Hyperchaotic Dynamic of Cournot-Bertrand Duopoly Game with Multi-Product and Chaos Control

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Abstract: - In real market, firms usually produce multi-product for some reasons, such as cost-saving advantages, catering for the diversity of consumer tastes and providing a barrier to entry. In this paper, Cournot-Bertrand duopoly game with two-product is presented. According to the different statuses of the products, the bounded rationality is expanded as the ordinary bounded rationality and the cautious bounded rationality. The bifurcation diagrams, the Lyapunov exponents, the attractors and the initial condition sensitiveness are investigated by numerical simulations. The simulation results show that the economic model is a novel four-dimensional hyperchaotic system which may appear flip-hopf bifurcation and wave shape chaos. The system is more complex and uncertain than ordinary chaos system, and it is disadvantage to firm's decision. At last, the nonlinear feedback control method is applied to the system, and achieves good effect. The results of this paper provide reference to practical industrial structure of this type, and are of theoretical and practical significance.

Key–Words: - Cournot, Bertrand, multi-product, hyperchaos, bifurcation, complexity

1 Introduction

The research of oligopoly game starts from Cournot model and Bertrand model. The classic Cournot model [1] uses output quantity as decision variable and the Bertrand model [2] uses price as decision variable for competition. Since then, a large number of literatures have gotten many achievements in the dynamic game of Cournot and Bertrand models [3-11]. As the development of research, some scholars have begun to pay attention to Cournot-Bertrand mixed game model where one oligopoly optimally adjusts output and the other one optimally adjusts its price [12-17].

However, in above studies, few oligopolies offer multi-product for game. The Cournot-Bertrand mixed model is based on the assumption that oligopolies each offer one product and adopt one single strategic variable (output or price). In the real economic activity, oligopolies often produce multiple products for obtaining cost-saving advantages, catering for the diversity of consumer tastes and providing a barrier to entry. For example, each oligopoly has some core and non-core products when they compete on produce, they make decisions for each product, and this is called two-product game.

At present, only a considerable lower number of studies devoted to output or price games with multi-product. Panzar and Willing [18] studied the role of cost factors (economy of scope) where some firms produced several products in 1981. Shaked and Sutton [19] discussed the fragmented equilibrium and concentrated equilibrium of multiple products firms, and the relationship between market size and equilibrium outcomes. Tan [20] concluded the concept of multi-product game. Tan [21] also studied the multi-product game of Cournot-Bertrand model and its equilibrium. Xiang and Cao [22] studied the two-product static game of Cournot-Bertrand model with incomplete information, when the two companies have some old and new alternative products.

In addition, above studies are confined in the aim of profit maximization. However, in real market, this aim is not fixed. For examples: when ownership and management right are separate, managers are probably to manipulate the benefit for their personal purpose than just pure profit maximization; under intervention of government policy or a special strategy of firm during certain period, the aim of profit maximization also may be adjusted.

This paper focuses on the complex dynamics of Cournot-Bertrand mixed model with two-product. Based on the previous studies [19-22], we extended the static model of two-product (the core product and the non-core product) to dynamic game for research: the core product plays as lead position; the non-core product plays as auxiliary position. The complex behaviors are proved by the numerical simulation.

The main contributions of this paper can be listed as follows: Firstly, multi-product is introduced into Cournot-Bertrand dynamic model. Secondly, according to the different statuses of products, the bounded rationality and cautious bounded rationality are used in model. At last, a novel hyperchaotic economic system of duopoly game is proposed because of multi-product. The study in terms of hyperchaotic characteristics provides the basis for enterprises decision in Cournot-Bertrand mixed model with multi-product.

The paper is organized as follows: in Section 2, we set up an elementary background framework; the model assumption and nomenclature are also described. In Section 3, the model is built. And some numerical simulations prove the complex feature of the model. In Section 4, nonlinear feedback control method is used for stabilizing the hyperchaotic economic model. In Section 5, some conclusions are presented.

2 The background and the assumption

2.1 The background analysis

In real market, firm usually produces multiple products in the same time. For example, HP produces a variety of models of printers to meet different customer demands. A firm produces two or more products, we call it multi-product firm. In this paper, we assume that each of the two firms produces two products, core product and non-core product. The core product represents firm's core value and core competence, and plays a leading role in the market. Firms first consider the maximization of core product's profit. The marketing strategy of non-core product cannot affect the profit maximization of core product. In fact, firms carry out the policy of protecting the profit maximization of core product. Firms choose to cut down a portion of profit of non-core product to protect the core product for getting much more long-run profits. This strategy is usually temporary for winning the long-term superiority of the market competition.

2.2 The assumption and the nomenclature

Our model is based on following assumptions and nomenclatures:

(1) There are two firms (firm 1 and firm 2) in the market. They produce and sell two substitute products in oligopolistic positions. The non-core product competes in output quantity and the core product competes in price.

