Equitable Colorings of Cartesian Product Graphs of Wheels with Complete Bipartite Graphs

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Abstract: By the sorting method of vertices, the equitable chromatic number and the equitable chromatic threshold of the Cartesian products of wheels with bipartite graphs are obtained.

Key–Words: Cartesian product, Equitable coloring, Equitable chromatic number, Equitable chromatic threshold

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer k and a real number x, let $[k] = \{1, 2, \cdots, k\}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the smallest integer not less than x and the largest integer not greater than x, respectively.

In recent years, many parameters and graph classes were studied. For example, in [24], Zuo showed that a conjecture holds for all unicyclic graphs and all bicyclic graphs, in [25], Xue, Zuo et al. studied the hamiltonicity and path t-coloring of Sierpińskilike graphs, in [27], Jin and Zuo gave the further ordering bicyclic graphs with respect to the Laplacian spectra radius, in [28], Lai et al. gave a survey for the more recent developments of the research on supereulerian graphs and the related problems, in [29], Jiang and Zhang studied Randomly M_t-decomposable multigraphs and M_2-equipackable multigraphs, and in [26], Xue and Zuo obtained the linear $(n-1)$-arboricity of $K_{n(m)}$.

Equitable coloring as a special vertex coloring on graphs was first introduced by Meyer[1]. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the work week to each route becomes the problem of 6-coloring of G. On practical grounds it might also be desirable to have an approximately equal number of routes run on each of the six days, so we have to color the graph in the equitable way.

We can find another application of equitable coloring in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. As we know, we can model this problem as coloring the vertices of a graph G whose nodes correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. For applications of equitable coloring such as scheduling and constructing timetables, please see [1, 5, 11, 12, 13].

A graph $G = (V, E)$ is equitably k-colorable if $V(G)$ cab be divided into k independent sets for which any two sets differ in size at most 1. The equitable chromatic number of a graph G, denoted by $\chi_=(G)$, is the minimum k such that G is equitably k-colorable. The equitable chromatic threshold of a graph G, denoted by $\chi^*_=(G)$, is the minimum t such that G is equitably k-colorable for all $k \geq t$. It is evident from the definition that

$$\chi(G) \leq \chi_=(G) \leq \chi^*_=(G)$$

for any graph G.

In [3], Lin and Chang obtained the exact values or upper bounds of the equitable chromatic number on Kronecker products of G and H, when G and H are complete graphs, bipartite graphs, paths or cycles, and in [4], they studied the equitable colorings of Cartesian product of paths and cycles, respectively, with bipartite graphs. In [16], Lih and Wu studied the equitable colorings of bipartite graphs, and in [17], Lih gave a good survey for this coloring. In [23], Zhu gave a survey for Hedetniemi's conjecture about the tensor product of graphs. The general problem of deciding if $\chi_=(G) \leq 3$ is NP-complete [10]. If, however, G has a regular or simplified structure we are sometimes able to provide a polynomial algorithm coloring it in the equitable way. For more details about this coloring, please see [1, 2, 6, 7, 8, 14, 20, 21, 22].

The Cartesian product of graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G\Box H$ with vertex set

$$\{(u, v) \mid u \in V_1, v \in V_2\}$$

and edge set

$$\left\{(u,v)(x,y) \mid \begin{array}{l} u = x \text{ with } vy \in E_2, \\ or\ v = y \text{ with } ux \in E_1 \end{array}\right\}.$$

Graph products are interesting and useful in many situations. For example, Sabbidussi [19] showed that any graph has the unique decomposition into prime factors under the Cartesian product. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

In the present paper, we study the equitable colorings of Cartesian products of wheels with complete bipartite graphs.

2 Main results

In the following, let s, l, m, n, n' be all nonnegative integers, and $W_{n'}$ represent the wheel with vertex set

$$V(W_{n'}) = \{x, x_1, x_2, \cdots, x_{n'}\}.$$

Let H be a complete bipartite graph with two parts

$$Y = \{y_1, y_2, \cdots, y_m\}$$

and

$$Z = \{z_1, z_2, \cdots, z_n\}$$

where $m \geq n$. In the following, we will study the equitable chromatic number and the equitable chromatic threshold of $W_{n'}\Box H$ according to the parity of n'. Clearly, if $n' = 2l + 1$, then

$$\chi(W_{n'}\Box H) \geq 4$$

since $\chi(W_{n'}) = 4$, so we have

$$\chi^*_=(W_{n'}\Box H) \geq \chi_=(W_{n'}\Box H)$$
$$\geq \chi(W_{n'}\Box H) \geq 4.$$

Theorem 1. *Suppose that $l \geq 1$ and $k \geq 4$. If $n' = 2l + 1$, then $W_{n'}\Box H$ is equitably k-colorable, hence*

$$\chi^*_=(W_{n'}\Box H) = 4.$$

Proof. The Cartesian product graph $W_{n'}\Box H$ is represented in Figure 1, where s is some fixed positive integer determined by the parity of m.

