Existence and Nonexistence of Global Solution for a Reaction-Diffusion Equation with Exponential Nonlinearity

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Abstract: In this work we consider the existence and decay estimate and nonexistence of global solution of reaction-diffusion equation with nonlinear exponential growth reaction terms.

Key–Words: reaction-diffusion equation, stable set, exponential reaction term, existence, global nonexistence

1 Introduction

In this paper we consider the existence and decay estimate and nonexistence of global solutions for the following initial boundary value problem

$$u_t - \Delta u = g(u), \quad x \in \Omega, \ t > 0, \tag{1}$$
$$u(x,t) = 0, \ x \in \partial\Omega, \quad t > 0, \tag{2}$$
$$u(x,0) = u_0(x), \ x \in \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $u_0(x)$ is a given data and $g(s)$ is a reaction term with exponential growth like e^{ks^2} at the infinity.

There is a vast literature on global existence and nonexistence of solutions to the reaction-diffusion equation with polynomial growth reaction terms. Here we just mention a few of them, for example, see the works [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. These papers deal with the questions of global existence, asymptotic behavior, blow-up in time and so forth as well as a variety of methods used to study these question.

In this paper we will assume that $g(s)$ is a reaction term with nonlinear exponential growth like e^{ks^2} at the infinity. When $g(u) = e^u$, model (1)–(3) was proposed by [9] and [10]. In this case, Fujita [11] studied the asymptotic stability of the solution, Peral and Vazquez [12] and Pulkkinen [13] considered the stability and blow-up of the solution. Tello [14] and Ioku [15] considered the Cauchy problem of heat equation with $g(u) \approx e^{u^2}$ for $|u| \geq 1$.

Recently, Alves and Cavalcanti [16] concerned with the nonlinear damped wave equation with exponential source. They proved the global existence as well as blow up of solutions in finite time, by taking the initial data inside the potential well [17, 18]. Moreover, they also got the optimal and uniform decay rates of the energy for global solutions.

Motivated by the ideas of [16], we concentrate our studying on the existence and uniform decay estimate of the energy and finite time blow-up property of problem (1)-(3). As far as we know, this is the first work in the literature that take into account the reaction-diffusion equation with exponential growth reaction term by potential well theory. The majority of the works in the literature makes use of the potential well theory when f possesses polynomial growth, for instance, see the following works: [6, 18, 19, 20, 21] and a long list of references therein. The ingredients used in our proof are essentially the Trudinger-Moser inequality (see [22, 23]). We establish the existence of solution and decay rates of the energy. The case of nonexistence results is also treated, where a finite time blow-up phenomenon is exhibited for finite energy solutions, by considering similar arguments due to [5], adapted for our context.

The remainder of our paper is organized as follows: in Section 2 we present the main assumptions and the results, Section 3, Section 4 and Section 5 are devoted to the proof of the main results.

2 Preliminaries and main results

In this section, we present the main results and some material needed for the proof of our results. Throughout this paper, we denote by $||.||$, $||.||_p$ $||.||_{H_0^1}$ the usual norms in space $L^2(\Omega)$, $L^p(\Omega)$ and $H_0^1(\Omega)$, respectively, and in $L^2(\Omega)$, the inner product is defined as

$$(u,v)=\int_\Omega u(x)v(x)dx.$$

For the exponential reaction term, a typical example is the functions

$$g(s)=|s|^{p-2}se^{k|s|^\alpha}$$

for some $p>2$, $k\geq 0$ and $1<\alpha<2$. For more general exponential reaction term, we assume that $g:\mathbb{R}\to\mathbb{R}$ is a C^1 function, for each $\beta>0$, there exists a positive constant C_β such that

$$|g(t)|\leq C_\beta e^{\beta t^2},\quad |g'(t)|\leq C_\beta e^{\beta t^2}. \tag{4}$$

Furthermore, we assume that the function $g(t)/t$ is increasing in $(0,\infty)$, and

$$\lim_{t\to 0}\frac{g(t)}{t}=0.$$

For each $\varepsilon>0$, $\beta>0$, there exists a positive constant a which depends on ε, β verifying

$$|g(t)|\leq \varepsilon|t|+a|t|^{p-1}e^{\beta t^2}, \tag{5}$$

$$|G(t)|\leq \tfrac{\varepsilon}{2}|t|^2+a|t|^pe^{\beta t^2}, \tag{6}$$

where

$$G(t)=\int_0^t g(s)ds, p>2$$

and there exists a positive constant $\theta>2$ such that

$$0<\theta G(t)<g(t)t,\quad t\in\mathbb{R}\backslash\{0\}. \tag{7}$$

Throughout this paper we will make repeated use of the Trudinger-Moser inequality which can be founded in [22, 23].

