Hybrid Proximal-Point Methods for Systems of Generalized Equilibrium Problems and Maximal Monotone Operators in Banach Spaces

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Abstract: In this paper, by using Bregman's technique, we introduce and study the hybrid proximal-point methods for finding a common element of the set of solutions to a system of generalized equilibrium Problems and zeros of a finite family of maximal monotone operators in reflexive Banach spaces. Strong convergence results of the proposed hybrid proximal-point algorithms are also established under some suitable conditions. As applications, the existence of solutions for a class of bilevel variational inequalities are established and some numerical examples are reported.

Key–Words: Equilibrium problem, maximal monotone operator, bilevel variational inequalities, Bregman distance, Bregman projection, totally convex function, Legendre function.

1 Introduction

It is well-known that the equilibrium problems have been important tools for solving the problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, fixed point problems and widely applied to physics, structural analysis, optimization, management science and economics (see, for example, [1, 2] and others). Various equilibrium problems were intensively investigated on the existence of their solutions and the behavior of solution set.

In this paper, without other specifications, let $\mathbb{R}$ be the set of real numbers, C be a nonempty closed and convex subset of a real reflexive Banach space E with the dual space E^*. The norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. Denote the domain of f by $\mathrm{dom}f$, i.e., $\mathrm{dom}f = \{x \in E : f(x) < +\infty\}$. The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

Let $T : C \to C$ be a nonlinear mapping and $F(T) = \{x \in C : Tx = x\}$ be the set of fixed points of T. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \ \forall\, x, y \in C.$$

Let $H_k : C \times C \to \mathbb{R}$ be a bifunction, $A_k : C \to E^*$ be a mapping and $\varphi_k : C \to \mathbb{R}$ be a real-valued function ($k = 1, 2, \cdots, m$). We consider the following system of generalized mixed equilibrium problem (for short,(SGMEP)):

Find $x \in C$ such that for all $y \in C$,

$$H_k(x, y) + \langle A_k x, y - x\rangle + \varphi_k(y) - \varphi_k(x) \geq 0. \quad (1)$$

Denote the solution set of (SGMEP) by Ω.

If $m = 1$, then (SGMEP) is reduced to the following generalized mixed equilibrium problem (for short, (GMEP)):

Find $x \in C$ such that for all $y \in C$,

$$H(x,y) + \langle Ax, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \qquad (2)$$

which was studied by Chang [47]. Denote the solution set of (2) by $EP(H, A, \varphi)$.

If $m = 1$, $A = 0$ and $\varphi = 0$, then (SGMEP) is reduced to the classical equilibrium problem proposed by Blum and Oettli [15]:

Find $x \in C$ such that

$$H(x,y) \geq 0 \ \ \forall y \in C, \qquad (3)$$

where $H : C \times C \to R$ is functional. Denote the set of solutions of (3) by $EP(H)$.

One of the most important and interesting topics in the theory of the equilibrium problems is to develop efficient and implementable algorithms for solving equilibrium problems and its generalizations (see, for example, [3, 4, 18, 19, 28, 37, 38] and others). Since the equilibrium problems have very close connections with both the fixed point problems and the variational inequalities problems, they became one of the hot topics in the related fields for the past few years (see, for example, [25, 26, 27, 39, 42, 43, 44, 45, 46] and others).

Let $M : E \to 2^{E^*}$ be a maximal monotone operator. If E is a Hilbert space, a classic method of solving $0 \in M(x)$ in a Hilbert space is the proximal point algorithm: for any starting point $x_0 \in E$, a sequence (x_n) in E generated by the iterative scheme

$$x_{n+1} = J_{r_n}(x_n) \ \ \forall n \geq 0, \qquad (4)$$

where (r_n) is a sequence in the interval $(0, 1)$, $J_r = (I + rM)^{-1}$ for all $r > 0$ is the resolvent operator for M and I is the identity operator on E.

This algorithm was first introduced by Martinet [32] and generally studied by Rockafellar [40] in the framework of Hilbert spaces. Especially, Rockafellar [40] proved that, if $M^{-1}(0) \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (4) weakly converges to an element of $M^{-1}(0)$. A natural question, posed by Rockafellar in [40], whether this convergence of the sequence (x_n) generated by (4) can be improved to strong convergence or not, which was answered in the negative by Güler [29, Corollary 5.1]. Thereafter, many efforts have been made to modify Rockafellar's proximal point algorithm in order to guarantee strong convergence.

Recently, by using Bregman's projection, Reich and Sabach [36] presented the following algorithms for finding common zeroes of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2, \cdots, N)$ in reflexive Banach space E:

$$\begin{cases} x_0 \in E, \\ y_n^i = \mathrm{Res}^f_{\lambda_n^i}(x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_0 \ \ \forall n \geq 0, \end{cases}$$

and

$$\begin{cases} x_0 \in E, \\ \eta_n^i = \xi_n^i + \frac{1}{\lambda_n^i}(\nabla f(y_n^i) - \nabla f(x_n)), \quad \xi_n^i \in A_i y_n^i, \\ \omega_n^i = \nabla f^*(\lambda_n^i \eta_n^i + \nabla f(x_n)), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \leq D_f(z, \omega_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_0 \ \ \forall n \geq 0, \end{cases}$$

where $(\lambda_n^i)_{i=1}^N \subseteq (0, +\infty)$, $(e_n)_{i=1}^N$ is an error sequence in E with $e_n^i \to 0$ and proj^f_C is the Bregman projection with respect to f from E onto a closed and convex subset C.

Further, under some suitable conditions, they obtained two strong convergence theorems of maximal monotone operators in reflexive Banach spaces, where D_f is the Bregman distance (see, Sect. 2, Definition 2) which was introduced by Bregman [17]. Since the Bregman distance is an elegant and effective technique in the process of designing and analyzing feasibility and optimization algorithms. Many authors applied Bregman's technique to design and analyze the iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed points of nonlinear mappings and so on (see, for example, [6, 7, 8, 10, 11, 12, 20, 22, 23, 33, 34, 37, 39, 41] and others).

The aim of this paper is devoted to construct the hybrid proximal-point methods for finding a common element of the set of solutions to the problem (SGMEP) and zeros of a finite family of maximal monotone operators in reflexive Banach spaces. Strong convergence results of the proposed hybrid proximal-point algorithms are established under some suitable conditions. As applications, we utilize our results to show the existence of solutions for a class of bilevel variational inequalities.

2 Preliminaries

Let C be a nonempty closed convex subset of a real reflexive Banach space E and $T : C \to C$ be a nonlinear mapping. A point $\omega \in C$ is called an *asymptotic fixed point* of T ([48]) if C contains a sequence (x_n) which converges weakly to ω such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. A point $\omega \in C$ is called an *strong asymptotic fixed point* of T ([48]) if C contains a sequence (x_n) which converges strongly to ω such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote the sets of asymptotic fixed points and strong asymptotic fixed points of T by $\hat{F}(T)$ and $\tilde{F}(T)$, respectively. When (x_n) is a sequence in E, we denote strong convergence of (x_n) to $x \in E$ by $x_n \to x$. For any $x \in \text{int}(\text{dom } f)$ and $y \in E$, the *right-hand derivative* of f at x in the direction y defined by

$$f'(x, y) := \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}.$$

f is called *Gâteaux differentiable* at x if, for all $y \in E$, $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists. In this case, $f'(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. f is called *differentiable* if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom } f)$. f is called *Fréchet differentiable* at x if this limit is attained uniformly for $\|y\| = 1$. We say that f is *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Legendre function $f : E \to (-\infty, +\infty]$ is defined in [9]. From [9], if E is a reflexive Banach space, then f is a Legendre function if and only if it satisfies the following conditions $(L1)$ and $(L2)$:

$(L1)$ The interior of the domain of f, int(dom f), is nonempty, f is Gâteaux differentiable on int(dom f) and dom f = int(dom f);

$(L2)$ The interior of the domain of f^*, int(dom f^*), is nonempty, f^* is Gâteaux differentiable on int(dom f^*) and dom f^* = int(dom f^*).

Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see, for example, [16]). This, by $(L1)$ and $(L2)$, implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1},\ \text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom } f^*)$$

and

$$\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom } f).$$

By Theorem 5.4 [9], the conditions $(L1)$ and $(L2)$ also yield that the functions f and f^* are strictly convex on the interior of their respective domains. From now on, we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre.

We first recall some basic results which are needed in our main results.

Assumption 1 *Let C be a nonempty closed convex subset of a uniformly convex and unifromly smooth Banach space E and $H : C \times C \to \mathbb{R}$ satisfy the following conditions $(C1) - (C4)$:*

(C1) $H(x, x) = 0$ for all $x \in C$;

(C2) H is monotone, i.e.,

$$H(x, y) + H(y, x) \le 0\ \ \forall x, y \in C;$$

(C3) for all $x, y, z \in C$,

$$\limsup_{t\to 0^+} H(tz + (1 - t)x, y) \le H(x, y);$$

(C4) for all $x \in C, H(x, \cdot)$ is convex and lower semicontinuous.

Definition 2 *([17, 21]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The function D_f : dom$f$ × int (dom f) $\to [0, +\infty)$, defined by*

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle,$$

is called the Bregman distance with respect to f.

Remark 3 *([39]) The Bregman distance has the following properties:*

(1) the three point identity, for any $x \in$ dom$f$ and $y, z \in$ int (dom f),

$$\begin{aligned} & D_f(x, y) + D_f(y, z) - D_f(x, z) \\ = & \ \langle \nabla f(z) - \nabla f(y), x - y\rangle; \end{aligned}$$

(2) the four point identity, for any $y, \omega \in$ dom$f$ and $x, z \in$ int (dom f),

$$\begin{aligned} & D_f(y, x) - D_f(y, z) - D_f(\omega, x) + D_f(\omega, z) \\ = & \ \langle \nabla f(z) - \nabla f(x), y - \omega\rangle. \end{aligned}$$

Definition 4 *([17]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection of $x \in$ int (dom f) onto the nonempty closed and convex set $C \subset$ dom$f$ is the necessarily unique vector $proj^f_C(x) \in C$ satisfying the following:*

$$D_f(proj^f_C(x), x) = inf\{D_f(y, x) : y \in C\}.$$

Remark 5 *(1) If E is a Hilbert space and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the Bregman projection $proj^f_C(x)$ is reduced to the metric projection of x onto C;*

(2) If E is a smooth Banach space and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the Bregman projection $proj^f_C(x)$ is reduced to the generalized projection $\Pi_C(x)$ (see, [48]) defined by

$$\phi(\Pi_C(x), x) = \min_{y\in C} \phi(y, x),$$

where $\phi(y, x) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2$ and J is the normalized duality mapping from E to 2^{E^*}.

Definition 6 *([20, 36]) Let C be a nonempty closed and convex set of dom$f$. The operator $T : C \to$ int (dom f) with $F(T) \neq \emptyset$ is called:*

(1) quasi-Bregman nonexpansive if

$$D_f(u, Tx) \leq D_f(u, x) \ \forall\, x \in C, u \in F(T);$$

(2) Bregman firmly nonexpansive if

$$\begin{aligned} & \langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \\ \leq \ & \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \ \forall\, x, y \in C. \end{aligned}$$

Remark 7 *([38, Lemma 1.3.2]) If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then, for any Bregman firmly nonexpansive T, $\hat{F}(T) = F(T)$. It is easy to see that the Bregman firmly nonexpansiveness implies the quasi-Bregman nonexpansiveness.*

Definition 8 *([28]) Let $H : C \times C \to \mathbb{R}$ be a bifunction. The f-resolvent of H is the operator $Res_H^f : E \to 2^C$ defined by*

$$\begin{aligned} & Res_H^f(x) = \{z \in C : H(z, y) \\ + \ & \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \ \forall\, y \in C\}. \end{aligned}$$

Definition 9 *([20]) Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:*

(1) totally convex at $x \in$ int (dom$f$) if its modulus of total convexity at x, that is, the function ν_f : int (dom$f$) $\times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x, t) := \mathit{inf}\{D_f(y, x) : y \in \mathrm{dom} f, \|y - x\| = t\}$$

is positive whenever $t > 0$;

(2) totally convex if it is totally convex at every point $x \in int(dom f)$;

(3) totally convex on bounded sets if $\nu_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function ν_f : int (dom$f$) $\times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B, t) := \mathit{inf}\{\nu_f(x, t) : x \in B \cap \mathrm{dom} f\}.$$

Definition 10 *([20, 36]) The function $f : E \to (-\infty, +\infty]$ is called:*

(1) cofinite if dom$f^* = E^*$;

(2) coercive if $\lim_{\|x\| \to +\infty}(f(x)/\|x\|) = +\infty$;

(3) sequentially consistent if, for any two sequences (x_n) and (y_n) in E, (x_n) is bounded and

$$\lim_{n \to \infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \to \infty} \|y_n - x_n\| = 0.$$

Definition 11 *([10]) Let $M : E \to 2^{E^*}$ be a maximal monotone operator. The f-resolvent of M is the operator $Res_M^f : E \to 2^E$ defined by*

$$Res_M^f(x) = (\nabla f + M)^{-1} \circ \nabla f(x).$$

Remark 12 *(1) From Proposition 3.8 (iv) in [10], we know that the f-resolvent of M is the operator Res_M^f is single-valued and the fixed point set of the resolvent Res_M^f is equal to the set of zeroes of the mapping M, i.e., $F(Res_M^f) = M^{-1}(0)$, and then $M^{-1}(0)$ is nonempty closed and convex;*

(2) If f is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then $F(Res_M^f) = \hat{F}(Res_M^f)$ (see [38]);

(3) From Proposition 2.7 in [36], $(Res_{\lambda M}^f(x), M_\lambda(x)) \in$ Graph(M) and $0 \in M(x)$ if and only if $0 \in M_\lambda(x)$ for all $x \in E$ and $\lambda > 0$, where Graph(M) is the graph of M and $M_\lambda : E \to E$ is the Yosida approximation defined by

$$M_\lambda(x) = \frac{1}{\lambda}(\nabla f(x) - \nabla f(Res_{\lambda M}^f(x))) \ \forall x \in E;$$

Moreover, if $M^{-1}(0) \neq \emptyset$, then

$$\begin{aligned} & D_f(u, Res_{\lambda M}^f(x)) + D_f(Res_{\lambda M}^f(x), x) \\ \leq \ & D_f(u, x) \ \forall\, \lambda > 0, u \in M^{-1}(0), \ x \in E. \end{aligned}$$

Lemma 13 *([36, Proposition 2.3]) If $f : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then f is cofinite.*

Lemma 14 *([22, Theorem 2.10]) Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then the following statements hold:*

(1) f is sequentially consistent if and only if it is totally convex on bounded sets;

(2) If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets;

(3) If f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain and the Fréchet derivative ∇f is uniformly continuous on bounded sets.