(2) The firm 1's output and price of core product are respectively q_{12} and p_{12} in period t , and the firm 1's output of non-core product is q_{11} in period t . The firm 2's output and price of core product are respectively q_{22} and p_{22} in period t , and the firm 2's output of non-core product is q_{21} in period t. The market price of non-core product is p_1.

d is the influence coefficient of the rival price to firm's customer needs. The influence coefficient of non-core product's price to firm 1 and firm 2 are respectively f_1 and f_2 . e is the product substitution parameter of the total market supply of core product to the market price of non-core product.

The output functions of core product in period t are respectively:

$$\begin{cases} q_{12} = a - p_{12} + dp_{22} + f_1 p_1, \\ q_{22} = b - p_{22} + dp_{12} + f_2 p_1, \\ a, b, d, f_1, f_2 > 0. \end{cases} \tag{1}$$

The price function of non-core product is:

$$\begin{aligned} &p_1 = m - n[(q_{11} + q_{21}) + e(q_{12} + q_{22})], \\ &m, n > 0, \end{aligned} \tag{2}$$

From Eqs. (1) and (2), we get Eq.(3):

$$\begin{aligned} p_1 = {} & (m - nq_{11} - nq_{21} - nea + nep_{12} - nedp_{22} \\ & - neb + nep_{22} - nedp_{12}) \frac{1}{1 + nef_2 + nef_1} \end{aligned} \tag{3}$$

(3) The firm 1's costs of non-core product and core product are respectively $c_{11}q_{11}$ and $c_{12}q_{12}$. The firm 2's costs of non-core product and core product are respectively $c_{21}q_{21}$ and $c_{22}q_{22}$. The profits of the two firms are respectively:

$$\begin{cases} \prod_1 = (p_1 - c_{11})q_{11} + (p_{12} - c_{12})q_{12} \\ \prod_2 = (p_1 - c_{21})q_{21} + (p_{22} - c_{22})q_{22} \end{cases} \tag{4}$$

That is,

$$\begin{aligned} &\prod_1 = [a - p_{12} + dp_{22} + (m - nq_{11} - nq_{21} - nea \\ &+ nep_{12} - nedp_{22} - neb + nep_{22} - nedp_{12}) \\ &\tfrac{f_1}{1+nef_2+nef_1}](p_{12} - c_{12}) + [(m - nq_{11} - nq_{21} \\ &- nea + nep_{12} - nedp_{22} - neb + nep_{22} - nedp_{12}) \\ &\tfrac{1}{1+nef_2+nef_1} - c_{11}]q_{11}; \end{aligned}$$

$$\prod_2 = [b - p_{22} + dp_{12} + (m - nq_{11} - nq_{21} - nea \\ +nep_{12} - nedp_{22} - neb + nep_{22} - nedp_{12}) \\ \frac{f_2}{1+nef_2+nef_1}](p_{22} - c_{22}) + [(m - nq_{11} - nq_{21} \\ -nea + nep_{12} - nedp_{22} - neb + nep_{22} - nedp_{12}) \\ \frac{1}{1+nef_2+nef_1} - c_{21}]q_{21}. \tag{5}$$

3 The model and numerical simulation

The core product is in the lead position and plays as a core role in the current market structure. The two firms take the basic strategy of core product protection.

3.1 The model

3.1.1 For the core product

The bounded rationality expectation which is based on the marginal profit is adopted in the decision of core product. The firms decide to increase (decrease) core product's price if they have positive (negative) marginal profits in the last period. The core product's marginal profits of firm 1 and firm 2 are respectively $\frac{\partial\Pi_1}{\partial p_{12}}$ and $\frac{\partial\Pi_2}{\partial p_{22}}$, the correlation equations are as follows:

$$\begin{cases} \frac{\partial\Pi_1}{\partial p_{12}} = \frac{(ne-ned)q_{11}}{1+nef_2+nef_1} + a - p_{12} + dp_{22} \\ +(p_{12} - c_{12})[-1 + \frac{f_1(ne-ned)}{1+nef_2+nef_1}] + (m \\ -nq_{11} - nq_{21} - nea + nep_{12} - nedp_{22} - neb \\ +nep_{22} - nedp_{12})\frac{f_1}{1+nef_2+nef_1} \\ \frac{\partial\Pi_2}{\partial p_{22}} = \frac{(ne-ned)q_{21}}{1+nef_2+nef_1} + b - p_{22} + dp_{12} \\ +(p_{22} - c_{22})[-1 + \frac{f_2(ne-ned)}{1+nef_2+nef_1}] + (m \\ -nq_{11} - nq_{21} - nea + nep_{12} - nedp_{22} - neb \\ +nep_{22} - nedp_{12})\frac{f_2}{1+nef_2+nef_1} \end{cases} \tag{6}$$

