Solvability of a coupled system of a fractional boundary value problem with fractional integral condition

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Abstract: In this paper, the following system of a fractional boundary value problem with fractional integral condition

$$ {}^CD^\alpha u(t) = f(t, v(t), {}^CD^p v(t)),\ \ {}^CD^\beta v(t) = g(t, u(t), {}^CD^q u(t)),\ \ t \in (0,1), $$

$$ u(0) = a_1 I^{\alpha-1}u(\eta),\ u(1) = b_1 I^{\alpha-1}u(\xi),\ v(0) = a_2 I^{\beta-1}v(\eta),\ v(1) = b_2 I^{\beta-1}v(\xi) $$

is considered, where $\alpha, \beta, p, q, \xi, \eta, a_1, a_2, b_1, b_2$ satisfy certain conditions. By using Schauder fixed point theorem, we establish sufficient conditions for the existence of solutions for the above system.

Key–Words: Integral condition; Fractional Caputo derivative; Coupled system; Schauder fixed point theorem.

1 Introduction

Fractional differential equations arise in various fields of science and engineering such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, biosciences, signal processing, systems control theory, electrochemistry, mechanics and diffusion processes. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, fractional differential equations have become a very important and useful area of mathematics over the last few decades. For details, see [1-9, 22-40] and the references therein.

Integral boundary conditions have various applications in applies fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so on. we refer the reader to [10-15] for more details of nonlocal and integral boundary conditions.

In recent years, some authors have studied fractional boundary value problem with integral boundary conditions. Bashir Ahmad and Juan J. Nieto [16] studied the existence results for a nonlinear fractional integrodifferential equation with integral boundary conditions

$$ {}^CD^q x(t) = f(t, x(t), \chi(x)(t)),\ 0 < t < 1,\ 1 < q \le 2, $$

$$ \alpha x(0) + \beta x'(0) = \int_0^1 q_1(x(s))ds, $$

$$ \alpha x(1) + \beta x'(1) = \int_0^1 q_2(x(s))ds. $$

where ${}^CD^q$ is the Caputo fractional derivative. $\alpha > 0,\ \beta \ge 0$ are real numbers.

$$ \chi(x)(t) = \int_0^t r(t,s)x(s)ds. $$

Bashir Ahmad, Sotiris K. Ntouyas, Ahmed Alsaedi [17] discussed the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1,2]$ with three point integral boundary conditions given by

$$ {}^CD^q x(t) = f(t, x(t)),\ 0 < t < 1,\ 1 < q \le 2, $$

$$ x(0) = 0, $$

$$ x(1) = \alpha \int_0^\eta x(s)ds,\ 0 < \eta < 1. $$

where ${}^CD^q$ denotes the Caputo fractional derivative. $\alpha \in R$ is such that $\alpha \ne \frac{2}{\eta^2}$.

Meiqiang Feng, Xuemei Zhang, Weigao Ge [18] studied the following higher-order singular boundary value problem of fractional differential equations

$$ D^\alpha_{0+}x(t) + g(t)f(t, x(t)) = 0,\ 0 < t < 1, $$

$$ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, $$

$$x(1) = \int_0^1 h(t)x(t)dt,$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order $n-1<\alpha<n,\ n\geq 3$.

With the wide application of fractional differential and integral, some authors began to study the fractional boundary value problem with fractional integral condition.

A. Guezane-Lakouda, R. Khaldi [19] considered a fractional boundary value problem generated by a fractional differential equation and a fractional integral condition

$$\begin{cases} {}^{C}D^{q}_{0+}u(t) = f(t, u(t), {}^{C}D^{\sigma}_{0+}u(t)) = 0, \\ \quad 0<t<1, \\ u(0) = 0, \\ u'(1) = I^{\sigma}_{0+}u(1), \end{cases}$$

where $1<q<2,\ 0<\sigma<1$.

Bashir Ahmad, Juan J Nieto [20] studied the existence and uniqueness of solutions for the following nonlinear fractional integro-differential equation

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t)), \\ \quad t\in[0,T],\ \alpha\in(1,2], \\ D^{\alpha-2}u(0^+) = 0, \\ D^{\alpha-1}u(0^+) = vI^{\alpha-1}u(\eta),\ 0<\eta<T, \end{cases}$$

where D^{α} denotes the standard Riemann-Liouville fractional derivative.