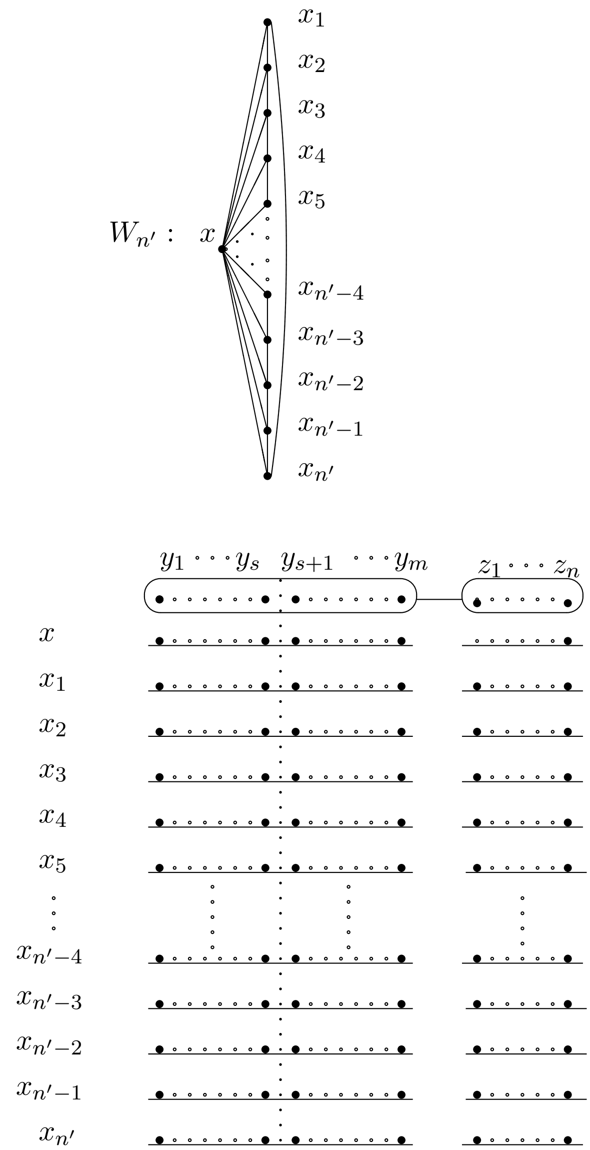

Figure 1. The Cartesian product $W_{n'}\Box H$

We will partition the vertex set of $W_{n'}\Box H$ into eight subsets as A, B, C, D, E, F, R, K, i. e.,

$$V(W_{n'}\Box H) =$$
$$A \cup B \cup C \cup D \cup E \cup F \cup R \cup K,$$

where

$$\begin{array}{ll} A: & (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m), \\ & (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_s), \end{array}$$

$$\begin{aligned} B: \quad & (x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), \\ & (x_5, z_1), (x_5, z_2), \ldots, (x_5, z_n), \ldots, \\ & (x_{n'-4}, z_1), (x_{n'-4}, z_2), \ldots, (x_{n'-4}, z_n), \\ & (x_{n'-2}, z_1), (x_{n'-2}, z_2), \ldots, (x_{n'-2}, z_n), \\ & (x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), \end{aligned}$$

$$\begin{aligned} C: \quad & (x_4, y_1), (x_4, y_2), \ldots, (x_4, y_s), \\ & (x_6, y_1), (x_6, y_2), \ldots, (x_6, y_s), \ldots, \\ & (x_{n'-3}, y_1), (x_{n'-3}, y_2), \ldots, (x_{n'-3}, y_s), \\ & (x_{n'-1}, y_1), (x_{n'-1}, y_2), \ldots, (x_{n'-1}, y_s), \end{aligned}$$

$$\begin{aligned} D: \quad & (x_2, y_{s+1}), (x_2, y_{s+2}), \ldots, (x_2, y_m), \\ & (x_1, y_1), (x_1, y_2), \ldots, (x_1, y_s), \end{aligned}$$

$$\begin{aligned} E: \quad & (x, z_1), (x, z_2), \ldots, (x, z_n), \\ & (x_4, y_{s+1}), (x_4, y_{s+2}), \ldots, (x_4, y_m), \\ & (x_6, y_{s+1}), (x_6, y_{s+2}), \ldots, (x_6, y_m), \ldots, \\ & (x_{n'-3}, y_{s+1}), (x_{n'-3}, y_{s+2}), \ldots, (x_{n'-3}, y_m), \\ & (x_{n'-1}, y_{s+1}), (x_{n'-1}, y_{s+2}), \ldots, (x_{n'-1}, y_m), \end{aligned}$$