3.1.2 For the non-core product

The non-core product's marginal profits of firm 1 and firm 2 are respectively $\frac{\partial\Pi_1}{\partial q_{11}}$ and $\frac{\partial\Pi_2}{\partial q_{21}}$. They are as follows:

$$\frac{\partial\Pi_1}{\partial q_{11}} = \frac{-nq_{11}}{1+nef_2+nef_1} - c_{11} - \frac{f_1 n(p_{12} - c_{12})}{1+nef_2+nef_1} + \\ (m - nq_{11} - nq_{21} - nea + nep_{12} - nedp_{22} - neb \\ +nep_{22} - nedp_{12})\frac{1}{1+nef_2+nef_1}$$

$$\frac{\partial\Pi_2}{\partial q_{21}} = \frac{-nq_{21}}{1+nef_2+nef_1} - c_{21} - \frac{f_2 n(p_{22} - c_{22})}{1+nef_2+nef_1} + \\ (m - nq_{11} - nq_{21} - nea + nep_{12} - nedp_{22} - neb \\ +nep_{22} - nedp_{12})\frac{1}{1+nef_2+nef_1} \tag{7}$$

In economics, the second order mixed derivative is of important practical implication: It means the effect on a certain product's marginal profit exerted by the other product's change. For example, the effect on the core product's marginal profit exerted by changing non-core product's output in firm 1 and firm 2 are respectively and $\frac{\partial\Pi_1^2}{\partial p_{12}\partial q_{11}}$ and $\frac{\partial\Pi_2^2}{\partial p_{22}\partial q_{21}}$. The correlation equations are as follows:

$$\begin{cases} \frac{\partial\Pi_1^2}{\partial p_{12}\partial q_{11}} = (m - nq_{11} - nq_{21} - nea + nep_{12} \\ -nedp_{22} - neb + nep_{22} - nedp_{12})\frac{1}{1+nef_2+nef_1} \\ +\frac{-nq_{11}}{1+nef_2+nef_1} - c_{11} - \frac{nf_1(p_{12}-c_{12})}{1+nef_2+nef_1} \\ \frac{\partial\Pi_2^2}{\partial p_{22}\partial q_{21}} = (m - nq_{11} - nq_{21} - nea + nep_{12} \\ -nedp_{22} - neb + nep_{22} - nedp_{12})\frac{1}{1+nef_2+nef_1} \\ +\frac{-nq_{21}}{1+nef_2+nef_1} - c_{21} - \frac{nf_2(p_{22}-c_{22})}{1+nef_2+nef_1} \end{cases} \tag{8}$$

3.1.3 The dynamic equations of the model

The dynamic model is as follows:

$$\begin{cases} q'_{11} = q_{11} + \alpha q_{11}\frac{\partial\Pi_1}{\partial q_{11}}\frac{\partial\Pi_1^2}{\partial p_{12}\partial q_{11}} & (9.1) \\ p'_{12} = p_{12} + \beta p_{12}\frac{\partial\Pi_1}{\partial p_{12}} & (9.2) \\ q'_{21} = q_{21} + \varepsilon q_{21}\frac{\partial\Pi_2}{\partial q_{21}}\frac{\partial\Pi_2^2}{\partial p_{22}\partial q_{21}} & (9.3) \\ p'_{22} = p_{22} + \gamma p_{22}\frac{\partial\Pi_2}{\partial p_{22}} & (9.4) \end{cases} \tag{9}$$

where $'$ denotes the unite-time advancement of variable. In this phase, two firms adopt the tactics of protecting core product, so the price strategy dominates the quantity strategy. We will give further explanation to the meaning of system (9):

(1) For the core product, the firms adjust current prices according to their marginal profits of previous period, as Eqs. (9.2) and (9.4) described. If the marginal profits are positive (negative), they will increase (decrease) their prices in the next period. This is what we call the bounded rational expectation.

(2) For the non-core product, the firms are more cautious about their output adjustment because of non-core product's subordinate role, we called it "cautious bounded rationality" in this paper. The economic intuition behind the Eqs. (9.1) and (9.3) is that the cautious bounded rational firm increases (or decreases) its output according to not only the marginal profit of the last period but also the effect on core

product's marginal profit exerted by non-core product's output adjustment.

Further interpretation is done: Only when the non-core product's marginal profit is positive ($\frac{\partial \Pi_1}{\partial q_{11}} > 0$ for firm 1, $\frac{\partial \Pi_2}{\partial q_{21}} > 0$ for firm 2), and the effect on core product's marginal profit exerted by non-core product's output adjustment is also positive ($\frac{\partial \Pi_1^2}{\partial p_{12} \partial q_{11}} > 0$ for firm 1, $\frac{\partial \Pi_2^2}{\partial p_{22} \partial q_{21}} > 0$ for firm 2), the output of non-core product will be increased in the next period. If not, non-core product's output will be decreased in the next period. Namely, if $\frac{\partial \Pi_1}{\partial q_{11}} \frac{\partial \Pi_1^2}{\partial p_{12} \partial q_{11}}$ ($\frac{\partial \Pi_2}{\partial q_{21}} \frac{\partial \Pi_2^2}{\partial p_{22} \partial q_{21}}$ for firm 2) is positive (negative), the output will be increased (decreased) in the next period.