Motivated by the above works, in this paper, we study the following coupled system of nonlinear fractional differential equations with fractional integral condition

$${}^{C}D^{\alpha}u(t) = f(t, v(t), {}^{C}D^{p}v(t)),\ 0<t<1, \quad (1)$$

$${}^{C}D^{\beta}v(t) = g(t, u(t), {}^{C}D^{q}u(t)),\ 0<t<1, \quad (2)$$

$$u(0) = a_1 I^{\alpha-1}u(\eta),\ u(1) = b_1 I^{\alpha-1}u(\xi), \quad (3)$$

$$v(0) = a_2 I^{\beta-1}v(\eta),\ v(1) = b_2 I^{\beta-1}v(\xi), \quad (4)$$

where ${}^{C}D^{\alpha}$, ${}^{C}D^{\beta}$ denote the Caputo fractional derivative, $1<\alpha,\beta\leq 2,\ 0<\eta<\xi<1,\ a_i\ (i=1,2)$ and $b_i\ (i=1,2)$ are arbitrary real constants, $p,q>0,\ \alpha-q\geq 1,\ \beta-p\geq 1$, and $f,g:\ [0,1]\times R\to R$ are given continuous functions. It is important to note that the nonlinear terms in the coupled system involve the fractional derivatives of the unknown functions.

As we all know, when the integral conditions are allowed to depend on the fractional integral $I^{\alpha-1}u$, difficulties arise immediately. In this paper, we use the properties of Caputo fractional derivative and Riemann-Liouville fractional integral to overcome the difficulties. To the best knowledge of the authors, no work has been done for the fractional system (1)- (4) by use of Schauder fixed point theorem. The aim of this paper is to fill the gap in the relevant literature.

2 Preliminaries and Lemmas

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results.

Definition 1 *If $f\in C[a,b]$ and $\alpha>0$, then the Riemann-Liouville fractional integral is defined as*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds.$$

Definition 2 *Let $\alpha>0,\ n=[\alpha]+1$. If $f\in AC^n[a,b]$, then the Caputo's fractional derivative of order α of f defined by*

$${}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)ds}{(t-s)^{\alpha+1-n}}ds,$$

exist almost everywhere on $[a,b]$ ($[\alpha]$ is the entire part of α).

Lemma 3 [21] *Let $\alpha,\beta>0$ and $n=[\alpha]+1$, then the following relations hold:*

$${}^{C}D^{\alpha}t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}t^{\beta-\alpha-1},\ \beta>n$$

and

$${}^{C}D^{\alpha}t^{k} = 0,\ \ k=0,1,2,\cdots,n-1.$$

Lemma 4 [21] *For $\alpha>0,\ f(t)\in C(0,1)$, then the homogenous fractional differential equation*

$${}^{C}D^{\alpha}f(t) = 0$$

has a solution

$$f(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},\ \ c_i\in R,$$

$$i=1,2,\cdots,n, n=[\alpha]+1.$$

Lemma 5 [21] *Let $\alpha>0$, then*

$$I^{\alpha}\,{}^{C}D^{\alpha}u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i\in R,\ \ i=1,2,\cdots,n, n=[\alpha]+1$.

Denote by $L'([0,1],R)$ the Banach space of Lebesgue integrable functions from $[0,1]$ into R with the norm $\|f\|_{L'} = \int_0^1 |f(t)|dt$.

The following Lemmas give some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative.

Lemma 6 [21] *Let* $p, q \geq 0,\ f \in L'[a,b]$. *Then*

$$I^p I^q f(t) = I^{p+q} f(t) = I^q I^p f(t)$$

and

$$^C D^q I^q f(t) = f(t),\ \ t \in [a,b].$$

Lemma 7 [21] *Let* $\beta > \alpha > 0,\ f \in L'[a,b]$. *Then*

$$^C D^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t),\ \ t \in [a,b].$$

Let $C(J)$ denote the space of all continuous functions defined on $J = [0,1]$. Let

$$X = \left\{ u(t) \mid u \in C(J);\ ^C D^q u \in C(J) \right\}$$

be a Banach space endowed with the norm

$$\|u\|_X = \max_{t\in J} |u(t)| + \max_{t \in J} |^C D^q u(t)|$$

where $1 < \alpha < 2,\ 0 < q \leq \alpha - 1$ and

$$Y = \left\{ v(t) \mid v \in C(J);^C D^p v \in C(J) \right\}$$

be a Banach space endowed with the norm

$$\|v\|_Y = \max_{t\in J} |v(t)| + \max_{t \in J} |^C D^p v(t)|$$

where $1 < \beta < 2,\ 0 < p \leq \beta - 1$.

Thus,$(X \times Y, \|\cdot\|_{X\times Y})$ is a Banach space with the norm defined by

$$\|(u,v)\|_{X\times Y} = \max\{\|u\|_X, \|v\|_Y\}$$

for $(u,v) \in X \times Y$.