$$\begin{aligned} F: \quad & (x_3, y_1), (x_3, y_2), \ldots, (x_3, y_s), \\ & (x_5, y_1), (x_5, y_2), \ldots, (x_5, y_s), \ldots, \\ & (x_{n'-4}, y_1), (x_{n'-4}, y_2), \ldots, (x_{n'-4}, y_s), \\ & (x_{n'-2}, y_1), (x_{n'-2}, y_2), \ldots, (x_{n'-2}, y_s), \\ & (x_1, y_{s+1}), (x_1, y_{s+2}), \ldots, (x_1, y_m), \\ & (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s), \\ & (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n), \\ & (x_3, y_{s+1}), (x_3, y_{s+2}), \ldots, (x_3, y_m), \\ & (x_5, y_{s+1}), (x_5, y_{s+2}), \ldots, (x_5, y_m), \ldots, \\ & (x_{n'-4}, y_{s+1}), (x_{n'-4}, y_{s+2}), \ldots, (x_{n'-4}, y_m), \\ & (x_{n'-2}, y_{s+1}), (x_{n'-2}, y_{s+2}), \ldots, (x_{n'-2}, y_m), \end{aligned}$$

$$\begin{aligned} R: \quad & (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \\ & (x_6, z_1), (x_6, z_2), \ldots, (x_6, z_n), \ldots, \\ & (x_{n'-3}, z_1), (x_{n'-3}, z_2), \ldots, (x_{n'-3}, z_n), \\ & (x_{n'-1}, z_1), (x_{n'-1}, z_2), \ldots, (x_{n'-1}, z_n), \end{aligned}$$

and

$$\begin{aligned} K: \quad & (x_{n'}, y_{s+1}), (x_{n'}, y_{s+2}), \ldots, (x_{n'}, y_m), \\ & (x, y_1), (x, y_2), \ldots, (x, y_s), \\ & (x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n). \end{aligned}$$

Clearly,

$$|A| = m,$$

$$|B| = ln,$$

$$|C| = (l-1)s,$$

$$|D| = m,$$

$$|E| = n + (l-1)(m-s),$$

$$|F| = lm + n,$$

$$|R| = (l-1)n,$$

and

$$|K| = m + n.$$

Now sort the vertices of $W_{n'} \Box H$ as

$$A, B, C, D, E, F, R, K$$

in the order shown. It is not difficult to verify that the cardinality of every greatest independent vertex set which contains the consecutive vertices in this sorting is at least

$$m + n + (l-1)s.$$

Let

$$\sigma_t = \left\lfloor \frac{(2l+2)(m+n) + t - 1}{k} \right\rfloor,$$

for $t \in [k]$. Since $l \geq 1$ and $k \geq 4$, we will deal with the problem in the following cases.

(1) If $k \geq 2l + 2$, then

$$\begin{aligned} \sigma_1 &= \left\lfloor \frac{(2l+2)(m+n)}{k} \right\rfloor \\ \leq \sigma_k &= \left\lfloor \frac{(2l+2)(m+n) + k - 1}{k} \right\rfloor \\ &= \left\lceil \frac{(2l+2)(m+n)}{k} \right\rceil \leq m + n. \end{aligned}$$

If $m = 2p + 1$ and $s = p + 1$, then it is clear that

$$\sigma_k \leq m + n \leq m + n + (l-1)(p+1).$$

If $m = 2p$ and $s = p$, then

$$\sigma_k \leq m + n \leq m + n + (l-1)p.$$

So, applying the vertex sorting above, the vertex set of $W_{n'} \Box H$ can be partitioned into k independent subsets with the cardinality

$$\sigma_1, \sigma_2, \ldots, \sigma_k,$$

respectively. Hence $W_{n'} \Box H$ is equitably k-colorable for all $k \geq 2l + 2$.

(2) For $4 \leq k < 2l + 2$, it is evident that

$$|A \cup B| = m + nl,$$

$$|A \cup B \cup C| = m + nl + (l-1)s,$$

and the cardinality of the greatest independent set consisting of consecutive vertices in

$$P = C \cup D \cup E \cup F \cup R \cup K$$

is at least $s + nl + l(m - s)$ or $n + ml$, where the order of vertices of these six parts are all not varied.

(2.1) Note that

$$\sigma_t = \left\lfloor \frac{(2l+2)(m+n)+t-1}{4} \right\rfloor,$$

where $t \in [4]$ for $k = 4$.