Lemma 1 *[22, 23] Let Ω be a bounded domain in $\mathbb{R}^2$. Then, for all $u\in H_0^1(\Omega)$, we have*

$$e^{\alpha|u|^2}\in L^1(\Omega)\qquad \forall\,\alpha>0, \tag{8}$$

and there exist positive constants L such that, for all $\alpha\leq 4\pi$,

$$\sup_{u\in H_0^1(\Omega),||\nabla u||\leq 1}\int_\Omega e^{\alpha|u|^2}dx=L<\infty. \tag{9}$$

Now we define some functionals as follows

$$E(u)=\frac{1}{2}||\nabla u||^2-\int_\Omega G(u)dx, \tag{10}$$

$$I(u)=||\nabla u||^2-\int_\Omega ug(u)dx. \tag{11}$$

And also we define so called "potential well", "unstable set" and "potential depth", respectively, as follows [18, 17]

$$W_1=\{u\in H_0^1\mid E(u)<d, I(u)>0\}, \tag{12}$$

$$W_2=\{u\in H_0^1\mid E(u)<d, I(u)<0\}, \tag{13}$$

$$d=\inf\{\sup_{\lambda\in\mathbb{R}}E(\lambda u)\mid u\in H_0^1\backslash\{0\}\}. \tag{14}$$

It is obviously that $g(u)$ satisfies the hypotheses of the Mountain Pass Theorem [24], then $d>0$.

Our main results read as follows. The first results is concerned with the local existence and uniqueness of weak solutions to (1)-(3).

Theorem 2 *Let $g(s)$ satisfies the above assumption, $u_0\in H_0^1(\Omega)$. Then the problem (1)-(3) admits a uniqueness local weak solution u for some $T>0$ such that*

$$u\in C([0,T];H_0^1(\Omega)),\quad u_t\in L^2([0,T];L^2(\Omega)).$$

Furthermore, if

$$\sup_{0\leq t\leq T}||\nabla u(.,t)||^2<+\infty, \tag{15}$$

then $T=\infty$.

Theorem 3 *Under the assumption of Theorem 2, assume that $u_0\in W$, $E(u_0)<d$, then the local solutions of (1)-(3) obtained in Theorem 2 can be extended to $[0.\infty)$, $u(.)\in W$, and there exist positive constants C and k such that the energy $E(t)$ satisfies the decay estimates, for large t,*

$$E(t)\leq Ce^{-kt}. \tag{16}$$

Theorem 4 *Assume further that $u_0\in W_2$ and $E(0)<d$, then the solutions of problem (1)-(3) blows up in a finite time.*

We also need the following lemmas.

Lemma 5 *[25] Let $\phi(t)$ be a non-increasing and non-negative function on $[0,\infty)$, such that*

$$\sup_{s\in[t,t+1]}\phi(s)\leq C(\phi(t)-\phi(t+1)),\ \ t>0, \tag{17}$$

then $\phi(t)\leq Ce^{-\omega t}$, where C,ω are positive constants depending on $\phi(0)$ and other known qualities.

Lemma 6 *[5] Suppose that a positive, twice differentiable function $H(t)$ satisfied on $t\geq 0$ the inequality*

$$H''(t)H(t)-(\beta+1)(H'(t))^2\geq 0, \tag{18}$$

where $\beta>0$, then there is a $t_1<t_2=\frac{H(0)}{\beta H'(0)}$ such that $H(t)\to\infty$ as $t\to t_1$.

In order to prove our main results we remind that by the embedding theorem for $u \in H_0^1$ there exists a constant C_0 depending on p and Ω only such that

$$||u||_p \leq C_0|| \bigtriangledown u||.$$

By multiplying equation (1) by u_t, integrating over Ω, using integration by parts, we get

$$E'(t) = -\int_\Omega u_t^2(x,t)dx \leq 0. \tag{19}$$

3 Proof of Theorem 2

In this section, we are going to prove the existence and uniqueness of the local solution for the problem (1)-(3) by the contraction mapping principle. We divide the proof in some lemmas.

First, we make the necessary estimates of solutions for the linearized equation

$$u_t - \Delta u = f(x,t),\ x \in \Omega,\ t > 0, \tag{20}$$
$$u(x,t) = 0,\ \ x \in \partial\Omega, \quad t > 0, \tag{21}$$
$$u(x,0) = u_0(x),\ \ x \in \Omega, \tag{22}$$

Lemma 7 *[26] Let $u_0 \in H_0^1(\Omega)$ and $f \in L^2(Q_T)$, where $Q_T = \Omega \times [0,T]$. Then the problem (20)-(22) admits a uniqueness weak solution u such that*

$$u \in C([0,T]; H_0^1(\Omega)), \quad u_t \in L^2(Q_T),$$

and

$$\sup_{0 \leq t \leq T} || \bigtriangledown u||^2 + ||u_t||^2_{L^2(Q_T)} \leq C(T)||u_0||^2 + ||f||^2_{L^2(Q_T)}, \tag{23}$$

where $C(T)$ is a positive constant.

Now we define the function space

$$X_T = \{w \in C([0,T]; H_0^1(\Omega)), w_t \in L^2(Q_T), \; w(x,t) = 0, (x,t) \in \partial\Omega \times (0,T)\}$$

equipped with the norm defined by

$$||w||^2_{X_T} = \sup_{0 \leq t \leq T} || \bigtriangledown w||^2 + ||w_t||^2_{L^2(Q_T)}.$$

It is easy to see that X_T is a Banach space. Let $M = ||u_0||^2$ and define the set

$$P(M;T) = \{w \mid w \in X_T, ||w||_{X_T} \leq M\}.$$

Obviously, $P(M;T)$ is a nonempty bounded closed convex subset of X_T for each $M, T > 0$.

