
&\leq (1-\theta_n\tau)^2\|x_n - x_{n-1}\|^2.
\end{aligned}$$

Indeed, we have, for $\bar{x} \in S$,

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\| \\
&= \|\mathbf{Proj}_C y_n - \mathbf{Proj}_C \bar{x}\| \\
&\leq \|y_n - \bar{x}\| \\
&= \| - \theta_n\mu F(\bar{x}) \\
&\quad +(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_n \\
&\quad -(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f)\bar{x}\| \\
&\leq \mu\theta_n\|F(\bar{x})\| \\
&\quad +(1-\theta_n\tau)\|\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_n \\
&\quad -\mathbf{Proj}_C(I-\gamma\nabla f)\bar{x}\| \\
&= \|\mu\theta_nF(\bar{x})\| \\
&\quad +(1-\theta_n\tau)\|\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_n \\
&\quad -\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})\bar{x} \\
&\quad +\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})\bar{x} - \mathbf{Proj}_C(I-\gamma\nabla f)\bar{x}\| \\
&\leq \theta_n\|\mu F(\bar{x})\| + (1-\theta_n\tau)\|x_n - \bar{x}\| + \gamma\alpha_n\|\bar{x}\| \\
&= (1-\theta_n\tau)\|x_n - \bar{x}\| \\
&\quad +\theta_n(\|\mu F(\bar{x})\| + \gamma\frac{\alpha_n}{\theta_n}\|\bar{x}\|) \\
&\leq (1-\theta_n\tau)\|x_n - \bar{x}\| \\
&\quad +\theta_n(\|\mu F(\bar{x})\| + \gamma B\|\bar{x}\|) \\
&\leq \max\Big\{\|x_n - \bar{x}\|, \frac{1}{\tau}(\|\mu F(\bar{x})\| + \gamma B\|\bar{x}\|)\Big\}.
\end{aligned}$$

By induction

$$\|x_n - \bar{x}\| \leq \max\Big\{\|x_0 - \bar{x}\|, \frac{\|\mu F(\bar{x})\| + \gamma B\|\bar{x}\|}{\tau}\Big\}.$$

In particular, $\{x_n\}_{n=0}^{\infty}$ is bounded.

(2) We prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Let M be a constant such that

$$M > \max\Big\{\sup_{\kappa,n\geq 0}\mu\|FV_{\alpha_k}x_n\|, \sup_{n\geq 0}\gamma\|x_n\|\Big\}.$$

We compute

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|\mathbf{Proj}_C y_n - \mathbf{Proj}_C y_{n-1}\| \\
&\leq \|y_n - y_{n-1}\| \\
&\leq \|(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_n \\
&\quad -(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_{n-1}\| \\
&\quad +\|(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_{n-1} \\
&\quad -(I-\mu\theta_{n-1}F)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n-1}})x_{n-1}\| \\
&\leq (1-\theta_n\tau)\|x_n - x_{n-1}\| \\
&\quad +\|(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_{n-1} \\
&\quad -(I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n-1}})x_{n-1} + \\
&\quad (I-\mu\theta_nF)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n-1}})x_{n-1} \\
&\quad -(I-\mu\theta_{n-1}F)\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n-1}})x_{n-1}\| \\
&\leq (1-\theta_n\tau)\|x_n - x_{n-1}\| \\
&\quad +\|V_{\alpha_n}x_{n-1} - V_{\alpha_{n-1}}x_{n-1}\| \\
&\quad +\|(I-\mu\theta_nF)V_{\alpha_{n-1}}x_{n-1} \\
&\quad -(I-\mu\theta_{n-1}F)V_{\alpha_{n-1}}x_{n-1}\| \\
&= (1-\theta_n\tau)\|x_n - x_{n-1}\| \\
&\quad +\|V_{\alpha_n}x_{n-1} - V_{\alpha_{n-1}}x_{n-1}\| \\
&\quad +\mu|\theta_n - \theta_{n-1}|\|FV_{\alpha_{n-1}}x_{n-1}\| \\
&\leq (1-\theta_n\tau)\|x_n - x_{n-1}\| + M|\theta_n - \theta_{n-1}| \\
&\quad +\|V_{\alpha_n}x_{n-1} - V_{\alpha_{n-1}}x_{n-1}\|
\end{aligned}$$