If $m = 2p+1$ and $s = p+1$, then we can choose an independent vertex set M_1 with the cardinality σ_1 in

$$A \cup B \cup C$$

in continues from the first vertex because $A \cup B \cup C$ is an independent set. Since

$$\begin{aligned} m + ln &= |A \cup B| \\ &\leq \sigma_1 \leq \sigma_4 \leq |A \cup B \cup C| \\ &= m + ln + (l-1)s, \end{aligned}$$

we have

$$(A \cup B) \subseteq M_1 \subseteq A \cup B \cup C.$$

Let

$$P' = V(W_{n'} \Box H) - M_1.$$

Then $P' \subseteq P$, and so the cardinality of the greatest independent set consisting of consecutive vertices in P' is at least

$$p + 1 + nl + lp$$

or

$$n + ml.$$

It is not difficult to verify that

$$\begin{aligned} \sigma_4 &= \left\lceil \frac{(l+1)(m+n)}{2} \right\rceil \\ &\leq \min\{p+1+nl+lp, n+ml\}. \end{aligned}$$

Therefore, we can partition the remaining vertices of P' into three independent sets M_2, M_3, and M_4 with the cardinality σ_2, σ_3, and σ_4, respectively. Hence $W_{n'} \Box H$ is equitably 4-colorable.

Similarly, for $m = 2p$ and $s = p$, we can show that $W_{n'} \Box H$ is equitably 4-colorable.

(2.2) For $5 \leq k \leq 2l+1$,

$$\sigma_k = \left\lceil \frac{(2l+2)(m+n)}{k} \right\rceil > m + n.$$

Let $m = 2p+1$ and $s = p+1$. Note that

$$A \cup B \cup C$$

is an independent set. Choose the first k' (≥ 1) independent vertex subsets

$$M_1, M_2, \cdots, M_{k'}$$

of

$$A \cup B \cup C$$

in the order with cardinality

$$\sigma_1, \sigma_2, \ldots, \sigma_{k'},$$

respectively, such that

$$|A \cup B \cup C| - \sigma_1 - \ldots - \sigma_{k'} < \sigma_{k'+1},$$

and then we will choose the remaining $k - k'$ independent vertex subsets in two subcases according to the position of the last vertex of $M_{k'}$.

(2.2.1) If the last vertex of $M_{k'}$ is $(x_{n'}, z_n)$ or belongs to C, then let

$$M' = V(W_{n'} \Box H) - M_1 \cup M_2 \cup \cdots \cup M_{k'},$$

and thus $M' \subseteq P$. By

$$\begin{aligned} \sigma_k &= \left\lceil \frac{(2l+2)(m+n)}{k} \right\rceil \\ &\leq \min\{p+1+nl+lp, n+ml\}, \quad (*) \end{aligned}$$

the vertex set M' can be partitioned into $k - k'$ independent sets

$$M_{k'+1}, M_{k'+2}, \cdots, M_k$$

according to the sorting, with the cardinality

$$\sigma_{k'+1}, \sigma_{k'+2}, \cdots, \sigma_k,$$

respectively. Hence $W_{n'} \Box H$ is equitably k-colorable.

(2.2.2) Suppose that the last vertex of $M_{k'}$ belongs to B but not equals $(x_{n'}, z_n)$. Let

$$B' = A \cup B - M_1 \cup M_2 \cup \cdots \cup M_{k'},$$

and

$$B'' = V(W_{n'} \Box H) - M_1 \cup M_2 \cup \cdots \cup M_{k'},$$

then we will choose the $(k'+1)$th independent set $M_{k'+1}$ in B'' with the cardinality $\sigma_{k'+1}$ as following.

(2.2.2.1) If

$$|B' \cup C| < \sigma_{k'+1} \leq |B' \cup C \cup D|,$$

then we can choose $M_{k'+1}$ after $M_{k'}$ since $B' \cup C \cup D$ is an independent set. Let

$$D' = V(W_{n'} \Box H) - M_1 \cup M_2 \cup \cdots \cup M_{k'+1}.$$

Then $D' \subset P$, so, by $(*)$, we can partition vertex set D' into the remaining $k - k' - 1$ independent set

$$M_{k'+2}, \cdots, M_k$$

with cardinality

$$\sigma_{k'+2}, \cdots, \sigma_k,$$

respectively. Hence $W_{n'} \Box H$ is equitably k-colorable.

(2.2.2.2) Suppose that

$$\sigma_{k'+1} > |B' \cup C \cup D|,$$

then we will choose the $(k'+1)$th independent set $M_{k'+1}$ with cardinality $\sigma_{k'+1}$ in B'' according to $|B'|$.