Lemma 8 *Let $w \in X_T, u_0 \in H_0^1(\Omega)$. Then the problem*

$$u_t - \Delta u = g(w), \quad x \in \Omega,\ t > 0, \tag{24}$$
$$u(x,t) = 0, x \in \partial\Omega, \quad t > 0, \tag{25}$$
$$u(x,0) = u_0(x),\ \ x \in \Omega, \tag{26}$$

has a uniqueness weak solution u such that

$$u \in C([0,T]; H_0^1(\Omega)), \quad u_t \in L^2(Q_T).$$

Proof: Noting that $w \in X_T$, by (5) and Holder inequality we have

$$\begin{aligned} & \int_0^T \int_\Omega |g(w)|^2 dxdt \\ \leq & \int_0^T \int_\Omega C_1[\varepsilon^2|w|^2 + a^2|w|^{2(p-1)}e^{2\beta w^2}]dxdt \\ \leq & \int_0^T C_1[\varepsilon^2||w||^2 + a^2||w||^{2(p-1)}_{4(p-1)}(\int_\Omega e^{4\beta w^2})^{\frac{1}{2}}]dt. \end{aligned} \tag{27}$$

where C_i and in the following are positive constants. Using Trudinger-Moser inequality (8) and embedding theorem, we obtain the following inequality

$$\begin{aligned} & \int_0^T \int_\Omega |g(w)|^2 dxdt \\ \leq & C_1 \int_0^T [\varepsilon^2 C_0^2|| \bigtriangledown w||^2 \\ & + a^2 C_0^{2(p-1)} C_2 || \bigtriangledown w||^{2(p-1)}]dt < \infty. \end{aligned} \tag{28}$$

We know $g(w) \in L^2(Q_T)$. Then the result follows from Lemma 7. □

We define the map S for $w \in X_T$ as follows

$$S(w) = u$$

where u is the solution of problem (24)-(26). Obviously, S maps X_T into X_T. Our goal is to show that S has a unique fixed point in $P(M;T)$.

Lemma 9 *Let $u_0 \in H_0^1(\Omega)$, then S maps $P(M;T)$ into $P(M;T)$ and $S : P(M;T) \to P(M;T)$ is strictly contractive if T is appropriately small relative to M.*

Proof: Since $w \in P(M;T) \subset X_T$, form (27), choosing $4\beta M^2 \leq 4\pi$, i.e., $\beta \leq \frac{\pi}{M^2}$, and using Trudinger-Moser inequality (9), we obtain

$$\begin{aligned} & \sup_{||\bigtriangledown w|| \leq M} \int_\Omega e^{4\beta w^2} dx \\ = & \sup_{||\bigtriangledown w|| \leq M} \int_\Omega e^{4\beta ||\bigtriangledown w||^2 (\frac{w}{||\bigtriangledown u||})^2} dx \\ \leq & \sup_{||\bigtriangledown w|| \leq M} \int_\Omega e^{4\beta M^2 (\frac{w}{||\bigtriangledown w||})^2} dx \leq L, \end{aligned} \tag{29}$$

Then, by (27), (29) and embedding theorem, we obtain the following inequality

$$\begin{aligned}&\int_0^T\int_\Omega |g(w)|^2dxdt\\ \le\ & C_1\int_0^T[\varepsilon^2C_0^2||\nabla w||^2\\ &\quad +a^2C_0^{2(p-1)}L||\nabla w||^{2(p-1)}]dt\\ \le\ & C_1(\varepsilon^2C_0^2M^2+a^2C_0^{2(p-1)}LM^{2(p-1)})T\\ <\ & \infty. \qquad (30)\end{aligned}$$

By Lemma 7 and (30), we deduce

$$\begin{aligned}&||u||^2_{X_T}=\sup_{0\le t\le T}||\nabla u||^2+||u_t||^2_{L^2(Q_T)}\\ \le\ & C(T)||u_0||^2 \qquad (31)\\ & +C(T)C_1(\varepsilon^2C_0^2+a^2C_0^{2(p-1)}LM^{2(p-2)})M^2T.\end{aligned}$$

If M and T satisfy

$$\begin{aligned}&M^2\ge 2C(1)||u_0||^2,\\ &T\le\min\{1,\frac{1}{K}\}, \qquad (32)\\ &K=2C(1)C_1(\varepsilon^2C_0^2+a^2C_0^{2(p-1)}LM^{2(p-2)})\end{aligned}$$

then by (32), we know

$$\sup_{0\le t\le T}||\nabla u||^2+||u_t||^2_{L^2(Q_T)}\le M^2.$$

Therefore, if (32) holds, S maps $P(M;T)$ into $P(M;T)$.