and

$$\begin{aligned}
&\|V_{\alpha_n}x_{n-1} - V_{\alpha_{n-1}}x_{n-1}\| \\
&= \|\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_n})x_{n-1} \\
&\quad -\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n-1}})x_{n-1}\| \\
&\leq \|(I-\gamma\nabla f_{\alpha_n})x_{n-1} - (I-\gamma\nabla f_{\alpha_{n-1}})x_{n-1}\| \\
&= \| - \gamma\nabla f_{\alpha_n}(x_{n-1}) + \gamma\nabla f_{\alpha_{n-1}}(x_{n-1})\| \\
&= \gamma|\alpha_n - \alpha_{n-1}|\|x_{n-1}\|. \qquad (16)
\end{aligned}$$

Combining (16) and (16), we can obtain

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq (1-\tau\theta_n)\|x_n - x_{n-1}\| \\
&\quad +M(|\theta_n - \theta_{n-1}| + |\alpha_n - \alpha_{n-1}|). \qquad (17)
\end{aligned}$$

Apply lemma 5 to (17) to conclude that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

(3) We prove that $\omega_w(x_n) \subset S$. Let $\hat{x} \in \omega_w(x_n)$ and assume that $x_{n_k} \rightharpoonup \hat{x}$ for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$. We may further assume that $\alpha_{n_k} \to 0$. Set

$$V := \mathbf{Proj}_C(I-\gamma\nabla f).$$

Notice that V is nonexpansive and $FixV = S$. It turns out that

$$\begin{aligned}
&\|x_{n_k} - Vx_{n_k}\| \\
&\leq \|x_{n_k} - V_{\alpha_{n_k}}x_{n_k}\| + \|V_{\alpha_{n_k}}x_{n_k} - Vx_{n_k}\| \\
&\leq \|x_{n_k} - x_{n_k+1}\| + \|x_{n_k+1} - V_{\alpha_{n_k}}x_{n_k}\| \\
&\quad +\|V_{\alpha_{n_k}}x_{n_k} - Vx_{n_k}\|
\end{aligned}$$

$$\begin{aligned}
&= \|x_{n_k} - x_{n_k+1}\| \\
&\quad +\|\mathbf{Proj}_C y_{n_k} - \mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n_k}})x_{n_k}\| \\
&\quad +\|\mathbf{Proj}_C(I-\gamma\nabla f_{\alpha_{n_k}})x_{n_k} \\
&\quad -\mathbf{Proj}_C(I-\gamma\nabla f)x_{n_k}\| \\
&\leq \|x_{n_k} - x_{n_k+1}\| + \theta_{n_k}\| - \mu F V_{\alpha_{n_k}} x_{n_k}\| \\
&\quad +\gamma\alpha_{n_k}\|x_{n_k}\| \\
&\leq \|x_{n_k} - x_{n_k+1}\| + M(\theta_{n_k} + \alpha_{n_k}) \to 0 \\
&\qquad\qquad as\ k \to \infty.
\end{aligned}$$

So lemma 6 guarantees that $\omega_w(x_n) \subset FixV = S$.

(4) We prove that $x_n \to x^*$ as $n \to \infty$, where x^* is the unique solution of the VI (13). First we observe that there is some $\hat{x} \in \omega_w(x_n) \subset S$. Such that

$$\limsup_{n\to\infty}\langle Fx^*, x_n - x^*\rangle = \langle Fx^*, \hat{x} - x^*\rangle \geq 0. \quad (18)$$