If $|B'| \geq n$, then let

$$E' = E - \{(x, z_1), (x, z_2), \ldots, (x, z_n)\}.$$

Clearly, $B' \cup C \cup D \cup E'$ is an independent set with the cardinality at least $n + ml$. So, in $B' \cup C \cup D \cup E'$, we can choose $M_{k'+1}$ in the order where the last vertex belongs to E'. Let

$$E'' = B' \cup C \cup D \cup E' - M_{k'+1}.$$

Set

$$F' = F - \{(x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n)\}.$$

Then $E'' \cup F'$ is such a vertex subset in which the size of the every greatest independent vertex set consisting of the consecutive vertices is at least ml. First we choose

$$\{(x, z_1), (x, z_2), \ldots, (x, z_n)\},$$

then we choose the remaining vertices of $M_{k'+2}$ in the ordering shown in $E'' \cup F'$ with the last vertex belongs to F'.

Let

$$F'' = \{(x, z_1), (x, z_2), \ldots, (x, z_n)\} \cup E'' \cup F' - M_{k'+2}.$$

Then $F'' \subseteq F$. Now we can choose M_k from $R \cup K$ in the inverse order from the last vertex of it with the cardinality σ_k since

$$\sigma_k \leq |M_1 \cup \cdots \cup M_{k'}| + 1 \leq |A \cup B|,$$

and

$$|R \cup K| = |A \cup B|.$$

Set

$$K' = R \cup K - M_k,$$

then

$$L = F'' \cup K' \cup \{(x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n)\}$$

is an independent set. Thus, we can choose the remaining $k - k' - 3$ independent sets in L, with the cardinality

$$\sigma_{k'+3}, \cdots, \sigma_{k-1},$$

respectively.

If $1 \leq |B'| < n$, then

$$B' \subseteq \{(x_{n'}, z_2), \ldots, (x_{n'}, z_n)\}.$$

Suppose that the first vertex in B' is $(x_{n'}, z_g)$, and set

$$J' = E - \{(x, z_1), \ldots, (x, z_n)\}.$$

Then

$$B' \cup C \cup D \cup \{(x, z_1), \ldots, (x, z_{g-1})\} \cup J'$$

is an independent set with the cardinality $n + ml$, so we can choose $M_{k'+1}$ in the sequence shown. If

$$\{(x, z_1), \ldots, (x, z_{g-1})\} - M_{k'+1}$$

is not empty or (x, z_{g-1}) is the last vertex in $M_{k'+1}$, set

$$J'' = B' \cup C \cup D \cup \{(x, z_1), \ldots, (x, z_{g-1})\} - M_{k'+1},$$

then

$$(J'' \cup \{(x, z_g), \ldots, (x, z_n)\} \cup J' \cup F \cup R \cup K) \subset P.$$

So we can choose the remaining $k - k' - 1$ independent sets

$$M_{k'+2}, \cdots, M_k$$

with the cardinality

$$\sigma_{k'+2}, \cdots, \sigma_k,$$

respectively. Thus $W_{n'} \Box H$ is equitably k-colorable.

If the last vertex of $M_{k'+1}$ belongs to J', then set

$$L = B' \cup C \cup D \cup \{(x, z_1), \ldots, (x, z_{g-1})\} \cup J' - M_{k'+1}.$$

We will choose the $(k'+2)$th independent set $M_{k'+2}$ with cardinality $\sigma_{k'+2}$ as following. Let

$$W' = F - \{(x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n)\}.$$

Then $L \cup W'$ is a set which contains the greatest independent consecutive vertex set with the cardinality ml. First, we choose

$$\{(x, z_g), \ldots, (x, z_n)\}$$
$$\cup \{(x_2, z_1), \ldots, (x_2, z_{g-1})\},$$

then we choose the remaining vertices of $M_{k'+2}$ in $L \cup W'$ in the order where the last vertex belongs to W'. Let the set of remaining vertices of W' be denoted by W''. Now we can choose M_k with cardinality σ_k in $R \cup K$ since

$$\sigma_k \leq |M_1 \cup M_2 \cdots \cup M_{k'}| + 1 \leq |A \cup B|,$$

and

$$|R \cup K| = |A \cup B| .$$

Let $K' = R \cup K - M_k$, then

$$M' = K' \cup \{(x_2, z_g), \ldots, (x_2, z_n)\} \cup W''$$

is an independent set. So we can choose the remaining independent sets $k - k' - 3$ from M' with the cardinality

$$\sigma_{k'+3}, \cdots, \sigma_{k-1},$$

respectively.

Similarly as the argument above, we can show that $W_{n'} \Box H$ is equitably k-colorable for $m = 2p$ and $s = p$ in all subcases of (2.2).

In a word, we have proved that $W_{n'} \Box H$ is equitably k-colorable for any $k \geq 4$, and

$$\chi^*_=(W_{n'} \Box H) = 4$$

for odd n'. Hence Theorem 1 holds.