Lemma 8 *Let* $h(t) \in C[0,1]$ *and* $1 < \alpha \leq 2$, *then the unique solution of the linear fractional boundary value problem*

$$^C D^\alpha u(t) = h(t),\ 0 < t < 1, \tag{5}$$

$$u(0) = a_1 I^{\alpha-1} u(\eta),\ u(1) = b_1 I^{\alpha-1} u(\xi), \tag{6}$$

where $0 < \eta < \xi < 1$ *is*

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ &+ \frac{a_1}{\triangle}\left(1 - \frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right) I^{2\alpha-1} h(\eta) \\ &+ \frac{a_1}{\triangle}\frac{\eta^\alpha}{\Gamma(\alpha+1)}(b_1 I^{2\alpha-1}h(\xi) - I^\alpha h(1)) \\ &+ \frac{t}{\triangle}\left(1 - \frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right) b_1 I^{2\alpha-1} h(\xi) \\ &- \frac{t}{\triangle}\left(1 - \frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right) I^\alpha h(1) \\ &- \frac{a_1 t}{\triangle}\left(1 - \frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right) I^{2\alpha-1} h(\eta), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \triangle &= \left(1 - \frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{a_1\eta^\alpha}{\Gamma(\alpha+1)} \\ &+ \left(1 - \frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\left(1 - \frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right). \end{aligned}$$

Proof: We may apply Lemma 5 to reduce (5) to an equivalent integral equation

$$u(t) = I^\alpha h(t) + C_1 + C_2 t.$$

Since

$$u(0) = C_1,\ u(\eta) = I^\alpha h(\eta) + C_1 + C_2\eta,$$

From (6), we deduce that

$$\begin{aligned} C_1 &= a_1 \frac{1}{\Gamma(\alpha-1)} \int_0^\eta (\eta - s)^{\alpha-2} u(s)ds \\ &= a_1 \frac{1}{\Gamma(\alpha-1)} \int_0^\eta (\eta-s)^{\alpha-2}(I^\alpha h(s) + C_1 + C_2 s)ds \\ &= a_1 I^{2\alpha-1} h(\eta) + \frac{a_1 C_1 \eta^{\alpha-1}}{\Gamma(\alpha)} + \frac{a_1 C_2 \eta^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{8}$$

Finally, condition $u(1) = b_1 I^{\alpha-1} u(\xi)$ implies that

$$\begin{aligned} & I^\alpha h(1) + C_1 + C_2 \\ &= b_1 I^{2\alpha-1} h(\xi) + \frac{b_1 C_1 \xi^{\alpha-1}}{\Gamma(\alpha)} + \frac{b_1 C_2 \xi^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{9}$$

So, expression (8),(9) imply that

$$\begin{aligned} C_1 &= \frac{a_1}{\triangle}\left(1 - \frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right) I^{2\alpha-1} h(\eta) \\ &+ \frac{a_1}{\triangle}\frac{\eta^\alpha}{\Gamma(\alpha+1)}(b_1 I^{2\alpha-1} h(\xi) - I^\alpha h(1)), \\ C_2 &= \frac{1}{\triangle}\left(1 - \frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right) b_1 I^{2\alpha-1} h(\xi) \end{aligned}$$

$$-\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)I^{\alpha}h(1)$$
$$-\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)I^{2\alpha-1}h(\eta).$$

This completes the proof. □

Similarly, the general solution to equation

$$^{C}D^{\beta}v(t)=h(t),\ 0<t<1,$$

$$v(0)=a_2I^{\alpha-1}v(\eta),\ v(1)=b_2I^{\alpha-1}v(\xi),$$

where $0<\eta<\xi<1$ is

$$\begin{aligned}v(t) &= \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}h(s)ds\\ &+\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta}}{\Gamma(\beta+1)}\right)I^{2\beta-1}h(\eta)\\ &+\frac{a_2}{\Lambda}\frac{\eta^{\beta}}{\Gamma(\beta+1)}(b_2I^{2\beta-1}h(\xi)-I^{\beta}h(1))\\ &+\frac{t}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)b_2I^{2\beta-1}h(\xi)\\ &-\frac{t}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)I^{\beta}h(1)\\ &-\frac{a_2t}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)I^{2\beta-1}h(\eta),\end{aligned}$$

where

$$\begin{aligned}\Lambda &= \left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)\frac{a_2\eta^{\beta}}{\Gamma(\beta+1)}\\ &+\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\left(1-\frac{b_2\xi^{\beta}}{\Gamma(\beta+1)}\right).\end{aligned}$$