There is still one thing should be concerned: in the program made by MATLAB, the situation about " $\frac{\partial \Pi_1}{\partial q_{11}} < 0$ and $\frac{\partial \Pi_1^2}{\partial p_{12} \partial q_{11}} < 0$ ($\frac{\partial \Pi_2}{\partial q_{21}} < 0$ and $\frac{\partial \Pi_2^2}{\partial p_{22} \partial q_{21}} < 0$ for firm 2)" is ruled out by the programming.

The decision-making process is depicted in Figure 1.

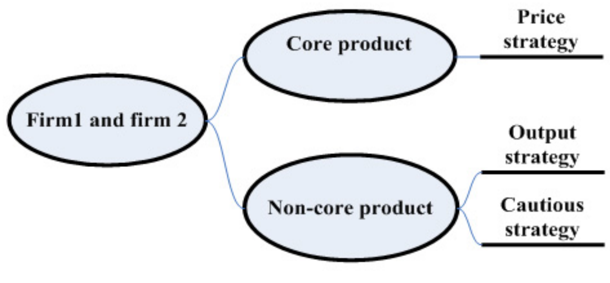

Figure 1: The graph of decision-making process.

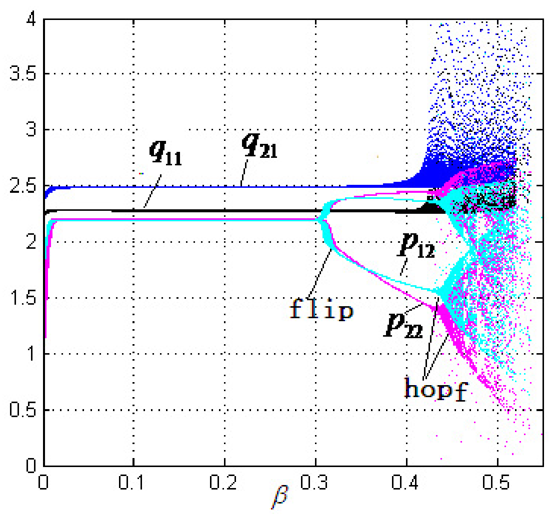

Figure 2: The flip-hopf bifurcation and wave shape chaos when $\alpha = 0.2, \varepsilon = 0.2, \gamma = 0.35$.

The expression of the dynamic system is as below:

$$\begin{cases} q'_{11} = q_{11} + \alpha q_{11}[\frac{-nq_{11}}{1+nef_2+nef_1} \\ -\frac{f_1 n(p_{12}-c_{12})}{1+nef_2+nef_1} + (m - nq_{11} - nq_{21} - nea \\ +nep_{12} - nedp_{22} - neb + nep_{22} - nedp_{12}) \\ \frac{1}{1+nef_2+nef_1} - c_{11}]^2 \\ p'_{12} = p_{12} + \beta p_{12}[\frac{(ne-ned)q_{11}}{1+nef_2+nef_1} \\ +(p_{12} - c_{12})(-1 + \frac{f_1(ne-ned)}{1+nef_2+nef_1}) + a - p_{12} \\ +dp_{22} + (m - nq_{11} - nq_{21} - nea + nep_{12} - \\ nedp_{22} - neb + nep_{22} - nedp_{12})\frac{f_1}{1+nef_2+nef_1}] \\ q'_{21} = q_{21} + \varepsilon q_{21}[\frac{-nq_{21}}{1+nef_2+nef_1} - \\ \frac{f_2 n(p_{22}-c_{22})}{1+nef_2+nef_1} + (m - nq_{11} - nq_{21} - nea \\ +nep_{12} - nedp_{22} - neb + nep_{22} - nedp_{12}) \\ \frac{1}{1+nef_2+nef_1} - c_{21}]^2 \\ p'_{22} = p_{22} + \gamma p_{22}[\frac{(ne-ned)q_{21}}{1+nef_2+nef_1} \\ +(p_{22} + c_{22})(-1 + \frac{f_2(ne-ned)}{1+nef_2+nef_1}) + b - p_{22} \\ +dp_{12} + (m - nq_{11} - nq_{21} - nea + nep_{12} - \\ nedp_{22} - neb + nep_{22} - nedp_{12})\frac{f_2}{1+nef_2+nef_1}] \end{cases} \quad (10)$$

3.2 The numerical simulation

The numerical simulation is an effective method to analyze high dimensional nonlinear dynamic system. We will evaluate the nonlinear dynamic characteristics of system (10) by bifurcation diagrams, Lyapunov exponents, phase portraits and time series diagrams. In general, for firms 1 and 2, the output and price adjustment factors are changeable, and all other factors are invariable, so these invariable factors are set values: $c_{11} = 0.5, c_{12} = 0.3, c_{21} = 0.4, c_{22} = 0.25, d = 0.8, f_1 = 0.1, a = 2, b = 2, e = 0.85$.