Theorem 2. *Suppose that $l \geq 2$, $n' = 2l$ and $k \geq 4$. Then $W_{n'} \Box H$ is equitably k-colorable.*

proof. Sort the vertices of $W_{n'} \Box H$ as following.

$$\begin{aligned}
&(x, y_1), (x, y_2), \ldots, (x, y_s), \\
&(x_2, z_1), (x_4, z_1), \ldots, (x_{2h}, z_1), \ldots, (x_{n'}, z_1), \\
&(x_2, z_2), (x_4, z_2), \ldots, (x_{2h}, z_2), \ldots, (x_{n'}, z_2), \\
&\ldots, \\
&(x_2, z_n), (x_4, z_n), \ldots, (x_{2h}, z_n), \ldots, (x_{n'}, z_n),
\end{aligned}$$

$$\begin{aligned}
&(x_1, y_{s+1}), (x_3, y_{s+1}), \ldots, (x_{2h-1}, y_{s+1}), \\
&\ldots, (x_{n'-1}, y_{s+1}), \\
&(x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_{2h-1}, y_{s+2}), \\
&\ldots, (x_{n'-1}, y_{s+2}), \\
&\ldots, \\
&(x_1, y_m), (x_3, y_m), \ldots, (x_{2h-1}, y_m), \\
&\ldots, (x_{n'-1}, y_m),
\end{aligned}$$

$$\begin{aligned}
&(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_s), \\
&(x_3, y_1), (x_3, y_2), \ldots, (x_3, y_s), \ldots, \\
&(x_{2h-1}, y_1), (x_{2h-1}, y_2), \ldots, (x_{2h-1}, y_s), \\
&\ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2) \ldots, (x_{n'-1}, y_s),
\end{aligned}$$

$$\begin{aligned}
&(x, z_1), (x, z_2), \ldots, (x, z_n), \\
&(x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, (x_{2h}, y_{s+1}), \\
&\ldots, (x_{n'}, y_{s+1}), \\
&(x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \ldots, \\
&(x_{n'}, y_{s+2}), \ldots, (x_2, y_m), \\
&(x_4, y_m), \ldots, \\
&(x_{2h}, y_m), \ldots, (x_{n'}, y_m), \\
&(x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), \\
&(x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), \ldots, \\
&(x_{2h}, y_1), (x_{2h}, y_2), \ldots, (x_{2h}, y_s), \ldots, \\
&(x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s),
\end{aligned}$$

$$\begin{aligned}
&(x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), \\
&(x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), \\
&\ldots, \\
&(x_{2h-1}, z_1), (x_{2h-1}, z_2), \ldots, (x_{2h-1}, z_n), \\
&\ldots, \\
&(x_{n'-1}, z_1), (x_{n'-1}, z_2), \ldots, (x_{n'-1}, z_n), \\
&(x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m),
\end{aligned}$$

where h is an integer and $1 \leq h \leq l$.

If $m = 2p + 1$, $p \geq 0$, and $s = p + 1$, then the cardinality of every greatest independent consecutive vertex set in this sorting is at least

$$lp + nl + p + 1$$

or

$$n + (l - 1)m + p.$$

If $m = 2p$, $p \geq 1$, and $s = p$, then the cardinality of every greatest independent set consisting of consecutive vertices in this ordering is at least

$$n + (l - 1)m + p$$

or

$$nl + pl + p.$$

Note that

$$\sigma_t = \left\lfloor \frac{(2l + 1)(m + n) + t - 1}{k} \right\rfloor,$$

where $t \in [k]$.

(1) Suppose that $k \geq 5$.

By $l \geq 2$, we have

$$\begin{aligned}
\sigma_1 &= \left\lfloor \frac{(2l + 1)(m + n)}{k} \right\rfloor \\
\leq \sigma_k &= \left\lfloor \frac{(2l + 1)(m + n) + k - 1}{k} \right\rfloor \\
&= \left\lceil \frac{(2l + 1)(m + n)}{k} \right\rceil \\
&\leq \left\lceil \frac{(2l + 1)(m + n)}{5} \right\rceil .
\end{aligned}$$

If $m = 2p + 1$, then

$$\sigma_k \leq \left\lceil \frac{(2l + 1)(m + n)}{5} \right\rceil$$

$$\leq \min\{lp + nl + p + 1, n + (l - 1)m + p\}.$$

If $m = 2p$, then

$$\sigma_k < \lceil \frac{(2l+1)(m+n)}{5} \rceil$$

$$\leq \min\{n + (l-1)m + p, nl + pl + p\}.$$

Thus, applying the vertex sorting above, the vertex set of $W_{n'} \square H$ can be partitioned into k independent sets with the cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Hence $W_{n'} \square H$ is equitably k-colorable.