We now compute

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\mathbf{Proj}_C y_n - \mathbf{Proj}_C x^*\|^2 \\
&\leq \|y_n - x^*\|^2 \\
&= \|(I-\mu\theta_n F)V_{\alpha_n}(x_n) - (I-\mu\theta_n F)Vx^* \\
&\quad -\theta_n\mu Fx^*\|^2 \\
&\leq \|(I-\mu\theta_n F)V_{\alpha_n}(x_n) - (I-\mu\theta_n F)Vx^*\|^2 \\
&\quad -2\theta_n\langle \mu Fx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\theta_n\tau)^2\|V_{\alpha_n}(x_n) - V_{\alpha_n}x^* \\
&\quad +V_{\alpha_n}x^* - Vx^*\|^2 \\
&\quad -2\mu\theta_n\langle Fx^*, x_{n+1} - x^*\rangle \\
&= (1-\theta_n\tau)^2(\|V_{\alpha_n}(x_n) - V_{\alpha_n}x^*\|^2 \\
&\quad +\|V_{\alpha_n}x^* - Vx^*\|^2 \\
&\quad +2\langle V_{\alpha_n}(x_n) - V_{\alpha_n}x^*, V_{\alpha_n}x^* - Vx^*\rangle) \\
&\quad -2\mu\theta_n\langle Fx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\theta_n\tau)^2(\|x_n - x^*\|^2 + \gamma^2\alpha_n^2\|x^*\|^2 \\
&\quad +2\gamma\alpha_n\|x^*\|\|x_n - x^*\|) \\
&\quad -2\mu\theta_n\langle Fx^*, x_{n+1} - x^*\rangle \\
&\leq (1-\theta_n\tau)^2\|x_n - x^*\|^2 \\
&\quad +\gamma^2\alpha_n^2\|x^*\|^2 + 2\gamma\alpha_n\|x^*\|\|x_n - x^*\| \\
&\quad -2\mu\theta_n\langle Fx^*, x_{n+1} - x^*\rangle \\
&= (1-2\theta_n\tau)\|x_n - x^*\|^2 \\
&\quad +\theta_n[2\gamma\frac{\alpha_n}{\theta_n}\|x^*\|\|x_n - x^*\| + \gamma^2\alpha_n\frac{\alpha_n}{\theta_n}\|x^*\|^2 \\
&\quad -2\langle \mu Fx^*, x_{n+1} - x^*\rangle + \theta_n\tau^2\|x_n - x^*\|^2] \\
&= (1-\bar{\theta_n})\|x_n - x^*\|^2 + \bar{\theta_n}\bar{\chi_n}. \qquad (19)
\end{aligned}$$

where$\bar{\theta_n} = 2\theta_n\tau$.

Therefore,

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\leq\ &(1-\bar{\theta_n})\|x_n - x^*\|^2 + \bar{\theta_n}\bar{\chi_n}.
\end{aligned}$$

$$\begin{aligned}
\bar{\chi_n} = \tfrac{1}{2\tau}[&2\gamma\tfrac{\alpha_n}{\theta_n}\|x^*\|\|x_n - x^*\| + \gamma^2\alpha_n\tfrac{\alpha_n}{\theta_n}\|x^*\|^2 \\
&-2\langle \mu Fx^*, x_{n+1} - x^*\rangle + \theta_n\tau^2\|x_n - x^*\|^2].
\end{aligned}$$

Applying lemma 5 to the inequality (19), together with (18), we get $\|x_n - x^*\| \to 0$ as $n \to \infty$.

4 Application

In this section, we give an application of Theorem 10 to the split feasibility problem (say SFP, for short), which was introduced by Censor and Elfving [12]. Since its inception in 1994, the split feasibility problem (SFP) has received much attention due to its applications in signal processing and image reconstruction, with particular progress in intensity-modulated radiation therapy.

The SFP can mathematically be formulated as the problem of finding a point x with the property

$$x \in C, \quad and \quad Ax \in Q, \qquad (20)$$

where C and Q are nonempty, closed and convex subset of Hilbert space H_1 and H_2, respectively. $A : H_1 \to H_2$ is a bounded linear operator.

It is clear that x^* is a solution to the split feasibility problem (21) if and only if $x^* \in C$ and $Ax^* - \mathbf{Proj}_Q Ax^* = 0$. We define the proximity function f by

$$f(x) = \frac{1}{2}\|Ax - \mathbf{Proj}_Q Ax\|^2,$$

and consider the constrained convex minimization problem

$$\min_{x\epsilon C} f(x) = \min_{x\epsilon C}\frac{1}{2}\|Ax - \mathbf{Proj}_Q Ax\|^2. \qquad (21)$$

Then x^* solves the split feasibility problem (21) if and only if x^* solves the minimization problem (22) with the minimize equal to 0. Byrne [15] introduced the so-called CQ algorithm to solve the (SFP).

$$x_{n+1} = \mathbf{Proj}_C(I - \gamma A^*(I - \mathbf{Proj}_Q)A)x_n,\ n \geq 0, \qquad (22)$$

where $0 < \gamma < 2/\|A\|^2$. He obtained that the sequence $\{x_n\}$ generated by (23) converges weekly to a solution of the (SFP).