3.2.1 The bifurcation

The bifurcation diagram provides a nice summary for the transition between different motions that can occur as one parameter is varied.

We can see that with increasing price adjustment factor β, Figure 2 represents the dynamic progress of system (10) from stable state to unstable state. Firstly, the price of core product is in stable state, then experiences Flip bifurcation, then, enters into Hopf bifurcation in 2-period points. This is called flip-hopf bifurcation.

While, the non-core product's output exhibits special dynamic phenomenon: At first, the output is a horizontal line in stable state for each firm. With increasing β, the non-core product's output develops into the unstable state in which the straight line becomes many curves and these curves over lap each

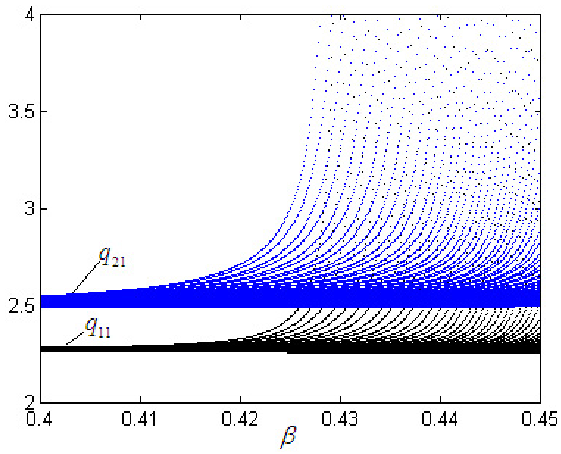

Figure 3: The enlarged wave shape chaos of outputs in Figure 2.

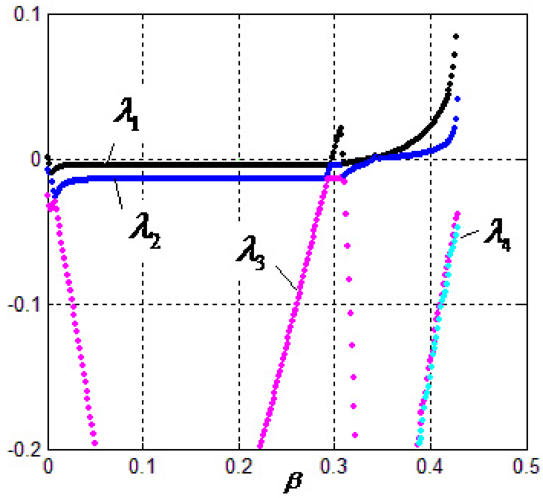

Figure 4: The Lyapunov exponents varying with β.

other. Namely, the output enters into complex fluctuation state. We call it wave shape chaos. This kind of chaos has appeared in nonlinear electrical field [23-25]. Figure 3 is the partial enlarged drawing of wave shape chaos in Figure 2.

3.2.2 The hyperchaotic characteristic

A hyperchaotic system is characterized as a chaotic system with at least two positive Lyapunov exponents, implying that its dynamic is expended in two or more different directions simultaneously. It means that hyperchaotic system has more complex dynamical behaviors. We assume that the Lyapunov exponents of system (10) are respectively $\lambda_1, \lambda_2, \lambda_3$ and λ_4. The stability of system (10) can be characterized with the Lyapunov exponents:

(1) For periodic orbits, $\lambda_1 = 0, \lambda_2, \lambda_3, \lambda_4 < 0$.

(2) For chaotic attractor, $\lambda_1 > 0, \lambda_2 = 0$, λ_3, $\lambda_4 < 0$.

(3) For hyperchaotic attractor, $\lambda_1, \lambda_2 > 0$.

The corresponding Lyapunov exponents are shown in Figure 4. The stability of the system is summarized as Table 1.

The Table 1 is in good agreement with Figure 4. For $0 < \beta \leq 0.2981$, the system is locally stable, i.e. in fixed point. With increasing β, the system becomes unstable and experience chaos ($0.2981 < \beta \leq 0.3096$), periodic orbits ($0.3096 < \beta \leq 0.3426$), chaos ($0.3426 < \beta \leq 0.3626$), at last enters into hyperchaos ($0.3626 < \beta \leq 0.43$).

3.2.3 The hyperchaotic attractor

Nowadays, more and more attentions have been cast on the research of novel chaotic or hyperchaotic attractor [26-28]. In this part, we will discuss the hyperchaotic attractor.

The hyperchaotic attractor is usually characterized as a chaotic attractor with two or more positive Lyapunov exponents [29]. The hyperchaotic attractor of system (10) with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.8943, 0.7691, -0.0005, -0.0154)$ is shown in Figure 5 from different angles. The attractor expands in two directions, and is more thicker than ordinary chaotic attractor [30]. In hyperchaotic state, the market is extremely complex and hard to predict. Because of the dramatical market fluctuates, it is unfavorable for the firm's plan and development.