Lemma 15 *([35, Proposition 2.1]) Let $f : E \to \mathbb{R}$ be a uniformly Fréchet differentiable and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 16 *([36, Lemma 3.1]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $(D_f(x_n, x_0))_{n=1}^{\infty}$ is bounded, then the sequence $(x_n)_{n=1}^{\infty}$ is also bounded.*

Lemma 17 *([36, Proposition 2.2]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and C be a nonempty closed convex subset of E. Suppose that the sequence (x_n) is bounded and any weak subsequential limit of (x_n) belongs to C. If $D_f(x_n, x_0) \leq D_f(proj_C^f(x_0), x_0)$ for any $n \geq 1$, then (x_n) converges strongly to $proj_C^f(x_0)$.*

Lemma 18 *([24, Proposition 2.17]) Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of $int\,(dom\, f)$ and $T : C \to C$ be a quasi-Bregman nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 19 *([24, Lemma 2.18]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and proper convex lower semicontinuous function. Then, for all $z \in E$,*

$$D_f(z, \nabla f^*(\sum_{i=1}^{N} t_i \nabla f(x_i))) \leq \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $(x_i)_{i=1}^N \subset E$ and $(t_i)_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 20 *([22, Corollary 4.4]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex on $int(domf)$. Let $x \in int(domf)$ and $C \subset int(domf)$ be a nonempty closed convex set. If $\hat{x} \in C$, then the following statements are equivalent:*

(1) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f;

(2) The vector $\hat{x}$ is the unique solution of the variational inequality:

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0 \ \ \forall\, y \in C;$$

(3) The vector $\hat{x}$ is the unique solution of the inequality:

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \ \ \forall\, y \in C.$$

Lemma 21 *([37, Lemmas 1 and 2]) Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function. Let C be a nonempty closed convex subset of $int\,(dom\, f)$. Assume that $H : C \times C \to \mathbb{R}$ satisfies Assumption 1. Then, for any $x \in E$, the following results hold:*

(1) there exists $z \in C$ such that

$$H(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \ \forall\, y \in C;$$

(2) the f-resolvent Res_H^f of (1.3) defined by Definition 2.4 has the following properties:

(a) Res_H^f is single valued and $dom(Res_H^f) = E$;

(b) Res_H^f is Bregman firmly nonexpansive;

(c) $EP(H)$ is a closed and convex subset of C and $EP(H) = F(Res_H^f)$;

(d) for all $x \in E$ and for all $u \in F(Res_H^f)$,

$$D_f(u, Res_H^f(x)) + D_f(Res_H^f(x), x) \leq D_f(u, x).$$

Proposition 22 *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function. Let C be a nonempty closed convex subset of $int\,(dom\, f)$. Assume that $H : C \times C \to \mathbb{R}$ satisfies Assumption 1, A is a continuous and monotone mapping, $\varphi : C \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function. Then, for any $x \in E$, the following statements hold:*

(1) there exists $z \in C$ such that

$$\begin{array}{l} H(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \\ +\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \ \forall\, y \in C; \end{array}$$

(2) if we define a mapping $Res_{H,A,\varphi}^f : E \to 2^C$ by

$$\begin{array}{rl} & Res_{(H,A,\varphi)}^f(x) \\ = & \{z \in C : H(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \\ + & \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0 \ \ \forall\, y \in C\}, \end{array}$$

the mapping $Res_{(H,A,\varphi)}^f$ has the following properties:

(a) $Res_{(H,A,\varphi)}^f$ is single valued and $dom(Res_{(H,A,\varphi)}^f) = E$;

(b) $Res_{(H,A,\varphi)}^f$ is Bregman firmly nonexpansive;

(c) $EP(H, A, \varphi)$ is a closed and convex subset of C and $EP(H, A, \varphi) = F(Res_{(H,A,\varphi)}^f)$;

(d) for all $x \in E$ and for all $u \in F(Res_{(H,A,\varphi)}^f)$,

$$\begin{array}{rl} & D_f(u, Res_{(H,A,\varphi)}^f(x)) + D_f(Res_{(H,A,\varphi)}^f(x), x) \\ \leq & D_f(u, x). \end{array}$$

Proof: For the sake of brevity, we define a function $\Gamma : C \times C \to \mathbb{R}$ by

$$\begin{array}{rcl} \Gamma(z, y) & = & H(z, y) + \langle Az, y - z\rangle \\ & & +\varphi(y) - \varphi(z) \ \ \forall\, y, z \in C. \end{array}$$

Since $H(y, y) = 0$ for all $y \in C$, we have

$$\begin{array}{rl} & \Gamma(y, y) \\ = & H(y, y) + \langle Ay, y - y\rangle + \varphi(y) - \varphi(y) \\ = & 0. \end{array}$$

By the monotonicity of A, for each $z, y \in C$, it follows that

$$\begin{aligned} & \Gamma(z,y)+\Gamma(y,z) \\ = \ & H(z,y)+\langle Az, y-z\rangle+\varphi(y)-\varphi(z) \\ & +H(y,z)+\langle Ay, z-y\rangle+\varphi(z)-\varphi(y) \\ = \ & H(z,y)+H(y,z)+\langle Az-Ay, y-z\rangle \\ \leq \ & 0. \end{aligned}$$

Since $H : C \times C \to \mathbb{R}$ is convex and lower semicontinuous with respect to the second argument, A is continuous and $\varphi : C \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function, we can conclude that the function $y \mapsto \Gamma(z, y)$ is convex and lower semicontinuous and so, for any $x, y, z \in C$,

$$\begin{aligned} & \limsup_{t\to 0^+}\Gamma(tx+(1-t)z,y) \\ = \ & \limsup_{t\to 0^+}[H(tx+(1-t)z,y)+\varphi(y) \\ & -\varphi(tx+(1-t)z)+\langle A(tx+(1-t)z), \\ & y-(tx+(1-t)z)\rangle] \\ \leq \ & \limsup_{t\to 0^+}H(tx+(1-t)z,y)+\varphi(y) \\ & +\limsup_{t\to 0^+}\langle A(tx+(1-t)z),y-(tx \\ & +(1-t)z)\rangle+\limsup_{t\to 0^+}(-\varphi(tx+(1-t)z)) \\ \leq \ & H(z,y)+\langle Az,y-z)\rangle+\varphi(y)-\varphi(z). \end{aligned}$$

Summing up the above arguments, the mapping Γ satisfies the conditions (C1)-(C4) of Assumption 1. By Lemma 21, we derive the desired conclusion. □

3 Main results

In this section, we construct a hybrid proximal-point method for finding a common element of the set of solutions to the problem (SGMEP) and zeros of a finite family of maximal monotone operators in reflexive Banach spaces. Further, we analyze the convergence of the sequence generated by the proposed hybrid proximal-point algorithms under some suitable conditions.

Algorithm 23 *(Hybrid proximal-point method)*

Step 1. Taking the initial point $x_1 \in C$ arbitrarily, let $C_1 = \{z \in C : D_f(z, u_1) \leq D_f(z, x_1)\}$ and go to the Step 2.

Step 2. For the current x_n, calculate z_n, y_n, u_n and C_{n+1}:

$$\begin{aligned} z_n &= Res^f_{\lambda_n^N M_N}\circ Res^f_{\lambda_n^{N-1} M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n), \\ y_n &= \nabla f^*(\alpha_n\nabla f(x_1)+(1-\alpha_n)\nabla f(z_n)), \\ u_n &= Res^f_{(H_m,A_m,\varphi_m)}\circ Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})}\circ \\ & \quad \cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n), \\ C_{n+1} &= \{z\in C_n : D_f(z,u_n)\leq \alpha_n D_f(z,x_1) \\ & \qquad +(1-\alpha_n)D_f(z,x_n)\}, \end{aligned}$$

and then calculate x_{n+1}:

$$x_{n+1} = proj^f_{C_{n+1}} x_1.$$

Step 3. Let $n := n + 1$ and go to Step 2.

