Semilinear nonlocal differential inclusions in Banach spaces

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Abstract: This paper is concerned with the existence of mild solutions to a class of semilinear differential inclusions with nonlocal conditions. By using the fixed point theory for multivalued maps, we get some general results on nonlocal differential inclusions, which include some recent results on nonlocal problems as special cases. An example of partial differential equations is provided to illustrate our results.

Key–Words: Differential inclusions, Nonlocal conditions, Fixed point theorems, Multivalued analysis, Mild solutions

1 Introduction

In this paper, we are concerned with the existence of mild solutions for the following nonlocal differential inclusions

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)), & t \in [0, b], \\ u(0) = g(u), \end{cases} \tag{1}$$

where $A : D(A) \subseteq X \to X$ is the densely defined generator of a strongly continuous semigroup $T(\cdot)$ in a Banach space $(X, \|\cdot\|)$, F is an upper Carathéodory multifunction, $g : C([0, b]; X) \to X$ is a given X-valued function.

The study of abstract nonlocal initial-value problems was first discussed by Byszewski [1, 2]. Since nonlocal initial condition $u(0) = g(u)$ has better effect in some physical problems than the classical initial condition $u(0) = u_0$, the theory of semilinear equations and inclusions with nonlocal conditions attracts the attention of many researchers. In the past several years, the theorems about existence, uniqueness and controllability of solutions of nonlocal differential and functional differential equations have been studied in [3-17]. Ntouyas and Tsamatos [4], Xue [5, 6] studied the following semilinear differential equation under Lipschitz or compact conditions on f and g,

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in (0, b], \\ u(0) = g(u), \end{cases}$$

where A is the infinitesimal generator of a compact semigroup $T(t)$, $t \geq 0$. Especially, the measure of noncompactness is used as an effective method to deal with the compactness of solution operators arising from nonlocal problem, see[6, 9, 10].

Cardinali and Rubbioni[18] proved the existence of mild solutions to semilinear evolution differential inclusions without nonlocal conditions when F is upper semicontinuous. V. Obukhovskii[19] applied the theory of integrated semigroups and the fixed point theory of condensing multivalued maps to obtain local and global existence results. Guo et al.[8], Chang et al.[13] got some existence and controllability results of differential inclusions with impulsive conditions. J. García-Falset[15] discusses the nonlinear inclusions with nonlocal initial conditions for the case when F is compact and g is condensing.

Motivated by the above works, we study the nonlocal differential inclusions (1) when multifunction F has not the compactness assumptions, which extend the results in [5] to differential inclusions scenario. Some general results are obtained when the nonlocal item g is Lipschitz continuous or compact, is not Lipschitz continuous and not compact, respectively. The cases that g is continuous in $C([0, b]; X)$ and g is continuous in $L^1([0, b]; X)$ are also included in our results. In Section 4, we discuss the nonlocal problem (1) under more general assumptions (see the hypotheses (b1), (b2)). The existence results in [11, 17] can be obtained as the corollaries of our main result, see Corollary 8 and Corollary 10.

This paper is organized as follows. In Section 2, we recall some concepts and facts about multivalued map and evolution system. In section 3, we obtain the existence of mild solutions for nonlocal problem (1) when Lipschitz conditions are satisfied. In section 4, we give the existence results when g is not Lipschitz and not compact. In section 5, we get the existence results when g is completely continuous. At last, an example is presented to illustrate the application of our results.

2 Preliminaries

In this section, we introduce some definitions and preliminary facts for multivalued analysis which will be useful in this paper.

Let X and Y be two Hausdorff topological spaces. We use the notation

$$P(Y) = \{A \in 2^Y : A \neq \phi\},$$
$$P_{cl}(Y) = \{A \in P(Y) : A \text{ closed}\},$$
$$P_b(Y) = \{A \in P(Y) : A \text{ bounded}\},$$
$$P_c(Y) = \{A \in P(Y) : A \text{ convex}\},$$
$$P_{cp}(Y) = \{A \in P(Y) : A \text{ compact}\},$$
$$P_{cp,c}(Y) = \{A \in P(Y) : A \in P_{cp}(Y) \bigcap P_c(Y)\}.$$

A multivalued map $F : X \to P(Y)$ is said to be convex(closed) valued if $F(x)$ is convex(closed) in Y for all $x \in X$. F is said to be compact if $F(B)$ is relatively compact for every $B \in P_b(X)$.

$F : X \to P(Y)$ is said to be upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of Y, and if for each open subset K of Y containing $F(x_0)$, there exists an open neighborhood Γ of x_0 such that $F(\Gamma) \subseteq K$.

The following conclusions are useful to get the upper semi-continuity of a multifunction F. Assume that $D \subset X$ and Fx is closed for all $x \in D$, then the following conclusions hold:

(i) If F is u.s.c. and D is closed, then F has a closed graph, i.e.,

$$x_n \to x, y_n \to y, y_n \in F(x_n) \text{ imply } y \in F(x).$$

(ii) If $\overline{F(D)}$ is compact and D is closed, then F is u.s.c. if and only if F has a closed graph.