For another, the existence of hyperchaotic attractor once again proves the existence of wave shape chaos.

The Lyapunov dimension of the attractor according to the Kaplan-Yorke conjecture [31] is defined as:

$$D = M + \frac{\sum\limits_{i=1}^{M} \lambda_i}{|\lambda_{M+1}|} > 4 \tag{11}$$

where M is the largest integer for which $\sum_{i=1}^{M} \lambda_i > 0$ and $\sum_{i=1}^{M+1} \lambda_i < 0$. D is larger than 4 , this means the system (10) exhibits a fractal structure and the occupied space is large and the structure is tight, which is shown as Figure 5.

3.2.4 The initial condition sensitiveness

The butterfly effect, namely initial condition sensitiveness is one of the important characteristics of chaos. A slight variation of initial condition will result in great difference of the ultimate system. This characteristic

Table 1: Stability of the system

β	λ_1	λ_2	λ_3	λ_4	System State
$0 < \beta \leq 0.2981$	-	-	-	-	Fixed Points
$0.2981 < \beta \leq 0.3096$	+	0	-	-	Chaotic
$0.3096 < \beta \leq 0.3426$	0	-	-	-	Periodic Orbits
$0.3426 < \beta \leq 0.3626$	+	0	-	-	Chaotic
$0.3626 < \beta \leq 0.43$	+	+	-	-	Hyperchaotic

is proved by Figure 6. The two initial values are (0.8, 0.8, 0.8) and (0.8+ Δx, 0.8, 0.8), Δx=0.001. The difference between them is very subtle. With the increase of game times, the difference becomes increasingly evident as shown in Figure 6.

4 Chaos control

According to above numerical simulations and analysis, we find that the dynamic model have more complex behaviors which are more disadvantageous for firms than the ordinary chaotic system. Chaos control is an important way to return system to stable state from chaotic. The nonlinear feedback method is simple but effective to control chaos and has been widely used in practice. The controlled model is as follows:

$$\begin{cases} q_{11}(t+k) = (1-\rho)A^k[q_{11}(t), p_{12}(t), q_{21}(t), \\ \qquad p_{22}(t)] + \rho q_{11}(t) \\ p_{12}(t+k) = (1-\rho)B^k[q_{11}(t), p_{12}(t), q_{21}(t), \\ \qquad p_{22}(t)] + \rho p_{12}(t) \\ q_{21}(t+k) = (1-\rho)C^k[q_{11}(t), p_{12}(t), q_{21}(t), \\ \qquad p_{22}(t)] + \rho p_{21}(t) \\ p_{22}(t+k) = (1-\rho)D^k[q_{11}(t), p_{12}(t), q_{21}(t), \\ \qquad p_{22}(t)] + \rho p_{22}(t) \end{cases} \tag{12}$$

where ρ is the control parameter. $k = 1$ means controlling Nash equilibrium point ($k = 2, 4$ mean controlling period 2, period 4 etc.). We make $k = 1$, the controlled model of system (10) is Eqs. (13).

$$\begin{cases} q'_{11} = (1-\rho)(q_{11} + \alpha q_{11}\frac{\partial \Pi_1}{\partial q_{11}}\frac{\partial \Pi_1^2}{\partial p_{12}\partial q_{11}}) + \rho q_{11} \\ p'_{12} = (1-\rho)(p_{12} + \beta p_{12}\frac{\partial \Pi_1}{\partial p_{12}}) + \rho p_{12} \\ q'_{21} = (1-\rho)(q_{21} + \varepsilon q_{21}\frac{\partial \Pi_2}{\partial q_{21}}\frac{\partial \Pi_2^2}{\partial p_{22}\partial q_{21}}) + \rho q_{21} \\ p'_{22} = (1-\rho)(p_{22} + \gamma p_{22}\frac{\partial \Pi_2}{\partial q_{22}}) + \rho p_{22} \end{cases} \tag{13}$$

From Figure 7 (a), we can see that the hyperchaotic model are controlled to the Nash equilibrium when $\rho > 0.26$. Comparing Figure 2 and Figure 7 (b) we find that the chaos is eliminated by nonlinear feedback control method with proper control parameter ($\rho = 0.6$). Experiments prove that the nonlinear feedback method is fit for controlling the hyperchaos in this paper.

5 Conclusion

In this paper, we present Cournot-Bertrand mixed duopoly game with two-product. The complex dynamic of the model is investigated by numerical simulations, such as bifurcation, Lyapunov exponents, hyperchaotic attractor and initial condition sensitiveness. At last, we stabilize the hyperchaotic system to Nash equilibrium with nonlinear feedback control method. Through the research, the results can be gained as follows:

(1) The experiment proved that Cournot-Bertrand duopoly game with core and non-core products in this paper is a hyperchaotic system in which flip-hopf bifurcation and wave shape chaos may occur. The market is more complex and unpredictable than ordinary chaotic system. It is more difficult for firms to adapt to the changes in drastic market.