The following result shows that the sequence (x_n) generated by Algorithm 23 strongly converges to some common element of the set of solutions to the problem (SGMEP) and zeros of a finite family of maximal monotone operators in reflexive Banach spaces.

Theorem 24 *Let C be a nonempty closed convex subset of a real reflexive Banach space E, $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Assume that, for each $k \in \{1, 2, \cdots, m\}$ and $i \in \{1, 2, \cdots, N\}$, $H_k : C \times C \to \mathbb{R}$ satisfies Assumption 1, A_k is a continuous and monotone mapping, $\varphi_k : C \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function and $M_i : E \to 2^{E^*}$ is a maximal monotone operator such that $(\bigcap_{i=1}^N M_i^{-1}(0)) \bigcap \Omega \neq \emptyset$. Let $\liminf_{n\to\infty} \lambda_n^i > 0$ for all $i \in \{1, 2, \cdots, N\}$ and $(\alpha_n) \subseteq [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence (x_n) generated by Algorithm 23 strongly converges to the point p^*, where $p^* = proj^f_{[(\bigcap_{i=1}^N M_i^{-1}(0))\bigcap\Omega]}(x_1)$.*

Proof: By Remark 12 and Proposition 22, $(\bigcap_{i=1}^N M_i^{-1}(0)) \bigcap \Omega$ is a nonempty closed and convex subset of E. Clearly, C_n is closed and convex for all $n \geq 1$. For the simplicity, let $U = (\bigcap_{i=1}^N M_i^{-1}(0)) \bigcap \Omega$.

Next, we split the rest proof of this theorem into the following steps.

Claim I. $U \subseteq C_n$ for all $n \geq 1$. Let $u \in U$. Then, by Proposition 22, Remark 12 and Lemma 19, we have

$$\begin{aligned} & D_f(u,u_n) \\ = \ & D_f(u, Res^f_{(H_m,A_m,\varphi_m)}\circ Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})} \\ & \circ\cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\ \leq \ & D_f(u, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})}\circ\cdots\circ \\ & Res^f_{(H_1,A_1,\varphi_1)}(y_n))-D_f(Res^f_{(H_m,A_m,\varphi_m)}\circ\cdots \end{aligned}$$

$$
\begin{aligned}
& \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n), Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})} \\
& \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\
\leq\ & D_f(u, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})} \circ \\
& \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\
& \cdots\cdots \\
\leq\ & D_f(u, y_n) \\
=\ & D_f(u, \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(z_n))) \\
\leq\ & \alpha_n D_f(u, x_1) + (1-\alpha_n) D_f(u, z_n) \\
=\ & \alpha_n D_f(u, x_1) + (1-\alpha_n) D_f(u, Res^f_{\lambda_n^N M_N} \circ \\
& Res^f_{\lambda_n^{N-1} M_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n)) \\
\leq\ & \alpha_n D_f(u, x_1) + (1-\alpha_n) \cdot \\
& [D_f(u, Res^f_{\lambda_n^{N-1} M_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n)) \\
& - D_f(Res^f_{\lambda_n^N M_N} \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n), \\
& Res^f_{\lambda_n^{N-1} M_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n))] \\
\leq\ & \alpha_n D_f(u, x_1) + (1-\alpha_n) D_f(u, Res^f_{\lambda_n^{N-1} M_{N-1}} \\
& \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n)) \\
& \cdots\cdots \\
\leq\ & \alpha_n D_f(u, x_1) + (1-\alpha_n) D_f(u, x_n).
\end{aligned}
$$

Let $n = 1$. Then $D_f(u, u_1) \leq D_f(u, x_1)$ and so one has $u \in C_1$. By induction, $u \in C_n$ for all $n \geq 1$. Therefore, it follows that $U \subseteq C_n$ for all $n \geq 1$. Since $U = (\bigcap_{i=1}^N M_i^{-1}(0)) \bigcap \Omega \neq \emptyset$, C_n is nonempty closed convex subset of C for all $n \geq 1$. So, (x_n) is well defined.

Claim II. (x_n) is bounded and $\lim_{n\to\infty} D_f(x_n, x_1)$ exists.

It follows from $x_n = \text{proj}^f_{C_n}(x_1) \in C_n$ and $x_{n+1} = \text{proj}^f_{C_{n+1}}(x_1) \in C_{n+1} \subseteq C_n$ that, for any $u \in U$,

$$D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1) \leq D_f(u, x_1). \tag{5}$$

Thus, $(D_f(x_n, x_1))$ is bounded and so are (x_n), $(u_n), (y_n)$ and (z_n). Then (5) implies that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Claim III. (x_n) is a Cauchy sequence.

Since $x_l \in C_l \subseteq C_n$ for all $l > n$ and $x_n = \text{proj}^f_{C_n}(x_1)$, by Lemma 20, one has

$$
\begin{aligned}
& D_f(x_l, \text{proj}^f_{C_n}(x_1)) + D_f(\text{proj}^f_{C_n}(x_1), x_1) \\
& \leq D_f(x_l, x_1)
\end{aligned}
$$

and so

$$D_f(x_l, x_n) \leq D_f(x_l, x_1) - D_f(x_n, x_1). \tag{6}$$

Taking $n \to \infty$ in (6), we have

$$
\begin{aligned}
& \lim_{n\to\infty} D_f(x_l, x_n) \\
& \leq \lim_{n\to\infty} (D_f(x_l, x_1) - D_f(x_n, x_1)) \\
& = 0,
\end{aligned}
$$

that is,

$$\lim_{n\to\infty} D_f(x_l, x_n) = 0.$$

Since f is totally convex on bounded subsets of E, by Definition 10 and Lemma 14, we have

$$\lim_{n\to\infty} \|x_l - x_n\| = 0, \tag{7}$$

i.e., (x_n) is a Cauchy sequence.

Claim IV. (x_n) strongly converges to a point of U.

By Claim III, (x_n) is a Cauchy sequence. Without loss of generality, let $x_n \to p \in C$. Since f is uniformly Fréchet differentiable on bounded subsets of E, from Lemma 14, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. This, together with (7), yields that

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0. \tag{8}$$

In view of $x_{n+1} = \text{proj}^f_{C_{n+1}} x_1$, we have $x_{n+1} \in C_{n+1}$ and

$$
\begin{aligned}
D_f(x_{n+1}, u_n) \leq\ & (1-\alpha_n) D_f(x_{n+1}, x_n) \\
& + \alpha_n D_f(x_{n+1}, x_1).
\end{aligned}
$$

Taking $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0. \tag{9}$$

As in the proof of Claim III, we do obtain that

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0$$

and so

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(u_n)\| = 0. \tag{10}$$

Noting that $\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0$$

and so $u_n \to p$ as $n \to \infty$. Let $u \in U$. It follows from Remark 3, (8) and (10) that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0$$

and

$$\begin{aligned}
& D_f(u,x_n)-D_f(u,u_n) \\
= & -D_f(x_n,u_n)+\langle \nabla f(u_n)-\nabla f(x_n), u-x_n\rangle \\
\leq & -f(x_n)+f(u_n)+\langle \nabla f(u_n), x_n-u_n\rangle \\
& +\langle \nabla f(u_n)-\nabla f(x_n), u-x_n\rangle.
\end{aligned}$$