Through this paper, let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0,b];X)$ the space of X-valued continuous functions on $[0,b]$ with the norm

$$\|x\|_C = \sup\{\|x(t)\|, t \in [0,b]\}$$

and by $L^1([0,b];X)$ the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\| \, \mathrm{d}t$. We define

$$S_F(u) \;=:\; \{f \in L^1([0,b];X) : f(t) \in F(t,u(t)) \text{ a.e. on } [0,b]\}$$

and say that F has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

Definition 1 *A function $u \in C([0,b];X)$ is said to be a mild solution of the problem* (1) *if it satisfies*

$$u(t) = T(t)g(u) + \int_0^t T(t-s)f(s)\,\mathrm{d}s, \;\; t \in [0,b],$$

where $f \in S_F(u)$.

We first give the hypothesis for the function F.

(HF) F is an upper Carathéodory multifunction, i.e., for every $x \in X$, $F(\cdot, x) : [0,b] \to P_{cp,c}(X)$ admits a strongly measurable selector; for a.e. $t \in [0,b]$, $F(t,\cdot) : X \to P_{cp,c}(X)$ is u.s.c.. And for every $u \in C([0,b];X)$, the set $S_F(u)$ is nonempty.

Lemma 2 ([20]) *Let X be a Banach space and F a multivalued map satisfying assumption* (HF). *Let*

$$\Gamma : L^1([0,b];X) \to C([0,b];X))$$

be a linear continuous map. Then the operator

$$\Gamma \circ S_F : C([0,b];X) \to P_{cl,c}(C([0,b];X)),$$
$$x \to (\Gamma \circ S_F)(x) := \Gamma(S_F(x)),$$

is a closed graph operator in $C([0,b];X) \times C([0,b],X)$.

Lemma 3 ([21]) *Let D be a nonempty, closed, convex subset of a completely Hausdorff locally convex linear topological space and let $G : D \to P(D)$ be an upper semicontinuous, compact map with $G(x)$ a nonempty, closed, convex subset of D. Then G has a fixed point in D.*

3 Lipschitz conditions

In this section, by using contraction principle, we prove the existence of nonlocal problem (1) when g is Lipschitz continuous in $C([0,b];X)$. We recall the Hausdorff distance $d_H(A,B)$ between set A and B, which is defined by

$$d_H(A,B) := \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where

$$d(a,B) = \inf\{d(a,b); b \in B\},$$
$$d(A,b) = \inf\{d(a,b); a \in A\}.$$

We let

$$M = \sup_{t\in[0,b]} \|T(t)\|.$$

A multivalued operator $F : X \to P_{cl}(X)$ is called

(i) $\delta-$Lipschitz if and only if there exists $\delta > 0$ such that $d_H(F(u), F(v)) \le \delta d(u, v)$ for all $u, v \in X$.

(ii) a contraction if and only if it is $\delta-$Lipschitz with $\delta < 1$.

Here we list the following hypotheses:

(a1) There exist constant $L > 0$, $K > 0$ such that

$$d_H(F(t,x), F(t,y)) \le L\|x-y\|, \tag{2}$$

for $t \in [0, b]$, a.e.,$x, y \in X$;

$$\|g(u) - g(v)\| \le K\|u - v\|_C, \tag{3}$$

for $u, v \in C([0, b]; X)$.

(a2) $MK + MbL < 1$.

Theorem 4 *Assume that the hypotheses* (HF), (a1) *and* (a2) *are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0, b]$.

Proof: Define the multivalued operator

$$G : C([0, b]; X) \to P(C([0, b]; X))$$

by

$$\begin{aligned} G(u) \;\; = \;\; & \{v \in C([0, b]; X) : v(t) = T(t)g(u) \\ & + \int_0^t T(t-s)f(s)\,\mathrm{d}s,\ f \in S_F(u)\}. \end{aligned}$$

It is easy to see that the fixed point of G is the mild solution of nonlocal problem (1). We will show that G has at least a fixed point due to the contraction principle.

Firstly, we show that G has a closed graph on $C([0, b]; X)$ with closed convex values. It is easy to check that G has convex values. In fact, let $v_1, v_2 \in Gu$. Then there exist $f_1, f_2 \in S_F(u)$ such that

$$v_1(t) = T(t)g(u) + \int_0^t T(t-s)f_1(s)\,\mathrm{d}s,$$

$$v_2(t) = T(t)g(u) + \int_0^t T(t-s)f_2(s)\,\mathrm{d}s.$$

For any given $\lambda \in [0, 1]$, we have

$$\begin{aligned} & \lambda v_1(t) + (1-\lambda)v_2(t) = T(t)g(u) \\ & + \int_0^t T(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]\,\mathrm{d}s. \end{aligned}$$

As the set $S_F(u)$ is convex in $L^1([0, b]; X)$, we get that

$$\lambda f_1 + (1-\lambda)f_2 \in L^1([0, b]; X),$$

and

$$\lambda f_1 + (1-\lambda)f_2 \in S_F(u).$$

So we can draw the conclusion that

$$\lambda v_1 + (1-\lambda)v_2 \in Gu,$$

i.e., G has convex values.