Consider the coupled system of integral equations

$$\begin{aligned}&u(t)=\\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,v(s),^{C}D^{p}v(s))ds\\ &+\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha}}{\Gamma(\alpha+1)}\right)I^{2\alpha-1}f(\eta,v(\eta),^{C}D^{p}v(\eta))\\ &+\frac{a_1}{\triangle}\frac{\eta^{\alpha}}{\Gamma(\alpha+1)}(b_1I^{2\alpha-1}f(\xi,v(\xi),^{C}D^{p}v(\xi))\\ &-I^{\alpha}f(1,v(1),^{C}D^{p}v(1)))\\ &+\frac{t}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)b_1I^{2\alpha-1}f(\xi,v(\xi),^{C}D^{p}v(\xi))\\ &-\frac{t}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)I^{\alpha}f(1,v(1),^{C}D^{p}v(1))\\ &-\frac{a_1t}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)I^{2\alpha-1}f(\eta,v(\eta),^{C}D^{p}v(\eta)),\end{aligned} \tag{10}$$

$$\begin{aligned}&v(t)=\\ &\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s,v(s),^{C}D^{q}v(s))ds\\ &+\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta}}{\Gamma(\beta+1)}\right)I^{2\beta-1}g(\eta,v(\eta),^{C}D^{q}v(\eta))\\ &+\frac{a_2}{\Lambda}\frac{\eta^{\beta}}{\Gamma(\beta+1)}(b_2I^{2\beta-1}g(\xi,v(\xi),^{C}D^{q}v(\xi)\\ &-I^{\beta}g(1,v(1),^{C}D^{q}v(1)))\\ &+\frac{t}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)b_2I^{2\beta-1}g(\xi,v(\xi),^{C}D^{q}v(\xi))\\ &-\frac{t}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)I^{\beta}g(1,v(1),^{C}D^{q}v(1))\\ &-\frac{a_2t}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)I^{2\beta-1}g(\eta,v(\eta),^{C}D^{q}v(\eta)).\end{aligned} \tag{11}$$

Lemma 9 *Assume that $f,g:\ [0,1]\times R\times R\to R$ are continuous functions. Then $(u,v)\in X\times Y$ is a solution of (1)- (4) if and only if $(u,v)\in X\times Y$ is a solution of (10), (11).*

Proof: The proof is immediate from Lemma 8, so we omit it.

Let us define an operator $T:\ X\times Y\to X\times Y$ as

$$T(u,v)(t)=(T_1v(t),T_2u(t)), \tag{12}$$

where

$$\begin{aligned}&T_1v(t)=\\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,v(s),^{C}D^{p}v(s))ds\\ &+\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha}}{\Gamma(\alpha+1)}\right)I^{2\alpha-1}f(\eta,v(\eta),^{C}D^{p}v(\eta))\\ &+\frac{a_1}{\triangle}\frac{\eta^{\alpha}}{\Gamma(\alpha+1)}(b_1I^{2\alpha-1}f(\xi,v(\xi),^{C}D^{p}v(\xi))\\ &-I^{\alpha}f(1,v(1),^{C}D^{p}v(1)))\\ &+\frac{t}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)b_1I^{2\alpha-1}f(\xi,v(\xi),^{C}D^{p}v(\xi))\\ &-\frac{t}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)I^{\alpha}f(1,v(1),^{C}D^{p}v(1))\\ &-\frac{a_1t}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)I^{2\alpha-1}f(\eta,v(\eta),^{C}D^{p}v(\eta)),\end{aligned} \tag{13}$$

$$\begin{aligned}&T_2u(t)=\\ &\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s,v(s),^{C}D^{q}v(s))ds\\ &+\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta}}{\Gamma(\beta+1)}\right)I^{2\beta-1}g(\eta,v(\eta),^{C}D^{q}v(\eta))\end{aligned}$$

$$
\begin{aligned}
&+\frac{a_2}{\Lambda}\frac{\eta^\beta}{\Gamma(\beta+1)}(b_2 I^{2\beta-1}g(\xi,v(\xi),^C D^q v(\xi)\\
&-I^\beta g(1,v(1),^C D^q v(1)))\\
&+\frac{t}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)b_2 I^{2\beta-1}g(\xi,v(\xi),^C D^q v(\xi))\\
&-\frac{t}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)I^\beta g(1,v(1),^C D^q v(1))\\
&-\frac{a_2 t}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)I^{2\beta-1}g(\eta,v(\eta),^C D^q v(\eta)).
\end{aligned} \tag{14}
$$

The continuity of f, g, it follows that T is continuous. Moreover, by Lemma 9, the fixed point of the operator T is the solution of (1)- (4).