Now we are going to prove that the map S is strictly contractive. Let $w_1,w_2\in P(M;T)$ be given. Set $u_1=Sw_1, u_2=Sw_2, u=u_1-u_2, w=w_1-w_2$. Then u satisfies

$$u_t-\Delta u=g(w_1)-g(w_2), x\in\Omega, t>0, \qquad (33)$$
$$u(x,t)=0, x\in\partial\Omega,\quad t>0, \qquad (34)$$
$$u(x,0)=0,\ \ x\in\Omega, \qquad (35)$$

By (4), for each $\beta>0$, there exists a constant C_3 such that

$$|g(w_1)-g(w_2)|\le C_3(e^{\beta w_1^2}+e^{\beta w_2^2})|w|. \qquad (36)$$

Hence

$$\begin{aligned}&||g(w_1)-g(w_2)||^2_{L^2(Q_T)}\\ =\ & \int_0^T\int_\Omega|g(w_1(t))-g(w_2(t))|^2dxdt\\ \le\ & C_4\int_0^T\int_\Omega(e^{2\beta w_1^2}+e^{2\beta w_2^2})|w(t)|^2dxdt\\ \le\ & C_4\int_0^T(\int_\Omega|w(t)|^4dx)^{\frac12}[(\int_\Omega(e^{4\beta w_1^2}dx))^{\frac12}\\ & +(\int_\Omega(e^{4\beta w_2^2}dx))^{\frac12}]dt\\ \le\ & C_5\int_0^T||w(t)||_4^2dt\\ \le\ & C_6\int_0^T||\nabla w(t)||^2dt\\ \le\ & C_7T\sup_{0\le t\le T}||\nabla w(t)||^2. \qquad (37)\end{aligned}$$

From Lemma 1, (37), we arrive at

$$\begin{aligned}&||u||^2_{X_T}=C(T)||g(w_1)-g(w_2)||^2_{L^2(Q_T)}\\ &\le C(T)C_7T\sup\nolimits_{0\le t\le T}||\nabla w(t)||^2. \qquad (38)\end{aligned}$$

If T satisfies

$$T\le\min\left\{1, C(1)C_7, \frac{1}{K}\right\}, \qquad (39)$$

where K is defined as in (32), then by (38), we get

$$\begin{aligned}&\sup_{0\le t\le T}||\nabla u||^2+||u_t||^2_{L^2(Q_T)}\\ \le\ & \frac12\sup_{0\le t\le T}||\nabla w||^2+||w_t||^2_{L^2(Q_T)}.\end{aligned}$$

The lemma is proved. □

Proof of Theorem 2: It follows from Lemma 9 and the contraction mapping principle that for appropriately chosen $T>0$, S has a unique fixed point $u(x,t)\in P(M;T)$ which is a weak solution of the problem (1)-(3).

Suppose that equation (15) holds and $T<+\infty$. For any $T'\in[0;T)$, we consider the following problem

$$v_t-\Delta v=g(v),\quad x\in\Omega, t>0, \qquad (40)$$
$$v(x,t)=0, x\in\partial\Omega,\quad t>0, \qquad (41)$$
$$v(x,T')=u(x,T'),\ \ x\in\Omega, \qquad (42)$$

By virtue of (15), $||\nabla u||^2$ is uniformly bounded in $T'\in[0,T)$, which allows us to choose $T''\in(0,T)$ such that for each $T'\in(0,T)$ the problem (40)-(42)

has a unique solution $v(x,t) \in X_{T''}$. The existence of such a T'' follows from Lemma 9 and the contraction mapping principle. In particular, (39) reveals that T'' can be selected independently of $T' \in [0,T)$. Set $T' = T - T''/2$, let v denote the corresponding solution of (40)-(42), and define $\widehat{u}(x,t) : \Omega \times [0, T + T''/2] \to \mathbb{R}$ by

$$\widehat{u}(x,t) = u(x,t), \text{ if } t \in [0,T'];$$
$$v(x, t - T'), \text{ if } t \in [T', T + T''/2].$$

By construction, $\widehat{u}(x,t)$ is a solution of (40)-(42) on $[0, T + T''/2)$, and by local uniqueness, $\widehat{u}$ extends u. This violates the maximality to $[0,T)$. Hence, if (15) holds, then $T = \infty$. This completes the proof. □

4 Proof of Theorem 3

In this section our goal is to prove Theorem 3. To this end, we begin this section by a result similar to [16].

Lemma 10 *Under the assumptions of Theorem 3, i.e. assume $u_0 \in W_1$, $E(u_0) < d$, then we have, for all $t \in [0, T_{max})$,*

$$u(t) \in W_1, \tag{43}$$

$$||\nabla u||^2 \le \frac{2\theta d}{\theta - 2}, \tag{44}$$

$$E(u(t)) \ge (\frac{1}{2} - \frac{1}{\theta})||\nabla u||^2. \tag{45}$$

Proof: Since $E(u(t))$ is decreasing by (19), then we have

$$E(u(t)) \le E(u_0) \le d, \tag{46}$$

which implies that $u(t) \in W_1$ for all $t \in [0, T_{max}]$ as in [16] arguing by contradiction. Then by (43) and (46), we have