In order to obtain strong convergence iterative sequence to solve the (SFP). We propose the following algorithm:

$$\begin{cases} y_n = (I-\mu\theta_n F)\mathbf{Proj}_C(I - \gamma(A^*(I - \mathbf{Proj}_Q)A \\ \quad +\alpha_n I))x_n, \\ x_{n+1} = \mathbf{Proj}_C y_n, \qquad\qquad n \geq 0, \end{cases} \qquad (23)$$

where the initial guess is $x_0 \in C$ and $F : C \to H$ is η-strongly monotone and κ-Lipschitzian with constants $\kappa > 0$, $\eta > 0$ such that $0 < \mu < 2\eta/\kappa^2$. We can show that the sequence $\{x_n\}$ generated by (24) converges strongly to a solution of the (SFP) (21) if the sequence $\{\theta_n\} \subset (0, 1)$ and the sequence $\{\alpha_n\}$ of parameters satisfy appropriate conditions.

Applying Theorem 10, we obtain the following result.

Theorem 11 *Assume that the split feasibility problem (21) is consistent. Let the sequence $\{x_n\}$ be generated by (24), where the sequence $\{\theta_n\} \subset (0, 1)$ and the sequence $\{\alpha_n\}$ satisfy the conditions (i)-(v). Then the sequence $\{x_n\}$ converges strongly to a solution of the split feasibility problem (21).*

Proof: By the definition of the proximity function f, we have

$$\nabla f(x) = A^*(I - \mathbf{Proj}_Q)Ax.$$

Hence, we obtain

$$\begin{aligned}
& \|\nabla f(x) - \nabla f(y)\|^2 \\
= \ & \|A^*(I - \mathbf{Proj}_Q)Ax - A^*(I - \mathbf{Proj}_Q)Ay\|^2 \\
= \ & \langle A^*(I - \mathbf{Proj}_Q)Ax - A^*(I - \mathbf{Proj}_Q)Ay, \\
& A^*(I - \mathbf{Proj}_Q)Ax - A^*(I - \mathbf{Proj}_Q)Ay\rangle \\
= \ & \langle (I - \mathbf{Proj}_Q)Ax - (I - \mathbf{Proj}_Q)Ay, \\
& AA^*(I - \mathbf{Proj}_Q)Ax - AA^*(I - \mathbf{Proj}_Q)Ay\rangle \\
\le \ & \|AA^*\|\|(I - \mathbf{Proj}_Q)Ax - (I - \mathbf{Proj}_Q)Ay\|^2 \\
= \ & \|AA^*\|\langle (I - \mathbf{Proj}_Q)Ax - (I - \mathbf{Proj}_Q)Ay, \\
& (I - \mathbf{Proj}_Q)Ax - (I - \mathbf{Proj}_Q)Ay\rangle \\
= \ & \|AA^*\|\langle Ax - Ay - (\mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay), \\
& (I - \mathbf{Proj}_Q)Ax - (I - \mathbf{Proj}_Q)Ay\rangle \\
= \ & \|AA^*\|(\langle x - y, A^*(I - \mathbf{Proj}_Q)Ax \\
& -A^*(I - \mathbf{Proj}_Q)Ay\rangle \\
& -\langle \mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay, \\
& (I - \mathbf{Proj}_Q)Ax - (I - \mathbf{Proj}_Q)Ay\rangle)
\end{aligned}$$

and

$$\begin{aligned}
& \langle \mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay, (I - \mathbf{Proj}_Q)Ax \\
& -(I - \mathbf{Proj}_Q)Ay\rangle \\
= \ & \langle \mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay, Ax - Ay \\
& -(\mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay)\rangle \\
= \ & \langle Ax - Ay, \mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay\rangle \\
& -\|\mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay\|^2 \\
\ge \ & \|\mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay\|^2 \\
& -\|\mathbf{Proj}_Q Ax - \mathbf{Proj}_Q Ay\|^2 \\
= \ & 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|\nabla f(x) - \nabla f(y)\|^2 \\
= \ & \|A^*(I - \mathbf{Proj}_Q)Ax - A^*(I - \mathbf{Proj}_Q)Ay\|^2 \\
\le \ & \|AA^*\|\langle x - y, A^*(I - \mathbf{Proj}_Q)Ax \\
& -A^*(I - \mathbf{Proj}_Q)Ay\rangle.
\end{aligned}$$