We give the growth conditions on f and g as:

(H_1) there exists a nonnegative function $a(t) \in L(0,1)$ such that

$$|f(t,x,y)| \le a(t) + \varepsilon_1|x|^{\rho_1} + \varepsilon_2|y|^{\rho_2},$$

$$\varepsilon_1,\ \varepsilon_2 > 0,\ 0 < \rho_1, \rho_2 < 1.$$

(H_2) there exists a nonnegative function $b(t) \in L(0,1)$ such that

$$|g(t,x,y)| \le b(t) + \delta_1|x|^{\sigma_1} + \delta_2|y|^{\sigma_2},$$

$$\delta_1,\ \delta_2 > 0,\ 0 < \sigma_1, \sigma_2 < 1.$$

For convenience, we introduce the following notations.

$$
\begin{aligned}
&\Lambda_1 =\\
&\frac{1}{\Gamma(\alpha+1)} + \left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right)\right|\frac{\eta^{2\alpha-1}}{\Gamma(2\alpha)}\\
&+\left|\frac{a_1}{\triangle}\right|\frac{b_1\eta^\alpha}{\Gamma(\alpha+1)}\frac{\xi^{2\alpha-1}}{\Gamma(2\alpha)} + \left|\frac{a_1}{\triangle}\right|\frac{\eta^\alpha}{\Gamma^2(\alpha+1)}\\
&+\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\xi^{2\alpha-1}}{\Gamma(2\alpha)}\\
&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(\alpha+1)}\\
&+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\eta^{2\alpha-1}}{\Gamma(2\alpha)}\\
&+\frac{1}{\Gamma(\alpha-q+1)} + \frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}\\
&\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\xi^{2\alpha-1}}{2\alpha-1}\\
&+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(\alpha+1)}\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\\
&+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}\\
&\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\eta^{2\alpha-1}}{2\alpha-1},
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\Lambda_2 =\\
&\frac{1}{\Gamma(\beta+1)} + \left|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^\beta}{\Gamma(\beta+1)}\right)\right|\frac{\eta^{2\beta-1}}{\Gamma(2\beta)}\\
&+\left|\frac{a_2}{\Lambda}\right|\frac{b_2\eta^\beta}{\Gamma(\beta+1)}\frac{\xi^{2\beta-1}}{\Gamma(2\beta)} + \left|\frac{a_2}{\Lambda}\right|\frac{\eta^\beta}{\Gamma^2(\beta+1)}\\
&+\left|\frac{b_2}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{\xi^{2\beta-1}}{\Gamma(2\beta)}\\
&+\left|\frac{1}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{1}{\Gamma(\beta+1)}\\
&+\left|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{\eta^{2\beta-1}}{\Gamma(2\beta)}\\
&+\frac{1}{\Gamma(\beta-p+1)} + \frac{1}{\Gamma(2-p)}\frac{1}{\Gamma(2\beta-1)}\\
&\left|\frac{b_2}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{\xi^{2\beta-1}}{2\beta-1}\\
&+\frac{1}{\Gamma(2-p)}\frac{1}{\Gamma(\beta+1)}\left|\frac{1}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\right|\\
&+\frac{1}{\Gamma(2-p)}\frac{1}{\Gamma(2\beta-1)}\\
&\left|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{\eta^{2\beta-1}}{2\beta-1},
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\mu =\\
&\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}a(s)ds\\
&+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right)\right|\\
&\frac{1}{\Gamma(2\alpha-1)}\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds\\
&+\left|\frac{a_1}{\triangle}\right|\frac{b_1\eta^\alpha}{\Gamma(\alpha+1)}\frac{1}{\Gamma(2\alpha-1)}\times\\
&\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds\\
&+\left|\frac{a_1}{\triangle}\right|\frac{\eta^\alpha}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}a(s)ds\\
&+\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2\alpha-1)}\times\\
&\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds\\
&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}a(s)ds\\
&+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2\alpha-1)}\times\\
&\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds\\
&+\frac{1}{\Gamma(\alpha-q)}\int_0^1(1-s)^{\alpha-q-1}a(s)ds
\end{aligned}
$$

$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(\alpha)}\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\times\int_0^1(1-s)^{\alpha-1}a(s)ds$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds, \quad (17)$$

$$\nu=\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}b(s)ds+\left|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^\beta}{\Gamma(\beta+1)}\right)\right|\times\frac{1}{\Gamma(2\beta-1)}\int_0^\eta(\eta-s)^{2\beta-2}b(s)ds$$
$$+\left|\frac{a_2}{\Lambda}\right|\frac{b_2\eta^\beta}{\Gamma(\beta+1)}\frac{1}{\Gamma(2\beta-1)}\int_0^\xi(\xi-s)^{2\beta-2}b(s)ds+\left|\frac{a_2}{\Lambda}\right|\frac{\eta^\beta}{\Gamma(\beta+1)}\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}b(s)ds$$
$$+\left|\frac{b_2}{\Lambda}\left(1-\frac{b_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{1}{\Gamma(2\beta-1)}\int_0^\xi(\xi-s)^{2\beta-2}b(s)ds+\left|\frac{1}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}b(s)ds$$
$$+\left|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)\right|\frac{1}{\Gamma(2\beta-1)}\int_0^\eta(\eta-s)^{2\beta-2}b(s)ds. \quad (18)$$