$$\int_\Omega (\frac{1}{2} g(u)u - G(u))dx \le d,$$

which together with (7) implies

$$\int_\Omega G(u)dx \le \frac{2d}{\theta - 2}. \tag{47}$$

Then (44) follows from (46) and (47). Next, we prove (45). Since $u_0 \in W_1$, from (43) we have $u \in W_1$ for all $t \in [0, T_{max})$. If $u = 0$, we easily get (45). If $I(u) > 0$, using (7), we have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}||\bigtriangledown u||^2 - \int_\Omega G(u)dx \\ &\ge \frac{1}{2}||\bigtriangledown u||^2 - \frac{1}{\theta}\int_\Omega ug(u)dx \\ &\ge (\frac{1}{2} - \frac{1}{\theta})||\bigtriangledown u||^2, \end{aligned}$$

which implies (45). Then the proof of Lemma 10 is complete. □

Proof of Theorem 3: To prove solution is global, it suffices to show that $||\nabla u||^2$ is bounded independent of t. It follows from Theorem 2 and (44) that we can extend the solution to problem (1)- (3) obtained in Theorem 2 to the whole interval $[0, \infty)$.

Next, we prove the decay estimates of the energy. From (19), we have for any $t > 0$

$$\int_t^{t+1} ||u_t||^2 ds = E(t) - E(t+1) = D^2(t). \tag{48}$$

Multiplying the equation (1) by u, integrating over $\Omega \times [t, t+1]$ and using integration by parts, we get

$$\begin{aligned} &\int_t^{t+1} (||\nabla u||^2 - \int_\Omega ug(u)dx)ds \\ = &\int_t^{t+1} (u, u_t)ds \le \int_t^{t+1} ||u|| ||u_t|| ds. \end{aligned} \tag{49}$$

Then we have

$$\begin{aligned} &2\int_t^{t+1} E(s)ds \\ = &\int_t^{t+1} (||\bigtriangledown u||^2 - \int_\Omega G(u)dx)ds \\ = &\int_t^{t+1} (||\bigtriangledown u||^2 - \int_\Omega ug(u)dx)ds \\ &+ \int_t^{t+1}\int_\Omega [ug(u) - 2G(u)]dxds. \end{aligned} \tag{50}$$

By (49), Young inequality, (45) and (48), we have for any $\eta > 0$

$$\begin{aligned} &\int_t^{t+1} (||\bigtriangledown u||^2 - \int_\Omega ug(u)dx)ds \\ \le &\int_t^{t+1} ||u|| ||u_t|| ds \\ \le &\int_t^{t+1} C_0||\bigtriangledown u|| ||u_t|| ds \\ \le &\int_t^{t+1} C_0\sqrt{\frac{2\theta}{\theta - 2}E(s)}||u_t|| ds \\ \le &\eta\int_t^{t+1} E(s)ds + \frac{2C_0^2\theta}{(\theta - 2)\eta}D^2(t). \end{aligned} \tag{51}$$

Using (7), (4) and Poincare inequality, we get

$$\begin{aligned}
&\int_t^{t+1}\int_\Omega [ug(u)-2G(u)]dxds\\
\leq\ & (1+\frac{2}{\theta})\int_t^{t+1}\int_\Omega ug(u)dxds\\
\leq\ & (1+\frac{2}{\theta}\int_t^{t+1}\int_\Omega [\varepsilon|u|^2+a|u|^p e^{\beta u^2}]dxds\\
\leq\ & (1+\frac{2}{\theta})\int_t^{t+1}[\varepsilon||u||^2\\
&\qquad +a||u||_{2p}^{2p}(\int_\Omega e^{2\beta u^2})^{\frac{1}{2}}dx]ds\\
\leq\ & (1+\frac{2}{\theta})\int_t^{t+1}[\varepsilon C_0^2||\bigtriangledown u||^2\\
&+aC_0^{2p}(\frac{2\theta d}{\theta-2})^{2(p-1)}||u||^2\times\\
&\left(\int_\Omega e^{2\beta\frac{2\theta d}{\theta-2}(\frac{u}{||\nabla u||^2})}\right)^{\frac{1}{2}}]ds. \qquad (52)
\end{aligned}$$

Choosing β such that $\frac{4\theta\beta d}{\theta-2} \leq 4\pi$, and using Trudinger-Moser inequality (9), we obtain

$$\begin{aligned}
&\int_t^{t+1}\int_\Omega [ug(u)-2G(u)]dxds\\
\leq\ & (1+\frac{2}{\theta})\int_t^{t+1}[\varepsilon C_0^2\frac{2\theta}{\theta-2}||\bigtriangledown u||^2\\
&+aC_0^{2p}(\frac{2\theta d}{\theta-2})^{2(p-1)}L||u||^2]ds\\
=\ & 2\varepsilon C_0^2\frac{\theta-2}{\theta+2}\int_t^{t+1}||\bigtriangledown u||^2ds\\
&+aC_0^{2p}(1+\frac{2}{\theta})(\frac{2\theta d}{\theta-2})^{2(p-1)}L\int_t^{t+1}||u||^2ds\\
\leq\ & \varepsilon C_8\int_t^{t+1}||\bigtriangledown u||^2ds+C_9\int_t^{t+1}||u||^2ds. \qquad (53)
\end{aligned}$$

Combining (45), (53), (51) with (50),we have

$$\begin{aligned}
&\int_t^{t+1}E(s)ds\\
\leq\ & (\varepsilon C_{10}+\eta C_{11})\int_t^{t+1}E(s)ds\\
&+C_9\int_t^{t+1}||u||^2ds++C_{12}D^2(t). \qquad (54)
\end{aligned}$$