Therefore, by Lemma 19 and Remark 12,

$$\begin{aligned}
& D_f(u_n,y_n) \\
\leq & D_f(u,y_n)-D_f(u,u_n) \\
= & D_f(u,\nabla f^*(\alpha_n\nabla f(x_1)+(1-\alpha_n)\nabla f(z_n))) \\
& -D_f(u,u_n) \\
\leq & \alpha_n D_f(u,x_1)+(1-\alpha_n)D_f(u,z_n) \\
& -D_f(u,u_n) \\
= & (1-\alpha_n)D_f(u, Res^f_{\lambda_n^N M_N}\circ Res^f_{\lambda_n^{N-1}M_{N-1}}\circ \\
& \cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))+\alpha_n D_f(u,x_1) \\
& -D_f(u,u_n) \\
\leq & \alpha_n D_f(u,x_1)+(1-\alpha_n)D_f(u,x_n) \\
& -D_f(u,u_n) \\
= & \alpha_n[D_f(u,x_1)-D_f(u,x_n)]+D_f(u,x_n) \\
& -D_f(u,u_n) \\
\leq & \alpha_n[D_f(u,x_1)-D_f(u,x_n)]-f(x_n) \\
& +f(u_n)+\langle \nabla f(u_n), x_n-u_n\rangle \\
& +\langle \nabla f(u_n)-\nabla f(x_n), u-x_n\rangle.
\end{aligned}$$

Since f is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E, it follows from Lemma 15 that f is continuous on E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*. Then we have

$$\begin{aligned}
& \lim_{n\to\infty} D_f(u_n,y_n) \\
\leq & \lim_{n\to\infty}\{\alpha_n[D_f(u,x_1)-D_f(u,x_n)]-f(x_n) \\
& +f(u_n)+\langle \nabla f(u_n), x_n-u_n\rangle \\
& +\langle \nabla f(u_n)-\nabla f(x_n), u-x_n\rangle\} \\
\leq & \lim_{n\to\infty}\{\alpha_n[D_f(u,x_1)-D_f(u,x_n)]-f(x_n) \\
& +f(u_n)\}+\lim_{n\to\infty}[\langle \nabla f(u_n), x_n-u_n\rangle \\
& +\langle \nabla f(u_n)-\nabla f(x_n), u-x_n\rangle] \\
= & 0.
\end{aligned}$$

Consequently, $\lim_{n\to\infty} D_f(u_n,y_n) = 0$ and so $\lim_{n\to\infty}\|u_n-y_n\|=0$. Furthermore, one has

$$\lim_{n\to\infty}\|\nabla f(u_n)-\nabla f(y_n)\|=0$$

and

$$\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(y_n)\|=0. \tag{11}$$

Again, from $u_n\to p$ and $\lim_{n\to\infty}\|u_n-y_n\|=0$ as $n\to\infty$, it follows that $y_n\to p$ as $n\to\infty$. Observe that, by Proposition 22,

$$\begin{aligned}
& D_f(u_n, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})}\circ \\
& Res^f_{(H_{m-2},A_{m-2},\varphi_{m-2})}\circ\cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\
\leq & D_f(u, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})}\circ \\
& Res^f_{(H_{m-2},A_{m-2},\varphi_{m-2})}\circ\cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\
& -D_f(u,u_n), \\
& \cdots\cdots, \\
& D_f(Res^f_{(H_1,A_1,\varphi_1)}(y_n),y_n) \\
& \leq D_f(u,y_n)-D_f(u, Res^f_{(H_1,A_1,\varphi_1)}(y_n)).
\end{aligned}$$

Thus, adding up the above inequalities, we have

$$\begin{aligned}
& D_f(u_n, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})}\circ\cdots\circ \\
& Res^f_{(H_1,A_1,\varphi_1)}(y_n))+\sum_{k=2}^{m-1} D_f(Res^f_{(H_k,A_k,\varphi_k)}\circ \\
& \cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n), Res^f_{(H_{k-1},A_{k-1},\varphi_{k-1})}\circ \\
& \cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\
& +D_f(Res^f_{(H_1,A_1,\varphi_1)}(y_n),y_n) \\
\leq & D_f(u,y_n)-D_f(u,u_n) \quad \text{(By Step I)} \\
\leq & \alpha_n[D_f(u,x_1)-D_f(u,x_n]+D_f(u,x_n) \\
& -D_f(u,y_n) \\
= & \alpha_n[D_f(u,x_1)-D_f(u,x_n]+f(y_n) \\
& -f(x_n)+\langle \nabla f(y_n), x_n-y_n\rangle \\
& +\langle \nabla f(y_n)-\nabla f(x_n), u-x_n\rangle.
\end{aligned}$$

Again, since $x_n\to p$ and $y_n\to p$ as $n\to\infty$, taking $n\to\infty$ in the above inequality, we obtain

$$\begin{aligned}
& \lim_{n\to\infty}\{D_f(u_n, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})}\circ\cdots \\
& \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n))+\sum_{k=2}^{m-1} D_f(Res^f_{(H_k,A_k,\varphi_k)}\circ \\
& \cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n), Res^f_{(H_{k-1},A_{k-1},\varphi_{k-1})}\circ \\
& \cdots\circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) \\
& +D_f(Res^f_{(H_1,A_1,\varphi_1)}(y_n),y_n)\} \\
\leq & \lim_{n\to\infty}\{\alpha_n[D_f(u,x_1)-D_f(u,x_n]+f(y_n) \\
& -f(x_n)+\langle \nabla f(y_n), x_n-y_n\rangle \\
& +\langle \nabla f(y_n)-\nabla f(x_n), u-x_n\rangle\} \\
= & 0.
\end{aligned}$$

Therefore, for $k = 2, 3, \cdots, m-1$, we have

$$\lim_{n\to\infty} D_f(u_n, Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) = 0,$$

$$\lim_{n\to\infty} D_f(Res^f_{(H_k,A_k,\varphi_k)} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n), Res^f_{(H_{k-1},A_{k-1},\varphi_{k-1})} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)) = 0$$

and

$$\lim_{n\to\infty} D_f(Res^f_{(H_1,A_1,\varphi_1)}(y_n), y_n) = 0.$$

Therefore, for $k = 2, 3, \cdots, m-1$, we have

$$\lim_{n\to\infty} \|u_n - Res^f_{(H_{m-1},A_{m-1},\varphi_{m-1})} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)\| = 0,$$

$$\lim_{n\to\infty} \|Res^f_{(H_k,A_k,\varphi_k)} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n) - Res^f_{(H_{k-1},A_{k-1},\varphi_{k-1})} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)\| = 0$$

and

$$\lim_{n\to\infty} \|Res^f_{(H_1,A_1,\varphi_1)}(y_n) - y_n\| = 0.$$

These, together with $u_n \to p$ and $y_n \to p$ as $n \to \infty$, show that

$$Res^f_{(H_k,A_k,\varphi_k)} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n) \to p$$

as $n \to \infty$ for each $k = 1, 2, \cdots, m$. Take into account $u_n = Res^f_{(H_m,A_m,\varphi_m)} \circ \cdots \circ Res^f_{(H_1,A_1,\varphi_1)}(y_n)$, we have

$$\begin{aligned} & H_m(u_n, y) + \langle A_m u_n, y - u_n\rangle + \varphi_m(y) \\ & -\varphi_m(u_n) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n\rangle \\ & \geq 0 \end{aligned}$$

for all $y \in C$. It follows from the monotonicity of A_m that

$$\langle A_m y - A_m u_n, y - u_n\rangle \geq 0 \ \forall\, y \in C,$$

and so

$$-\langle A_m u_n, y - u_n\rangle \geq \langle A_m y, u_n - y\rangle \ \forall\, y \in C. \quad (12)$$