That is,

$$\begin{aligned}
& \langle x - y, \nabla f(x) - \nabla f(y)\rangle \\
= \ & \langle x - y, A^*(I - \mathbf{Proj}_Q)Ax \\
& -A^*(I - \mathbf{Proj}_Q)Ay\rangle \\
\ge \ & \frac{1}{\|AA^*\|}\|A^*(I - \mathbf{Proj}_Q)Ax \\
& -A^*(I - \mathbf{Proj}_Q)Ay\|^2 \\
= \ & \frac{1}{\|A\|^2}\|\nabla f(x) - \nabla f(y)\|^2.
\end{aligned}$$

Hence, ∇f is $\frac{1}{\|A\|^2}$-ism. Set $f_{\alpha_n}(x) = f(x) + \frac{\alpha_n}{2}\|x\|^2$. Consequently,

$$\begin{aligned}
& \nabla f_{\alpha_n}(x) \\
= \ & \nabla f(x) + \alpha_n I(x) \\
= \ & A^*(I - \mathbf{Proj}_Q)Ax + \alpha_n x.
\end{aligned}$$

Then the iterative scheme (24) is equivalent to

$$\begin{cases} y_n = (I - \mu\theta_n F)\mathbf{Proj}_C(I - \gamma\nabla f_{\alpha_n})x_n, \\ x_{n+1} = \mathbf{Proj}_C y_n, \qquad n \ge 0, \end{cases} \tag{24}$$

where the initial guess is $x_0 \in C$ and the parameters $\{\theta_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{\alpha_n\}_{n=0}^{\infty}$ satisfy the above conditions (i)-(v). Due to Theorem 10, we have the conclusion immediately.

5 Numerical Result

In this section, we consider the following simple example to illustrate the effectiveness, realization, and convergence of the algorithm in Theorem 11.

Example 12 *In part 4, we assume that $H_1 = H_2 = \mathbb{R}^3$. Take $F = I$ with Lipschitz constant $\kappa = 1$ and strongly monotone constant $\eta = 1$. Give the parameters $\theta_n = \frac{1}{n+2}$, $\alpha_n = \frac{1}{(n+2)^2}$ for every $n \ge 0$. Fix $\mu = \gamma = \frac{1}{2}$. In the split feasibility problem (SFP), we take*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix}$$

$$b = \begin{pmatrix} 5 \\ -7 \\ -17 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The SFP can be formulated as the problem of finding a point x^* with the property

$$x^* \in C \;\; and \;\; Ax^* \in Q$$

where $C = H_1, Q = \{b\} \subset H_2$.

This implies that x^* is the solution of system of linear equations $Ax = b$, and $x^* = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$.

Then by Theorem 4.1, the sequence $\{x_n\}$ is generated by

$$\begin{aligned} & x_{n+1} \\ = \;& (I - \frac{1}{2(n+2)} I) \\ & P_C \left(x_n - \frac{1}{2} A^* A x_n + \frac{1}{2} A^* b - \frac{1}{2(n+2)^2} x_n \right). \end{aligned}$$

As $n \to \infty$, we have $\{x_n\} \to x^* = (2, -5, 3)^T$.

Table 1: $x_0 = (0, 1, 1)^T$

n(iterative number)	Error(n)
55	9.85×10^{-3}
550	9.899×10^{-4}
2200	9.9885×10^{-5}

Table 2: $x_0 = (0, 1, 1)^T$

n(iterative number)	x_n(iterative point)
55	$(1.9905, -4.9901, 2.9905)^T$
550	$(1.9989, -4.9990, 2.9985)^T$
2200	$(1.9999, -4.9998, 2.9998)^T$

From the computer programming point of view, the algorithms are easier to implement in this paper.

6 Conclusion

Methods for solving constrained convex minimization problem have been extensively studied in Hilbert space. But to the best of our knowledge, in this paper, it would probably be the first time in the literature that we use the idea of regularization to establish a different hybrid method for finding a minimizer of constrained convex minimization problems and also prove some strong convergence theorems. Finally, we apply this algorithm to the split feasibility problem.

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