Taking ε, η suitably small, we obtain

$$\int_t^{t+1}E(s)ds \leq C_{12}D^2(t)+C_{13}\int_t^{t+1}||u||^2ds. \qquad (55)$$

In order to estimate the last term on the RHS of (55), we make use of the inequality for all lager T

$$\int_0^T ||u||^2ds \leq C(T,E(0))\int_0^T ||u_t||^2ds. \qquad (56)$$

to be prove later (see Lemma 11). It follows from (55), (48) and (56) that

$$\int_t^{t+1}E(s)ds \leq C_{14}D^2(t). \qquad (57)$$

Noting that $E(t)$ is non-increasing and (48), we have

$$E(t)=E(t+1)+D^2(t) \leq C_{15}\int_t^{t+1}E(s)ds+D^2(t) \qquad (58)$$

and from (57) we have

$$E(t) \leq C_{16}D^2(t).$$

Since $E(t)$ is nonincreasing, using Nakao's theorem (Lemma 5), we conclude that there exist two positive constants C and k such that

$$E(t) \leq Ce^{-kt}$$

for t suitably large. Then the exponential decay of the energy is obtained. The proof of Theorem 3 is complete. □

Now let us obtain inequality (56), the method is essentially from Lemma 3.3 in [16].

Lemma 11 *Assume that the assumptions of Theorem 3 hold. Then, for all $T > T_0$, there exists a positive constant $C(T_0, E(0))$ such that the weak solution u of (1)-(3) satisfies (56).*

Proof: We argue by contradiction. Let us suppose that (56) is not verified and let $u_k(0)$ be a sequence of initial data and $u_k(0) \to u_0$ strongly in H_0^1 where the corresponding solutions $\{u_k\}_{k\in\mathbb{N}}$ of (1)- (3) with initial energy $E_k(0)$, verifies

$$\lim_{k\to\infty}\frac{\int_0^T ||u_k||^2ds}{\int_0^T ||u_{kt}||^2ds}=+\infty, \qquad (59)$$

that is

$$\lim_{k\to\infty}\frac{\int_0^T ||u_{kt}||^2ds}{\int_0^T ||u_k||^2ds}=0. \qquad (60)$$

Since the initial data are taken satisfying the Assumptions of Theorem 3, then, $E_k(0) < d$ for all $k \in \mathbb{N}$. Since $E_k(t) \leq E_k(0) < d$, for all $k \in \mathbb{N}$, we obtain a subsequence, still denoted by $\{u_k\}$ from now on, which verifies the convergence:

$$u_k \to u \text{ weakly in } H^1(Q_T), \tag{61}$$
$$u_k \to u \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \tag{62}$$
$$u_{kt} \to u_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{63}$$

Employing compactness results we also deduce that

$$u_k \to u \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{64}$$

which implies, from the continuity of g, that

$$g(u_k) \to g(u) \quad a.e.\ Q_T$$

Making use of (44) and Trudinger-Moser inequality, similar to [16] we deduce that for fixed $p > 1$

$$g(u_k) \to g(u) \text{ weakly in } L^p(Q_T).$$

We also observe that from (60) and (62)

$$\lim_{k\to\infty} \int_0^T ||u_{kt}||^2 ds = 0. \tag{65}$$

We will divide the proof into two cases, namely, $u \neq 0$ or $u = 0$.

Since u_k is a sequence of solutions to problem (1) -(3) it satisfies

$$u_{kt} - \Delta u_k = g(u_k), \quad x \in \Omega, t > 0, \tag{66}$$
$$u(x,t) = 0, x \in \partial\Omega, \quad t > 0, \tag{67}$$

If $u \neq 0$, then from the above convergence, passing to the limit in (66) we deduce

$$-\Delta u = g(u), \quad x \in \Omega, t > 0,$$
$$u_t(x,t) = 0, x \in \Omega, \quad t > 0,$$
$$u(x,t) = 0, x \in \partial\Omega, \quad t > 0.$$

From the above problem we deduce, for all $t \in [0,T]$, that

$$|| \bigtriangledown u||^2 - \int_\Omega ug(u)dx = I(t) = 0,$$

which is a contradiction since $u(t) \in W_1/\{0\}$, that is $I(t) = I(u(t)) > 0$.

Now, we define

$$c_k = [\int_0^T ||u_k||^2 dt]^{\frac{1}{2}}, \tag{68}$$
$$\overline{u}_k = \frac{u_k}{c_k}, \tag{69}$$
$$\overline{E}_k(t) = \frac{1}{2}||\overline{u}_{tk}||^2 - \frac{1}{2}|| \bigtriangledown \overline{u}_k||^2. \tag{70}$$

If $u = 0$, then from (64) we deduce that

$$c_k \to 0 \quad as \quad k \to +\infty,$$
$$\int_0^T ||\overline{u}_k||^2 dt = 1, \tag{71}$$
$$\overline{E}_k(t) \le \frac{E_k(t)}{c_k^2}. \tag{72}$$