(2) If $k = 4$, then

$$\sigma_t = \left\lfloor \frac{(2l+1)(m+n)+t-1}{4} \right\rfloor$$

for $t \in [4]$.

(2.1) Suppose that $m \geq n+2$.

For $m = 2p+1$, it is obvious that

$$\sigma_4 \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil$$

$$\leq \min\{lp + nl + p + 1, n + (l-1)m + p\}.$$

For $m = 2p$, clearly,

$$\sigma_4 \leq \left\lceil \frac{(2l+1)(m+n)}{4} \right\rceil$$

$$\leq \min\{n + (l-1)m + p, nl + pl + p\}.$$

Hence $W_{n'} \square H$ is equitably 4-colorable.

(2.2) Suppose that $m \leq n+1$.

Since the cardinality of the greatest independent set which contains consecutive vertices from the first one in the sorting is

$$l(m+n) + s - ls,$$

we can choose the independent set M_1 with cardinality σ_1 from beginning of the sorting. Clearly, the last vertex of M_1 is $(x_{n'-2}, z_n)$, or $(x_{n'}, z_n)$, or (x_i, y_j) where $i \in \{1, 3, \ldots, n'-1\}$ and $j \in [s+1, m]$ by $m \geq n$. Furthermore, the cardinality of the greatest independent set K which contains consecutive vertices beginning from the next one of the last vertex in M_1 in the sorting is $lm + n$, and then we can choose the independent set M_2 from the first vertex in K. By

$$l(m+n) + \frac{m+n}{2} - 1$$

$$\leq \sigma_1 + \sigma_2$$

$$\leq l(m+n) + \frac{m+n}{2},$$

we will show that the following result holds.

Claim The last vertex of M_2, denoted by (u, v), is $(x_{n'-1}, y_s)$, or (x, z_q) for some $q \in [1, n]$, where $(m, n) \neq (1, 1)$.

Note that

$$\sigma_i = \left\lfloor \frac{(2l+1)(m+n)+i-1}{4} \right\rfloor$$

for $i \in [1, 4]$, and

$$n \leq m \leq n+1.$$

Let

$$A = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), \\ (x_2, z_1), (x_4, z_1), \ldots, \\ (x_{2h}, z_1), \ldots, (x_{n'}, z_1), \\ (x_2, z_2), (x_4, z_2), \ldots, \\ (x_{2h}, z_2), \ldots, (x_{n'}, z_2), \\ \ldots, \\ (x_2, z_n), (x_4, z_n), \ldots, \\ (x_{2h}, z_n), \ldots, (x_{n'}, z_n) \end{array} \right\}$$

and

$$B = \left\{ \begin{array}{l} (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots, \\ (x_{2h-1}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), \\ (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, \\ (x_{2h-1}, y_{s+2}), \ldots, (x_{n'-1}, y_{s+2}), \\ \ldots, \\ (x_1, y_m), (x_3, y_m), \ldots, \\ (x_{2h-1}, y_m), \ldots, (x_{n'-1}, y_m), \\ (x_1, y_1), (x_1, y_2), \ldots, (x_1, y_s), \\ (x_3, y_1), (x_3, y_2), \ldots, (x_3, y_s), \\ \ldots, \\ (x_{2h-1}, y_1), (x_{2h-1}, y_2), \\ \ldots, (x_{2h-1}, y_s), \\ \ldots, \\ (x_{n'-1}, y_1), (x_{n'-1}, y_2), \\ \ldots, (x_{n'-1}, y_s), \\ (x, z_1), (x, z_2), \ldots, (x, z_n) \end{array} \right\}.$$

It is obvious that

$$|A| = nl + s,$$

and

$$|B| = lm + n.$$

We will show that the claim in the following cases.

Case 1. $m = n$.

Clearly,

$$\sigma_1 = \left\lfloor \frac{(2l+1)n}{2} \right\rfloor$$

in this case.

If $m = 2p$, then the last vertex of M_1 is $(x_{n'}, z_n)$ by

$$\sigma_1 = \left\lfloor \frac{(2l+1)n}{2} \right\rfloor = nl + \frac{n}{2}$$

and

$$|A| = nl + \frac{n}{2}.$$

Thus the last vertex in M_2 is $\left(x, z_{\frac{n}{2}}\right)$ since $\sigma_1 = \sigma_2$, and B is an independent set.

If $m = 2p + 1$, then

$$\sigma_1 = nl + \frac{n-1}{2},$$

and

$$|A| - \sigma_1 = 1,$$

so the last vertex of M_1 is $(x_{n'-2}, z_n)$. Since

$$\sigma_2 = \sigma_1 = nl + \frac{n-1}{2},$$

and

$$\begin{aligned} &|B \cup \{(x_{n'}, z_n)\} - \{(x, z_n)\}| \\ &= ml + n = nl + n, \end{aligned}$$

the last vertex in M_2 is $(x_{n'-1}, y_s)$ or (x, z_q) for some $q \in [1, n-1]$.