By Assumption 1 and (12), we have

$$\begin{aligned} & \langle \nabla f(u_n) - \nabla f(y_n), y - u_n\rangle \\ \geq\ & -(H_m(u_n, y) + \langle A_m u_n, y - u_n\rangle \\ & +\varphi_m(y) - \varphi_m(u_n)) \\ \geq\ & H_m(y, u_n) + \langle A_m y, u_n - y\rangle \\ & +\varphi_m(u_n) - \varphi_m(y). \end{aligned}$$

Consequently, one can conclude that

$$\begin{aligned} & H_m(y, p) + \langle A_m y, p - y\rangle + \varphi_m(p) - \varphi_m(y) \\ \leq\ & \liminf_{n\to\infty} \{H_m(y, u_n) + \langle A_m y, u_n - y\rangle \\ & +\varphi_m(u_n) - \varphi_m(y)\} \\ \leq\ & \liminf_{n\to\infty} \langle \nabla f(u_n) - \nabla f(y_n), y - u_n\rangle \\ \leq\ & \liminf_{n\to\infty} \|\nabla f(u_n) - \nabla f(y_n)\| \cdot \|y - u_n\| \\ =\ & 0. \end{aligned}$$

For any $y \in C$ and $t \in (0, 1]$, let $y_t = ty + (1-t)p \in C$. Then

$$H_m(y_t, p) + \langle A_m y_t, p - y_t\rangle + \varphi_m(p) - \varphi_m(y_t) \leq 0.$$

By Proposition 22, one has

$$\begin{aligned} & 0 \\ =\ & H_m(y_t, y_t) + \langle A_m y_t, y_t - y_t\rangle \\ & +\varphi_m(y_t) - \varphi_m(y_t) \\ \leq\ & t(H_m(y_t, y) + \langle A_m y_t, y - y_t\rangle + \varphi_m(y) \\ & -\varphi_m(y_t)) + (1-t)(H_m(y_t, p) + \\ & \langle A_m y_t, p - y_t\rangle + \varphi_m(p) - \varphi_m(y_t)) \\ \leq\ & t(H_m(y_t, y) + \langle A_m y_t, y - y_t\rangle \\ & +\varphi_m(y) - \varphi_m(y_t)), \end{aligned}$$

that is,

$$\begin{aligned} & H_m(y_t, y) + \langle A_m y_t, y - y_t\rangle \\ & +\varphi_m(y) - \varphi_m(y_t) \geq 0. \end{aligned}$$

Moreover, one has

$$\begin{aligned} & 0 \\ \leq\ & \limsup_{t\to 0^+} [H_m(y_t, y) + \langle A_m y_t, y - y_t\rangle \\ & +\varphi_m(y) - \varphi_m(y_t)] \\ \leq\ & H_m(p, y) + \langle A_m p, y - p\rangle + \varphi_m(y) - \varphi_m(p), \end{aligned}$$

namely, for all $y \in C$,

$$H_m(p, y) + \langle A_m p, y - p\rangle + \varphi_m(y) - \varphi_m(p) \geq 0.$$

Similarly, for each $k = 1, 2, \cdots, m-1$, we do conclude that for all $y \in C$,

$$H_k(p, y) + \langle A_k p, y - p\rangle + \varphi_k(y) - \varphi_k(p) \geq 0.$$

As a consequence, one has $p \in \Omega$.

Now, we show that $p \in \bigcap_{i=1}^N M_i^{-1}(0)$. Note that

$$\begin{aligned} & D_f(z_n, Res^f_{\lambda_n^{N-1} M_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n)) \\ \leq\ & D_f(u, Res^f_{\lambda_n^{N-1} M_{N-1}} \circ \cdots \circ Res^f_{\lambda_n^1 M_1}(x_n)) \end{aligned}$$

$$
\begin{aligned}
&-D_f(u,z_n)\\
&\cdots\cdots\\
\leq\ & D_f(u,x_n)-D_f(u,z_n),\\
&D_f(Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n),\\
&Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
\leq\ & D_f(u,Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
&-D_f(u,Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
&\cdots\cdots\\
\leq\ & D_f(u,x_n)-D_f(u,Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\\
&Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)),\\
&\cdots\cdots,\\
&D_f(Res^f_{\lambda_n^1 M_1}(x_n),x_n)\\
&\leq D_f(u,x_n)-D_f(u,Res^f_{\lambda_n^1 M_1}(x_n))\\
&\leq D_f(u,x_n).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\|\nabla f(y_n)-\nabla f(z_n)\|\\
=\ & \|\alpha_n\nabla f(x_1)+(1-\alpha_n)\nabla f(z_n)-\nabla f(z_n)\|\\
=\ & \alpha_n\|\nabla f(x_1)-\nabla f(z_n)\|,
\end{aligned}
$$

letting $n\to\infty$ in the above equality, we derive that

$$
\begin{aligned}
&\lim_{n\to\infty}\|\nabla f(y_n)-\nabla f(z_n)\|\\
=\ & \lim_{n\to\infty}\alpha_n\|\nabla f(x_1)-\nabla f(z_n)\|=0.
\end{aligned}
$$

Since f is totally convex on bounded subsets of E, it follows from Definition 10 and Lemma 14 that $\lim_{n\to\infty}\|y_n-z_n\|=0$. From (11) and $y_n\to p$ as $n\to\infty$, $\lim_{n\to\infty}\|x_n-z_n\|=0$ and $z_n\to p$ as $n\to\infty$. In view of

$$
\begin{aligned}
&D_f(z_n,Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
\leq\ & D_f(u,x_n)-D_f(u,z_n)\\
=\ & f(z_n)-f(x_n)+\langle\nabla f(z_n),x_n-z_n\rangle\\
&+\langle\nabla f(z_n)-\nabla f(x_n),u-x_n\rangle,
\end{aligned}
$$

taking $n\to\infty$ in the above inequality, we have $\lim_{n\to\infty}D_f(z_n,Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))$

$$
\begin{aligned}
\leq\ & \lim_{n\to\infty}\{f(z_n)-f(x_n)+\langle\nabla f(z_n),x_n-z_n\rangle\\
&+\langle\nabla f(z_n)-\nabla f(x_n),u-x_n\rangle\}\\
=\ & 0.
\end{aligned}
$$

Moreover, one has

$$\|z_n-Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)\|\to 0$$

and so

$$\|x_n-Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)\|\to 0. \quad (13)$$

Thus, from (13) and $z_n\to p$ as $n\to\infty$,

$$Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)\to p$$

as $n\to\infty$. Noticing that

$$
\begin{aligned}
&D_f(Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n),\\
&\qquad Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
\leq\ & D_f(u,x_n)-D_f(u,Res^f_{\lambda_n^{N-1}M_{N-1}}\circ\\
&\qquad Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)),\\
&D_f(Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n),\\
&\qquad Res^f_{\lambda_n^{N-3}M_{N-3}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
\leq\ & D_f(u,x_n)-D_f(u,Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\\
&\qquad Res^f_{\lambda_n^{N-3}M_{N-3}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)),\\
&\cdots\cdots,\\
&D_f(Res^f_{\lambda_n^1 M_1}(x_n),x_n)\\
\leq\ & D_f(u,x_n)-D_f(u,Res^f_{\lambda_n^1 M_1}(x_n)).
\end{aligned}
$$

Similarly, one can derive that

$$
\begin{aligned}
&\|x_n-Res^f_{\lambda_n^{N-2}M_{N-2}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)\|\to 0,\\
&\|x_n-Res^f_{\lambda_n^{N-3}M_{N-3}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)\|\to 0,\\
&\cdots\cdots,\\
&\|x_n-Res^f_{\lambda_n^1 M_1}(x_n)\|\to 0.
\end{aligned}
$$