Recalling (55) and (19) similar to its proof, we obtain, for some $M_0,\ T > M_0$

$$E(T) \le M_0[\int_0^T ||u_t||^2 ds + \int_0^T ||u||^2 ds]$$

where M_0 depends on θ, d, and employing the integral of (19), we can write

$$E(T) \le E(0) \le M_1[\int_0^T ||u_t||^2 ds + \int_0^T ||u||^2 ds] \tag{73}$$

for all $t \in (0,T)$, with T large enough. (72) and (73) give us

$$\overline{E}_k(t) \le \frac{E_k(t)}{c_k^2} \le M_1[\frac{\int_0^T ||u_{kt}||^2 ds}{\int_0^T ||u_k||^2 ds} + 1]. \tag{74}$$

From (60) and (74) we conclude that there exists a positive constant M such that for all $t \in [0,T]$ and for all $k \in \mathbb{N}$,

$$\overline{E}_k(t) \le \frac{E_k(t)}{c_k^2} \le M,$$

that is, for all $t \in [0,T]$ and for all $k \in N$,

$$\frac{1}{2}||\overline{u}_{tk}||^2 - \frac{1}{2}|| \bigtriangledown \overline{u}_k||^2 \le M. \tag{75}$$

For a subsequence $\{\overline{u}_k\}$ we obtain

$$\overline{u}_k \to \overline{u} \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \tag{76}$$
$$\overline{u}_k \to \overline{u} \text{ strongly in } L^\infty(0,T;L^2(\Omega)), \tag{77}$$
$$\overline{u}_{kt} \to \overline{u}_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \tag{78}$$

We observe that from (60) we deduce

$$\lim_{k\to\infty} \int_0^T ||\overline{u}_{kt}||^2 ds = 0. \tag{79}$$

In addition $\overline{u}_k$ satisfies the equation

$$\overline{u}_{kt} - \Delta\overline{u}_k = \frac{g(u_k)}{c_k}, \quad x \in \Omega, t > 0, \tag{80}$$
$$\overline{u}(x,t) = 0, x \in \partial\Omega, \quad t > 0, \tag{81}$$

Similar the proof of (3.66) in [16], we have

$$\lim_{k\to\infty}\int_0^T\int_\Omega \frac{g(u_k)}{c_k}dxds = 0. \qquad (82)$$

Then, from the above convergence we can pass to the limit in (80), when k goes to infinity to obtain

$$-\Delta\overline{u}_k = 0, \quad x\in\Omega; \overline{u}(x,t) = 0, x\in\partial\Omega \qquad (83)$$

showing that $u = 0$ which is a contradiction with (71). So, the proof is completed. □

5 Proof of Theorem 4

In this section, we shall prove Theorem 4 by adapting the concavity method.

Lemma 12 *[16] Assume that $u_0 \in W_2$ and $E(0) < d$, then it holds that*

$$u(t) \in W_2, \ \ for \ \ t\in[0, T_{max}), \qquad (84)$$
$$||\bigtriangledown u||^2 \geq 2d, \ \ for \ \ t\in[0,T_{max}). \qquad (85)$$

Proof of Theorem 4: Assume by contradiction that the solution is global. Then for any $T > 0$ we consider the function $F(t) : [0, T] \to R^+$ defined by

$$F(t) = \int_0^t ||u||^2 ds + (T-t)||u_0||^2 + \rho(t+\beta)^2, (86)$$

where β, T, ρ are positive constants which will be fixed later(see Levine[5]). Direct computations show that

$$\begin{aligned} F'(t) &= ||u||^2 - ||u_0||^2 + 2\rho(t+\beta), \\ &= 2\int_0^t (u(s), u_t(s))ds + 2\rho(t+\beta), (87) \end{aligned}$$

from (1) and integration by parts. Therefore, due to equation (1), (7) and (85)

$$\begin{aligned} F''(t) &= -2||\bigtriangledown u||^2 + 2\int_\Omega ug(u)dx + 2\rho \\ &\geq -2||\bigtriangledown u||^2 + 2\theta\int_\Omega G(u)dx + 2\rho \\ &= (\theta-2)||\bigtriangledown u||^2 - 2\theta E(t) + 2\rho \\ &= (\theta-2)||\bigtriangledown u||^2 - 2\theta E(0) \\ &\quad +2\theta\int_0^t\int_\Omega u_t^2(x,s)dxds + 2\rho \\ &\geq 2(\theta-2)d - 2\theta E(0) \\ &\quad +2\theta\int_0^t\int_\Omega u_t^2(x,s)dxds + 2\rho. \quad (88) \end{aligned}$$

Now let $2\theta\rho = 2(\theta-2)d - 2\theta E(0) > 0$. Then

$$F''(t) \geq 2\theta\rho + 2\theta\int_0^t\int_\Omega u_t^2(x,s)dxds. \qquad (89)$$

We also note that

$$\begin{aligned} &F(0) = T||u_0||^2 + \rho\beta^2 > 0, \\ &F'(0) = 2\rho\beta > 0, \\ &F''(t) \geq 2\theta\rho > 0, \ t\geq 0. \end{aligned}$$