(2) In this paper, the bounded rationality is extended as the bounded rationality and the cautious bounded rationality under the situation of multi-product.

(3) By nonlinear feedback method, chaos is eliminated, and the systems are controlled in equilibrium state.

The research has practical significance. We hope that the results of these investigations shed some light on multi-product duopoly dynamic game, which is still far from what has been achieved to date for real economic structure.

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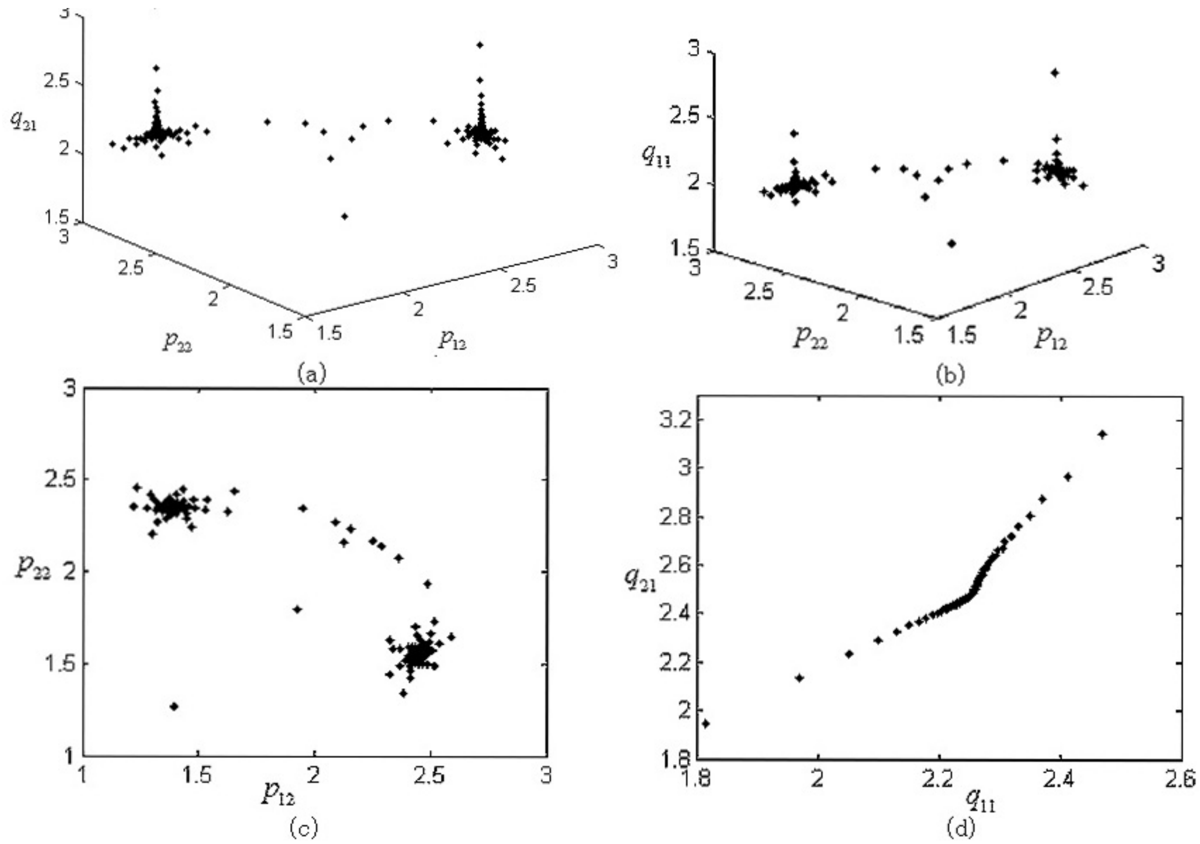

Figure 5: The hyperchaotic attractor with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.8943, 0.7691, -0.0005, 0.0154) and \beta = 0.4341$.