Therefore, for each $i=1,2,\cdots,N$, $Res^f_{\lambda_n^i M_i}\circ Res^f_{\lambda_n^{i-1}M_{i-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)\to p$ as $n\to\infty$. Furthermore, from $\liminf_{n\to\infty}\lambda_n^i>0$ for all $i\in\{1,2,\cdots,N\}$ and Remark 12, we have

$$
\begin{aligned}
&M_{\lambda_n^i}(Res^f_{\lambda_n^{i-1}M_{i-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
=\ & \frac{1}{\lambda_n^i}(\nabla f(Res^f_{\lambda_n^{i-1}M_{i-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\\
&-\nabla f(Res^f_{\lambda_n^i M_i}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n)))
\end{aligned}
$$

and so
$\lim_{n\to\infty}\|M_{\lambda_n^i}(Res^f_{\lambda_n^{i-1}M_{i-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\|$
$=\lim_{n\to\infty}\frac{1}{\lambda_n^i}\|\nabla f(Res^f_{\lambda_n^{i-1}M_{i-1}}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))-\nabla f(Res^f_{\lambda_n^i M_i}\circ\cdots\circ Res^f_{\lambda_n^1 M_1}(x_n))\|$

$= 0$.
Again, from Remark 12, we know that, for each $i \in \{1, 2, \cdots, N\}$,
$(Res^f_{\lambda^i_n M_i} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n),$
$M_{\lambda^i_n}(Res^f_{\lambda^{i-1}_n M_{i-1}} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n)))$
$\in$ Graph(M_i).
For any $(x, \tau) \in$ Graph(M_i), by the monotonicity of M_i, one has
$\langle Res^f_{\lambda^i_n M_i} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n) - x,$
$M_{\lambda^i_n}(Res^f_{\lambda^{i-1}_n M_{i-1}} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n)) - \tau\rangle \geq 0.$
Then we have

$$\begin{aligned}
& \langle x - Res^f_{\lambda^i_n M_i} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n), \tau\rangle \\
\geq\ & \langle x - Res^f_{\lambda^i_n M_i} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n), \\
& \quad M_{\lambda^i_n}(Res^f_{\lambda^{i-1}_n M_{i-1}} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n))\rangle \\
\geq\ & \|M_{\lambda^i_n}(Res^f_{\lambda^{i-1}_n M_{i-1}} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n))\| \cdot \\
& \quad (-\|x - Res^f_{\lambda^i_n M_i} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n)\|).
\end{aligned}$$

Note that

$$\|M_{\lambda^i_n}(Res^f_{\lambda^{i-1}_n M_{i-1}} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n))\| \to 0.$$

Then

$$\begin{aligned}
& \langle x - p, \tau\rangle \\
=\ & \lim_{n\to\infty} \langle x - Res^f_{\lambda^i_n M_i} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n), \tau\rangle \\
\geq\ & 0.
\end{aligned}$$

By the maximal monotonicity of M_i, $p \in M_i^{-1}(0)$ and so, $p \in \bigcap_{i=1}^N M_i^{-1}(0)$. Thus $p \in U$.

Finally, we assert that $p = p^*$. Indeed, from $p^* = \text{proj}^f_U(x_1) \in U$ and $x_{n+1} = \text{proj}^f_{C_{n+1}}(x_1)$, we have

$$D_f(x_{n+1}, x_1) \leq D_f(p^*, x_1).$$

It follows from Lemma 17 that $x_{n+1} \to p^*$ as $n \to \infty$ and so $x_n \to p^*$ as $n \to \infty$. Moreover, one has $p = p^* = \text{proj}^f_U(x_1)$. Therefore, the sequence (x_n) converges strongly to the point p^*. □

The following results can be directly obtain from Theorem 24.

Corollary 25 *Let C be a nonempty closed convex subset of a real reflexive Banach space E, $f : E \to R$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Assume that, for each $k \in \{1, 2, \cdots, m\}$ and $i \in \{1, 2, \cdots, N\}$, $H_k : C \times C \to R$ satisfies Assumption 1 and $M_i : E \to 2^{E^*}$ is a maximal monotone operator such that $(\bigcap_{i=1}^N M_i^{-1}(0)) \bigcap (\bigcap_{k=1}^m EP(H_k)) \neq \emptyset$. Let (x_n) be a sequence generated by the following algorithm:*

$$\begin{cases}
x_1 \in C, \\
z_n = Res^f_{\lambda^N_n M_N} \circ \cdots \circ Res^f_{\lambda^1_n M_1}(x_n), \\
y_n = \nabla f^*(\alpha_n \nabla f(x_1) + (1 - \alpha_n)\nabla f(z_n)), \\
u_n = Res^f_{H_m} \circ Res^f_{H_{m-1}} \circ \cdots \circ Res^f_{H_1}(y_n), \\
C_1 = \{z \in C : D_f(z, u_1) \leq D_f(z, x_1)\}, \\
C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq \alpha_n D_f(z, x_1) \\
\qquad\qquad +(1 - \alpha_n) D_f(z, x_n)\}, \\
x_{n+1} = proj^f_{C_{n+1}} x_1 \ \ \forall\, n \geq 1,
\end{cases}$$

where $\liminf_{n\to\infty} \lambda^i_n > 0$ for all $i \in \{1, 2, \cdots, N\}$ and $(\alpha_n) \subseteq [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence (x_n) strongly converges to the point $proj^f_B(x_1)$, where $B = (\bigcap_{i=1}^N M_i^{-1}(0)) \bigcap (\bigcap_{k=1}^m EP(H_k))$ and $proj^f_B(x_1)$ is the Bregman projection of C onto B.

Proof: From Lemma 21 and Theorem 24, we obtain the desired result. □

Remark 26 *(1) From Theorem 24 and Corollary 25, we also can derive the corresponding results on common solutions to systems of variational inequalities, systems of optimization problems and maximal monotone operators;*

(2) If E is a uniformly convex and uniformly smooth Banach space and $f(x) = \frac{\|x\|^2}{2}$ for all $x \in E$, Theorem 24 and Corollary 25 still hold.

4 Applications

In this section, we apply the main results in Section 3 to study the following bilevel variational inequalities (BVI):

Find $x \in S$ such that

$$\langle V(x), y - x\rangle \geq 0 \ \ \forall\, y \in E, \tag{14}$$

where $V : E \to E^*$ is a continuous and monotone mapping, S is the set of solutions to the following variational inequality problem:

Find $x \in E$ such that

$$h(x, y - x) \geq 0 \ \ \forall\, y \in E, \tag{15}$$

where $h(x, y - x) = g(y) - g(x)$ for all $y \in E$ and $g : E \to R$ is a proper convex and lower semicontinuous function. In the sequel, we assume that (BVI) is solvable and denote the solution set of the problem (BVI) by K.

It is easy to see that the problem (BVI) is equivalent to the following problem:

Find a common solution of the following problems (16) with (17):

$$\min_{x \in E} \quad g(x), \tag{16}$$

and

Find $x \in E$ such that

$$\langle V(x), y-x \rangle \geq 0 \ \forall y \in E. \tag{17}$$

Since $g : E \to R$ is a proper convex and lower semicontinuous function, ∂g is maximal monotone, where ∂g is the subdifferential of g in the sense of convex analysis. It is well known that $x \in E$ is a solution of the problem (16) if and only if $x \in \partial g^{-1}(0)$. Denote the sets of solutions to the problems (16) and (17) by W and D, respectively. Then we have $W \bigcap D = K \neq \emptyset$.