Therefore $F(t)$ and $F'(t)$ are both positive. It is clearly that

$$F(t) \geq \int_0^t ||u||^2 ds + \rho(t+\beta)^2. \qquad (90)$$

Thus for all $(\xi,\eta)\in R^2$, from (86)- (90) follows

$$\begin{aligned} & F(t)\xi^2 + F'(t)\xi\eta + \frac{1}{2\theta}F''(t)\eta^2 \\ \geq\ & (\int_0^t ||u||^2 ds + \rho(t+\beta)^2)\xi^2 \\ & +2\xi\eta\int_0^t (u,u_t)ds + 2\rho(t+\beta)\xi\eta \\ & +\rho\eta^2 + \eta^2\int_0^t ||u_t||^2 ds \geq 0, \end{aligned}$$

which implies $(F'(t))^2 - \frac{1}{\theta}F(t)F''(t) \leq 0$. That is

$$F(t)F''(t) - \theta(F'(t))^2 \geq 0.$$

Then we complete the proof by standard concavity method(Lemma 6). □

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References:

[1] P. Quittner, Ph. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States,* Birkhauser Advanced Texts, 2007.

[2] J. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford,* 25, 1977, pp. 473-486.

[3] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations,* Plenum Press, New York, 1992.

[4] M. Fila, Boundedness of global solutions of nonlinear diffusion equations, *J. Diff. Equ.*, 98, 1992, pp. 226–240.

[5] H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$, *Arch. Rat. Mech. Anal.*, 51, 1973, pp. 371-386.

[6] M. Tsutsumi, Existence and nonexistence of global solutions for nonlinear parabolic equations, *Publ. Res. Inst. Math. Sci.*, 8, 1972, pp. 211-229.

[7] Z. Tan, The reaction-diffusion equation with Lewis function and critical Sobolev exponent. *J. Math. Anal. Appl.*, 272, 2002, pp. 480-495.

[8] L. E. Payne, P. W. Schaefer, Lower bounds for blow-up time in parabolic problems under Dirichlet conditions, *J. Math. Anal. Appl.*, 328, 2007, pp. 1196-1205.

[9] D. A. Frank-Kamenetskii, *Diffusion and heat transfer in chemical kinetics*, Plenum Press, New York, 1969.

[10] I. M. Gel'fand, Some problems in the theory of quasilinear equations, *Uspehi Mat. Nauk.* 14,1959, no.2, pp. 87–158; *English transi. Amer. Math. Soc. Transi.*, 29, 1963, pp. 295-381.

[11] H. Fujuita, On the nonlinear equations $\Delta u + e^u = 0$ and $\partial v/\partial t = \Delta v + e^v$, *Bulltein of the Amer. Math. Society*, 75, 1969, pp. 132-135.

[12] I. Peral and J. L. Vazquez, On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term, *Arch. Rational Mech. Anal.*, 129, 1995, pp. 201-224.

[13] A. Pulkkinen, Blow-up profiles of solutions for the exponential reaction-diffusion equation, *Mathematical Meth. Appl. Sci.*, 34, 2011, pp. 20111-2030.

[14] J. Ignacio Tello, Stability of steady states of the Cauchy problem for the exponential reactionCdiffusion equation, *J. Math. Anal. Appl.*, 324, 2006, pp.381-396.

[15] N. Ioku, The Cauchy problem for heat equations with exponential nonlinearity. *J. Differential Equations,* 251, 2011, pp. 1172–1194.

[16] C. O. Alves, M. M. Cavalcanti, On existence, uniform decay rates and blow up for solutions of the 2-D wave equation with exponential source, *Calc. Var.*, 34, 2009, pp. 377-411.

[17] D. H. Sattinger, On global solutions of nonlinear hyperbolic equations, *Arch. Rat. Mech. Anal.*, 30, 1968, pp. 148-172.

[18] R. Iekhata, T. Suzuki, Stable and unstable sets for evolution equations of parabolic and hyperbolic type, *Hiroshima Math. J.*, 26, 1996, pp. 475-491.

[19] H. A. Levine, S. R. Park, J. Serrin, Global existence and nonexistence theorems for quasilinear evolution equations of formally parabolic type, *J. of Diff. Equa.*, 142, 1998, pp. 212-229.

[20] S. A. Messaoudi, A decay result for a quasilinear parabolic system type, *Progress in Nonlinear Differential Equations and their Applications,* 63, 2005, pp. 43-50.

[21] H. Chen, G. W. Liu, Global existence and nonexistence theorems for semiilinear parabolic equations with conical degeneration, *J. of Pseudo-Diff. Oper. Appl.*, 3, 2012, pp. 329-349.

[22] J. Moser, A sharp form of an inequality by N.Trudinger, *Indiana University Mathematics Journal,* 20, 1971, pp. 1077-1092.

[23] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *Journal of Mathematics and Mechanics,* 17, 1967, pp. 473-483.

[24] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *Journal of Functional Analysis,* 14, 1973, pp. 349-381.

[25] M. Nakao, K. Ono, Global existence to the Cauchy problem of the semi-linear evolution equations with a nonlinear dissipation, *Funkcial Ekvae.*, 38, 1995, pp. 417-431.

[26] S. Wang, *Soblev Space and Partial Differential equations,* Science Press, Beijing, 2009 (in Chinese).