Case 2. $m = n + 1$.

If $m = 2p$, then

$$\begin{aligned} \sigma_1 &= \left\lfloor \frac{(2l+1)(m+n)}{4} \right\rfloor \\ &= \left\lfloor \frac{(2l+1)(2n+1)}{4} \right\rfloor \\ &= \left\lfloor nl + \frac{n+1}{2} + \frac{2l-1}{4} \right\rfloor, \end{aligned}$$

and

$$|A| = nl + \frac{n+1}{2},$$

so $\sigma_1 \geq |A|+1$ and the last vertex of M_1 is (x_i, y_j) for $l \geq 3$, where $i \in \{1, 3, ..., n'-1\}$ and $j \in [s+1, m]$. For $l = 2$, $\sigma_1 = |A|$, then the last vertex of M_1 is $(x_{n'}, z_n)$.

If $m = 2p + 1$, then

$$\begin{aligned} \sigma_1 &= \left\lfloor \frac{(2l+1)(m+n)}{4} \right\rfloor \\ &= \left\lfloor \frac{(2l+1)(2n+1)}{4} \right\rfloor \\ &= \left\lfloor nl + \frac{n}{2} + \frac{2l+1}{4} \right\rfloor, \end{aligned}$$

and

$$|A| = nl + \frac{n}{2} + 1.$$

If $l \geq 4$, then

$$\sigma_1 \geq |A| + 1,$$

so the last vertex of M_1 is some (x_i, y_j), where $i \in \{1, 3, ..., n'-1\}$ and $j \in [s+1, m]$. If $l \in \{2, 3\}$, then $\sigma_1 = |A|$, so the last vertex of M_1 is $(x_{n'}, z_n)$.

If $n \geq 2$, similarly as Case 1, we can obtain that the last vertex of M_2 is $(x_{n'-1}, y_s)$ or (x, z_q) for some $q \in [1, n-1]$.

If $n = 1$, then $m = 2$,

$$|A \cup B - \{(x, z_1), ..., (x, z_n)\}| = 3l + 1,$$

and

$$\left\lfloor \frac{3l+1}{2} \right\rfloor \geq |A|,$$

so

$$A \cup B - \{(x, z_1), ..., (x, z_n)\}$$

can be equitably partitioned into two independent sets, i. e., the last vertex of M_2 is $(x_{n'-1}, y_s)$. Hence the claim holds. □

Now we deal with the problem according to the value of n.

(2.2.1) $n \geq 2$.

Since

$$\sigma_1 + \sigma_2 \geq l(m+n) + \frac{m}{2},$$

we have

$$\sigma_1 + \sigma_2 \geq l(m+n) + s,$$

and then

$$(u, v) = (x_{n'-1}, y_s)$$

or

$$(u, v) = (x, z_q)$$

for some $q \in [1, n-1]$. Thus, the cardinality of the greatest independent set consisting of consecutive vertices from the next vertex of (u, v) is

$$n + l(m-s) + ls = lm + n \geq \sigma_3$$

by

$$\sigma_3 \leq \frac{(2l+1)(m+n)+2}{4}.$$

Hence we can choose independent sets M_3, M_4 with cardinality σ_3, σ_4, respectively.

(2.2.2) $n = 1$.

Note that

$$1 \leq m \leq n + 1 = 2.$$

Since

$$\sigma_1 + \sigma_2 \leq l(m+n) + \frac{m+n}{2}$$
$$< l(m+n) + s + n,$$

we have

$$(u, v) = (x_{n'-1}, y_s),$$

and then the size of the greatest independent set consisting of consecutive vertices from the next vertex of $(x_{n'-1}, y_s)$ is

$$lm + 1 - s = 2l \geq 3l/2 + 1 = \sigma_3$$

for $m = 2$. So we can obtain the result as (2.2.1) similarly.

If $m = 1$, then the vertex partition

$$(x, y_1), (x_2, z_1), (x_4, z_1), \ldots, (x_{n'}, z_1);$$
$$(x_1, y_1), (x_3, y_1), \ldots, (x_{n'-1}, y_1);$$
$$(x, z_1), (x_2, y_1), (x_4, y_1), \ldots, (x_{n'-2}, y_1);$$
$$(x_{n'}, y_1), (x_1, z_1), (x_3, z_1), \ldots, (x_{n'-1}, z_1),$$

is an equitably 4-coloring of $W_{n'} \Box H$.

In a word, we have proved Theorem 2.

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