Theorem 27 *Assume that $f : E \to R$ is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Let (x_n) be the iterative sequence generated by*

$$\begin{cases} x_1 \in E, \\ z_n = Res^f_{\lambda_n \partial g^{-1}}(x_n), \\ y_n = \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(z_n)), \\ u_n = Res^f_V(y_n), \\ C_1 = \{z \in E : D_f(z, u_1) \leq D_f(z, x_1)\}, \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq \alpha_n D_f(z, x_1) \\ \qquad\qquad +(1-\alpha_n) D_f(z, x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}} x_1 \ \forall n \geq 1, \end{cases}$$

where $\lambda_n > 0$, $(\alpha_n) \subseteq [0,1]$ and, for all $x \in E$,

$$\begin{aligned} & Res^f_V(x) \\ = \ & \{z \in E : \langle V(z), y-z \rangle \\ & +\langle \nabla f(z) - \nabla f(x), y-z \rangle \geq 0 \ \forall y \in E\}. \end{aligned}$$

Assume that $\liminf_{n\to\infty} \lambda_n > 0$ and $(\alpha_n) \subseteq [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence (x_n) strongly converges to the point $proj^f_K(x_1)$.

Proof: Let $H(x,y) = \langle V(x), y-x \rangle$ and $M(x) = \partial g(x)$ for all $x, y \in E$. Then M is a maximal monotone operator on E. From the continuity and monotonicity of V, it follows that H satisfies the conditions (C1)-(C4) of Assumption 1. By Corollary 25, the sequence (x_n) strongly converges to the point $\mathrm{proj}^f_K(x_1)$. □

In Theorem 27, let g be a constant function. Then Theorem 27 reduces to an iterative sequence for finding a solution of the classical variational inequalities (17).

Theorem 28 *Let $f : E \to R$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E. Let (x_n) be the iterative sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = \nabla f^*(\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(x_n)), \\ u_n = Res^f_V(y_n), \\ C_1 = \{z \in E : D_f(z, u_1) \leq D_f(z, x_1)\}, \\ C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq \alpha_n D_f(z, x_1) \\ \qquad\qquad +(1-\alpha_n) D_f(z, x_n)\}, \\ x_{n+1} = proj^f_{C_{n+1}} x_1 \ \forall n \geq 1, \end{cases}$$

where $(\alpha_n) \subseteq [0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence (x_n) strongly converges to the point $proj^f_D(x_1)$.

Proof: It directly follows from Theorem 27. □

Example 29 *Let $E = R = (-\infty, +\infty)$. Let $f(x) = \frac{1}{2}x^2$ and $V(x) = \frac{1}{5}x$ for all $x \in E$. Then find $x \in E$ such that*

$$\langle V(x), y-x \rangle \geq 0 \ \forall y \in E. \tag{18}$$

It is easy to see that f is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E and V is continuous. Now, we verify that V is monotone. For any $x, y \in E$,

$$\begin{aligned} \langle V(x) - V(y), x-y \rangle &= \langle \frac{1}{5}x - \frac{1}{5}y, x-y \rangle \\ &= \frac{1}{5}\|x-y\|^2 \geq 0, \end{aligned}$$

that is, V is monotone on E. For each $x \in E$, by (18), we have

$$\begin{aligned} Res^f_V(x) = \{z \in E : \langle \frac{1}{5}z, y-z \rangle \\ +\langle z-x, y-z \rangle \geq 0 \ \forall y \in E\}. \end{aligned}$$

Moreover, one has $Res^f_V(x) = \frac{5}{6}x$. It is easy to see that the solution set $D \neq \emptyset$ since 0 is obviously a solution. So all the conditions of Theorem 28 are satisfied.

Let $\alpha_n = 0$ for all $n \geq 1$. Use C programming language, take $x_1 = 1$ arbitrarily and let the tolerance $\epsilon = 10^{-6}$. The selected values of $\{u_n\}$ and $\{x_n\}$ computed by computer programs of the iterative sequence $\{x_n\}$ in Theorem 28 are listed below Table 1.

Remark 30 *(1) Table 1 illustrates that both of the sequences (x_n) and (u_n) strongly converge to 0 which is a solution of (18);*

Table 1: Selected values of (x_n) and (u_n) when $\alpha_n = 0$

Iter n	x_n	u_n
1	1.000000	0.833333
2	0.916667	0.763889
3	0.840278	0.700231
4	0.770255	0.641879
10	0.456986	0.380822
19	0.208845	0.174038
31	0.073509	0.061258
50	0.014072	0.011727
80	0.001034	0.000863
100	0.000182	0.000511
120	0.000032	0.000027
145	0.000004	0.000003
150	0.000003	0.000002
160	0.000001	0.000001
165	0.000001	0.000001
170	0.000000	0.000000

(2) Compared with many well-known iterative algorithms in the literature, we remove that $Q_{n+1} = \{z \in C_n \cap Q_n : \langle \nabla f(x_1) - \nabla f(x_n), z - x_n \rangle \leq 0\}$. Indeed, if $\alpha_n = 0$ for all $n \geq 1$, one can conclude that

$$C_1 = \{z \in E : z \leq \frac{1}{2}(u_1 + x_1) = \frac{11}{12}x_1\},$$
$$C_{n+1} = \{z \in C_n : z \leq \frac{1}{2}(u_n + x_n) = \frac{11}{12}x_n\}$$

and

$$Q_1 = E, \quad Q_{n+1} = \{z \in E : z \leq x_n\}.$$

Therefore, $C_n \subset Q_n$ for all $n \geq 1$.

Note that Theorem 27 and Theorem 28 do not require that $\sum \alpha_n = +\infty$. In the sequel, we consider the influence of the condition $\sum \alpha_n = +\infty$ to the iterative sequence (x_n).

Let $\alpha_n = \frac{1}{n+1}$ for all $n \geq 1$. Take $x_1 = 1$ arbitrarily and let the tolerance $\epsilon = 10^{-6}$. A part of the values of (x_n) and (u_n) computed by computer programs are listed below Table 2.

Remark 31 *(1) From Table 2, we know that the convergence speed of the iterative sequence (x_n) is slow when $\alpha_n = \frac{1}{n+1}$ such that $\sum \alpha_n = +\infty$;*

(2) The above tests show that the new iterative sequence without the condition $\sum \alpha_n = +\infty$ converges much faster in this case as shown in Table 1 and Table 2;

(3) If we are only interested in solving a classical variational inequality, then we could apply Theorem 28 to work for this purpose.

Table 2: Selected values of (x_n) and (u_n) when $\alpha_n = \frac{1}{n+1}$

Iter n	x_n	u_n
1	1.000000	0.833333
2	0.916667	0.787037
3	0.870643	0.752485
4	0.834474	0.722983
10	0.725862	0.610502
19	0.635998	0.545165
31	0.564184	0.481503
50	0.494305	0.420184
100	0.397989	0.336625
400	0.239006	0.200753
1800	0.126163	0.105540
3010	0.099997	0.083525
13190	0.050000	0.041853
19000	0.041960	0.035009
19825	0.041108	0.034297
19826	0.041107	0.034296
19827	0.041106	0.034296
19828	0.041105	0.034294
...	...	...

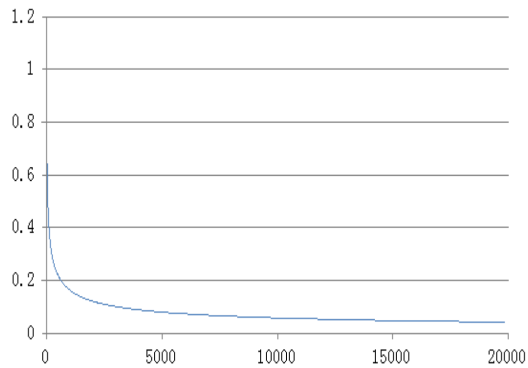

Fig. 1. The convergent process of the sequence (x_n) when $\alpha_n = \frac{1}{n+1}$

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