Define a ball B in the Banach space $X\times Y$ as

$$B=\Big\{(u(t),v(t))|(u(t),v(t))\in X\times Y, \|(u(t),v(t))\|_{X\times Y}\le R,\ t\in J\Big\},$$

where

$$R\ge\max\Big\{(3\Lambda_1\varepsilon_1)^{\frac{1}{1-\rho_1}},(3\Lambda_1\varepsilon_2)^{\frac{1}{1-\rho_2}},(3\Lambda_2\delta_1)^{\frac{1}{1-\sigma_1}},(3\Lambda_2\delta_2)^{\frac{1}{1-\sigma_2}},3\mu,3\nu\Big\}.$$

3 Main result

Theorem 10 *Assume that the hypotheses (H_1) and (H_2) hold. Then there exists a solution for the coupled system (1)-(4).*

Proof: Firstly, we prove that $T:\ B\to B$, using the Lemmas 6, 7, we obtain

$$|T_1v(t)|\le\frac{1}{\Gamma(\alpha)}\Big[\int_0^t(t-s)^{\alpha-1}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^t(t-s)^{\alpha-1}ds\Big]$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right)\right|\frac{1}{\Gamma(2\alpha-1)}\Big[\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^\eta(\eta-s)^{2\alpha-2}ds\Big]$$
$$+\left|\frac{a_1}{\triangle}\right|\frac{b_1\eta^\alpha}{\Gamma(\alpha+1)}\frac{1}{\Gamma(2\alpha-1)}\Big[\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^\xi(\xi-s)^{2\alpha-2}ds\Big]$$
$$+\left|\frac{a_1}{\triangle}\right|\frac{\eta^\alpha}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}\Big[\int_0^1(1-s)^{\alpha-1}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^1(1-s)^{\alpha-1}ds\Big]$$
$$+\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2\alpha-1)}\Big[\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^\xi(\xi-s)^{2\alpha-2}ds\Big]$$
$$+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(\alpha)}\Big[\int_0^1(1-s)^{\alpha-1}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^1(1-s)^{\alpha-1}ds\Big]$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2\alpha-1)}\Big[\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^\eta(\eta-s)^{2\alpha-2}ds\Big]$$
$$\le\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}a(s)ds+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^\alpha}{\Gamma(\alpha+1)}\right)\right|\frac{1}{\Gamma(2\alpha-1)}\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds$$
$$+\left|\frac{a_1}{\triangle}\right|\frac{b_1\eta^\alpha}{\Gamma(\alpha+1)}\frac{1}{\Gamma(2\alpha-1)}\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds+\left|\frac{a_1}{\triangle}\right|\frac{\eta^\alpha}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}a(s)ds$$

$$+\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right| \frac{1}{\Gamma(2\alpha-1)}\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds$$
$$+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}a(s)ds$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right| \frac{1}{\Gamma(2\alpha-1)}\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds$$
$$+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\Big[\frac{1}{\Gamma(\alpha+1)}+ \left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha}}{\Gamma(\alpha+1)}\right)\right|\frac{\eta^{2\alpha-1}}{\Gamma(2\alpha)}$$
$$+\left|\frac{a_1}{\triangle}\right|\frac{b_1\eta^\alpha}{\Gamma(\alpha+1)}\Big]\frac{\xi^{2\alpha-1}}{\Gamma(2\alpha)}$$
$$+\left|\frac{a_1}{\triangle}\right|\frac{\eta^\alpha}{\Gamma^2(\alpha+1)}$$
$$+\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\xi^{2\alpha-1}}{\Gamma(2\alpha)}$$
$$+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(\alpha+1)}$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\eta^{2\alpha-1}}{\Gamma(2\alpha)}\Big]$$

and

$$|^CD^qT_1v(t)|\le^C D^qI^\alpha f(t,v(t),^C D^pv(t))$$
$$+\Big[\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|I^{2\alpha-1}f(\xi,v(\xi),^C D^pv(\xi))$$
$$+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|I^\alpha f(1,v(1),^C D^pv(1))$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|I^{2\alpha-1}f(\eta,v(\eta),^C D^pv(\eta))\Big]$$
$$\times^C D^q t$$
$$\le I^{\alpha-q}f(t,v(t),^C D^pv(t))$$
$$+\frac{1}{\Gamma(2-q)}\Big[\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|$$
$$\times I^{2\alpha-1}f(\xi,v(\xi),^C D^pv(\xi))$$
$$+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|I^\alpha f(1,v(1),^C D^pv(1))$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|I^{2\alpha-1}f(\eta,v(\eta),^C D^pv(\eta))\Big]$$