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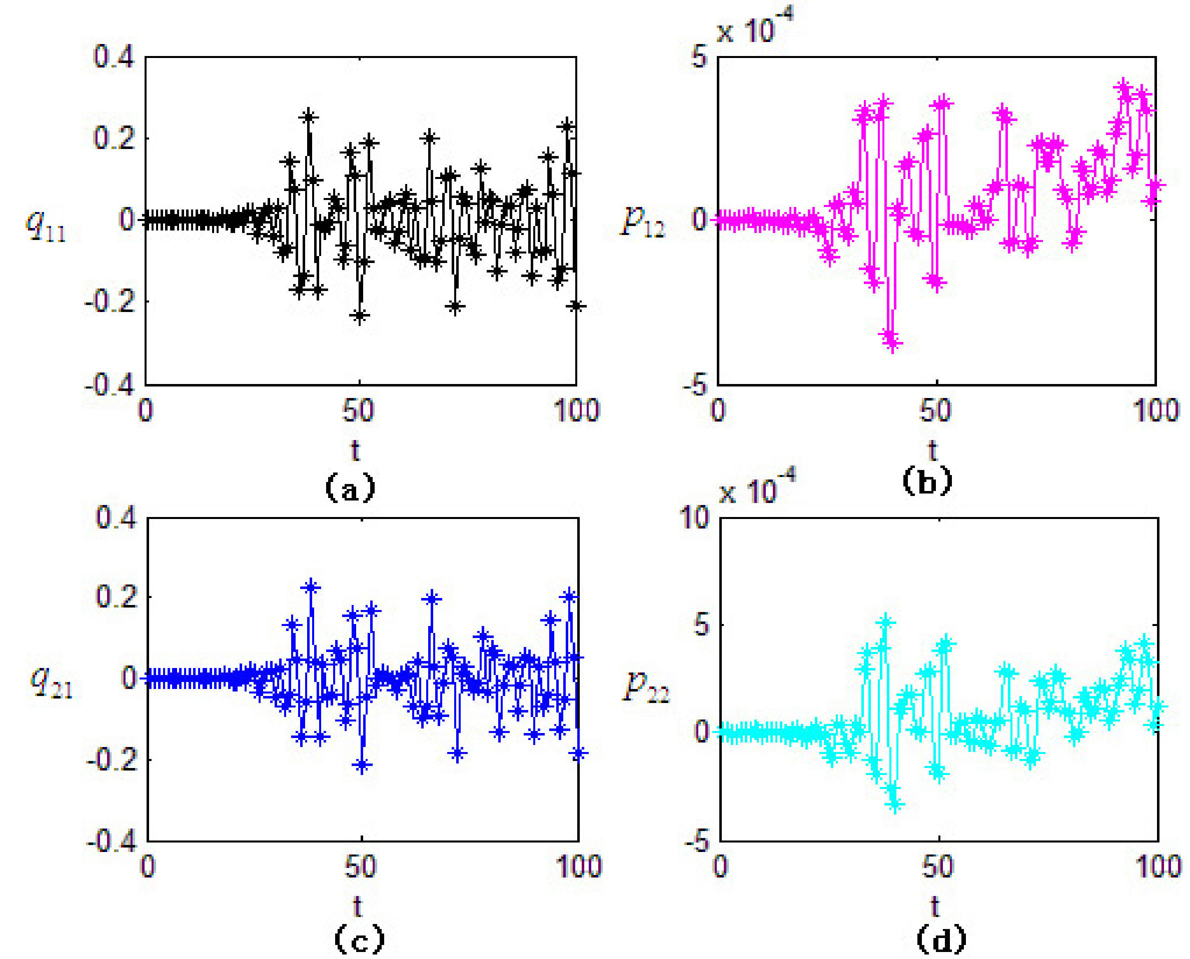

Figure 6: Sensitivity to initial conditions of (0.8, 0.8, 0.8)and (0.8+Δx,0.8,0.8).

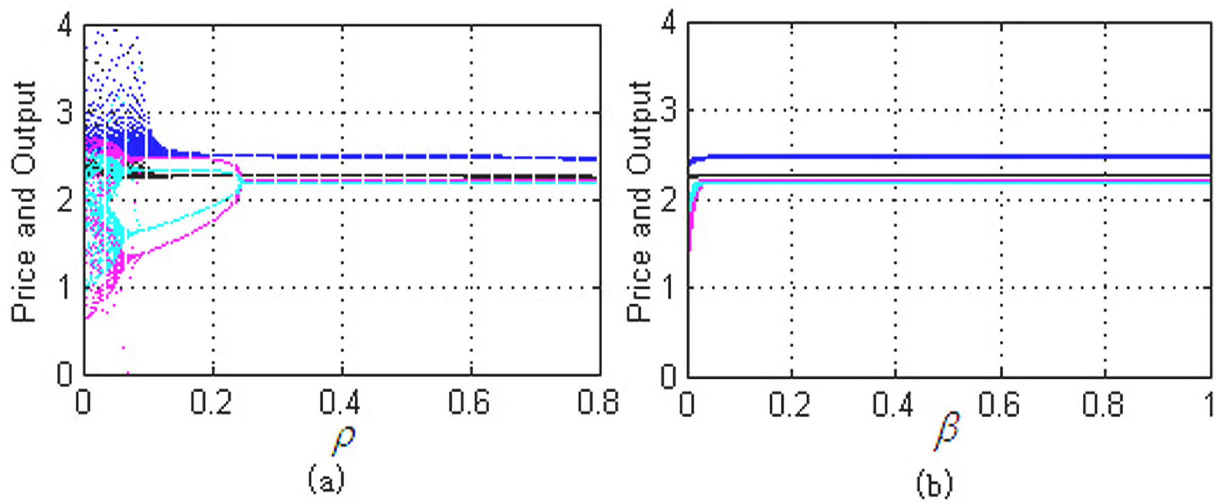

Figure 7: The bifurcation under control.

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