$$\le\frac{1}{\Gamma(\alpha-q)}\Big[\int_0^t(t-s)^{\alpha-q-1}a(s)ds$$
$$+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^t(t-s)^{\alpha-q-1}ds\Big]$$
$$+\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2-q)}$$
$$\frac{1}{\Gamma(2\alpha-1)}\Big[\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds$$
$$+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^\xi(\xi-s)^{2\alpha-2}ds\Big]$$
$$+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2-q)}$$
$$\frac{1}{\Gamma(\alpha)}\Big[\int_0^1(1-s)^{\alpha-1}a(s)ds$$
$$+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^1(1-s)^{\alpha-1}ds\Big]$$
$$+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{1}{\Gamma(2-q)}$$
$$\frac{1}{\Gamma(2\alpha-1)}\Big[\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds$$
$$+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\int_0^\eta(\eta-s)^{2\alpha-2}ds\Big]$$
$$\le\frac{1}{\Gamma(\alpha-q)}\int_0^1(1-s)^{\alpha-q-1}a(s)ds$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}$$
$$\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\int_0^\xi(\xi-s)^{2\alpha-2}a(s)ds$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(\alpha)}\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|$$
$$\int_0^1(1-s)^{\alpha-1}a(s)ds$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}$$
$$\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\int_0^\eta(\eta-s)^{2\alpha-2}a(s)ds$$
$$+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})$$
$$\Big[\frac{1}{\Gamma(\alpha-q+1)}+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}$$
$$\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\xi^{2\alpha-1}}{2\alpha-1}$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(\alpha+1)}\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\right|$$
$$+\frac{1}{\Gamma(2-q)}\frac{1}{\Gamma(2\alpha-1)}$$

$$\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\frac{\eta^{2\alpha-1}}{2\alpha-1}\Bigg].$$

Thus,

$$\begin{aligned}&\|T_1v(t)\|_X=\max_{t\in J}|T_1v(t)|+\max_{t\in J}|^CD^qT_1v(t)|\\&\le\mu+(\varepsilon_1|R|^{\rho_1}+\varepsilon_2|R|^{\rho_2})\Lambda_1\\&\le\tfrac{R}{3}+\tfrac{R}{3}+\tfrac{R}{3}=R.\end{aligned}$$

Similarly, it can be show that

$$\|T_2u(t)\|_Y\le\nu+(\delta_1|R|^{\sigma_1}+\delta_2|R|^{\sigma_2})\Lambda_2\le R.$$

Hence, we conclude that $\|T(u,v)\|_{X\times Y}\le R$. Since, $T_1v(t),\ T_2u(t),\ ^CD^qT_1v(t),\ ^CD^pT_2u(t)$ are continuous on J, therefore, $T:\ B\to B$.

Now we prove that T is a completely continuous operator. For that, we denote

$$M=\max_{t\in J}|f(t,v(t),^CD^pv(t)|,$$

$$N=\max_{t\in J}|g(t,u(t),^CD^qu(t)|.$$

For $(u,v)\in B,\ t_1,\ t_2\in J,\ (t_1<t_2)$,

$$\begin{aligned}&|T_1v(t_2)-T_1v(t_1)|\\&\le\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,v(s),^CD^pv(s))ds\right.\\&\left.-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,v(s),^CD^pv(s))ds\right|\\&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)b_1I^{2\alpha-1}f(\xi,v(\xi),^CD^pv(\xi))\right|\\&\times|t_2-t_1|\\&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)I^{\alpha}f(1,v(1),^CD^pv(1))\right|\\&\times|t_2-t_1|\\&+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)I^{2\alpha-1}f(\eta,v(\eta),^CD^pv(\eta))\right|\\&\times|t_2-t_1|\\&\le M\left[\left|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right|\right.\\&\left.+\left|\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right|\right]\\&+\left[\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{\xi^{2\alpha-1}}{\Gamma(2\alpha)}\right|\right.\\&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{\Gamma(\alpha+1)}\right|\\&\left.+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{\eta^{2\alpha-1}}{\Gamma(2\alpha)}\right|\right]M|t_2-t_1|.\end{aligned}$$

Clearly, the right hand side of the above inequality tends to zero independently of v as $t_2\to t_1$,

$$\begin{aligned}&|^CD^qT_1v(t_2)-^CD^qT_1v(t_1)|\\&\le I^{\alpha-q}f(t_2,v(t_2),^CD^pv(t_2))\\&-I^{\alpha-q}f(t_1,v(t_1),^CD^pv(t_1))\\&+\left[\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)b_1I^{2\alpha-1}f(\xi,v(\xi),^CD^pv(\xi))\right|\right.\\&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)I^{\alpha}f(1,v(1),^CD^pv(1))\right|\\&\left.+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)I^{2\alpha-1}f(\eta,v(\eta),^CD^pv(\eta))\right|\right]\\&\times\frac{t_2^{1-q}-t_1^{1-q}}{\Gamma(2-q)}\\&\le\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}f(s,v(s),^CD^pv(s))ds\right.\\&\left.-\int_0^{t_1}\frac{(t_1-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}f(s,v(s),^CD^pv(s))ds\right|\\&+\left[\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)b_1I^{2\alpha-1}f(\xi,v(\xi),^CD^pv(\xi))\right|\right.\\&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)I^{\alpha}f(1,v(1),^CD^pv(1))\right|\\&\left.+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)I^{2\alpha-1}f(\eta,v(\eta),^CD^pv(\eta))\right|\right]\\&\times\frac{t_2^{1-q}-t_1^{1-q}}{\Gamma(2-q)}\\&\le M\left[\left|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}ds\right|\right.\\&\left.+\left|\int_0^{t_1}\frac{(t_2-s)^{\alpha-q-1}-(t_1-s)^{\alpha-q-1}}{\Gamma(\alpha)}ds\right|\right]\\&+\frac{M}{\Gamma(2-q)}\left[\left|\frac{b_1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{\xi^{2\alpha-1}}{\Gamma(2\alpha)}\right|\right.\\&+\left|\frac{1}{\triangle}\left(1-\frac{a_1\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{\Gamma(\alpha+1)}\right|\\&\left.+\left|\frac{a_1}{\triangle}\left(1-\frac{b_1\xi^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{\eta^{2\alpha-1}}{\Gamma(2\alpha)}\right|\right](t_2^{1-q}-t_1^{1-q}).\end{aligned}$$

Obviously, the right hand side of the above inequality tends to zero independently of v as $t_2\to t_1$. Thus, it follows by the Arzela-Ascoli theorem that T_1 is completely continuous.

Similarly, it can be proved that

$$|T_2u(t_2)-T_2u(t_1)|$$

$$\leq N\Big[\Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}ds\Big| + \Big|\int_0^{t_1}\frac{(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}}{\Gamma(\beta)}ds\Big|\Big] + \Big[\Big|\frac{b_2}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\frac{\xi^{2\beta-1}}{\Gamma(2\beta)}\Big| + \Big|\frac{1}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\frac{1}{\Gamma(\beta)}\Big| + \Big|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)\frac{\eta^{2\beta-1}}{\Gamma(2\beta)}\Big|\Big]N|t_2-t_1|.$$

$$|^C D^p T_2 u(t_2) -^C D^p T_2 u(t_1)| \leq N\Big[\Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds\Big| + \Big|\int_0^{t_1}\frac{(t_2-s)^{\beta-p-1}-(t_1-s)^{\beta-p-1}}{\Gamma(\beta)}ds\Big|\Big] + \frac{N}{\Gamma(2-p)}\Big[\Big|\frac{b_2}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\frac{\xi^{2\beta-1}}{\Gamma(2\beta)}\Big| + \Big|\frac{1}{\Lambda}\left(1-\frac{a_2\eta^{\beta-1}}{\Gamma(\beta)}\right)\frac{1}{\Gamma(\beta+1)}\Big| + \Big|\frac{a_2}{\Lambda}\left(1-\frac{b_2\xi^{\beta-1}}{\Gamma(\beta)}\right)\frac{\eta^{2\beta-1}}{\Gamma(2\beta)}\Big|\Big](t_2^{1-p}-t_1^{1-p}).$$

So, it follows by the Arzela-Ascoli theorem, we conclude that T_2 is completely continuous. Therefore, the operator T is a completely continuous operator. Hence, by Schauder fixed point theorem, there exists a solution of (1)-(4). This completes the proof.

4 Conclusion

This paper is motivated from some recent papers treating the fractional boundary value problem with fractional integral condition. We first give some notations, recall some concepts and preparation results. Second, some sufficient conditions for the existence of solutions of coupled system of a fractional boundary value problem with fractional integral condition are established by applying fixed point theorem. Our results complements previous work in the area of fractional integral condition of fractional order. When the integral conditions are allowed to depend on the fractional integral $I^{\alpha-1}u$, difficulties arise immediately. In this paper, we use the properties of Caputo fractional derivative and Riemann-Liouville fractional integral to overcome the difficulties. To the best knowledge of the authors, no work has been done for the fractional system (1)- (4) by use of Schauder fixed point theorem.

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