Now we show that G has a closed graph. Let $(u_m)_{m\in\mathrm{N}}$ and $(v_m)_{m\in\mathrm{N}} \subset C([0, b]; X)$ satisfy

$$u_m \to u,\ v_m \in G(u_m),\ v_m \to v,$$

in $C([0, b]; X)$. Then there exists a sequence $\{f_m\}_{m=1}^{\infty} \subset L^1([0, b]; X)$, $f_m \in S_F(u_m)$ for $m \ge 1$, such that

$$v_m(t) = T(t)g(u_m) + \int_0^t T(t-s)f_m(s)\,\mathrm{d}s,$$

for all $t \in [0, b]$. Consider the linear operator

$$\Gamma : L^1([0, b]; X) \to C([0, b]; X)$$

defined as

$$(\Gamma f)(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

Obviously, Γ is linear and continuous. Then from Lemma 2, we get that $\Gamma \circ S_F(\cdot)$ is a closed graph operator. Moreover, we have

$$v_m(\cdot) - T(\cdot)g(u_m) \in \Gamma \circ S_F(u_m).$$

Since $u_m \to u$ and $v_m \to v$, we obtain that

$$v(\cdot) - T(\cdot)g(u) \in \Gamma \circ S_F(u),$$

that is,

$$v(t) - T(t)g(u) = \int_0^t T(t-s)f(s)\,\mathrm{d}s,$$

for some $f \in S_F(u)$. Therefore, G has a closed graph. Hence G has closed values on $C([0, b]; X)$.

It remains to prove that G is a contraction operator. Let $u_1, u_2 \in C([0, b]; X)$ and $v_1 \in G(u_1)$. Then there exists $h_1 \in S_F(u_1)$, such that

$$v_1(t) = T(t)g(u_1) + \int_0^t T(t-s)h_1(s)\,\mathrm{d}s,\ \ t \in [0, b].$$

By (2), we have that

$$d_H(F(t, u_1(t)), F(t, u_2(t))) \le L\|u_1(t) - u_2(t)\|,$$

for $t \in [0, b]$. There exists $\psi \in F(t, u_2(t))$ such that

$$\|h_1(t) - \psi\|_X \le L\|u_1(t) - u_2(t)\|_X.$$

The multivalued map $\Theta : [0, b] \to P(X)$ defined by

$$\Theta(t) = \{\psi \in X : \|h_1(t) - \psi\|_X \leq L\|u_1(t) - u_2(t)\|_X\}$$

is measurable. Hence there exists h_2 measurable such that

$$h_2(t) \in \Theta(t) \bigcap F(t, u_2(t))$$

for each $t \in [0, b]$. Then $h_2(t) \in F(t, u_2(t))$ and

$$\|h_1(t) - h_2(t) \leq L\|u_1(t) - u_2(t)\|, \quad t \in [0, b]. \quad (4)$$

Set

$$v_2(t) = T(t)g(u_2) + \int_0^t T(t-s)h_2(s)\,\mathrm{d}s, \ \ t \in [0, b].$$

Then

$$\begin{aligned} & v_1(t) - v_2(t) \\ = \ & T(t)[g(u_1) - g(u_2)] \\ & + \int_0^t T(t-s)[h_1(s) - h_2(s)]\,\mathrm{d}s. \end{aligned}$$

Since $\|T(t)\| \leq M$, it follows from assumption (a1) and (4)

$$\begin{aligned} & \|v_1(t) - v_2(t)\| \\ \leq \ & \|T(t)[g(u_1) - g(u_2)]\| \\ & + \int_0^t \|T(t-s)\| \, \|h_1(s) - h_2(s)\|\,\mathrm{d}s \\ \leq \ & MK\|u_1 - u_2\|_C + MbL\|u_1 - u_2\|_C \\ \leq \ & (MK + MbL)\|u_1 - u_2\|_C. \end{aligned}$$

Interchanging the role of v_1 and v_2, we see that

$$d_H(Gu_1, Gu_2) \leq (MK + MbL)\|u_1 - u_2\|_C.$$

From assumption (a2) we get that G is a contraction operator. At last, we apply Nadler's Theorem [22] to deduce that G has a fixed point u_0, which is a solution of nonlocal problem (1). This completes the proof. □

4 g is not Lipschitz and not compact

In this section, we give the existence of mild solutions for the nonlocal problem (1) when the nonlocal item g is not Lipschitz and not compact in $C([0, b]; X)$, which extends many previous results in this area.

We also consider the map G on $C([0, b]; X)$ defined by

$$\begin{aligned} (Gu)(t) \ = \ & \Big\{v \in C([0, b]; X) : v(t) = T(t)g(u) \\ & + \int_0^t T(t-s)f(s)\,\mathrm{d}s, \ f \in S_F(u)\Big\}, \end{aligned}$$

with

$$(G_1u)(t) = T(t)g(u),$$

$$\begin{aligned} (G_2u)(t) \ = \ & \Big\{v \in C([0, b]; X) : v(t) \\ & = \int_0^t T(t-s)f(s)\,\mathrm{d}s, f \in S_F(u)\Big\}. \end{aligned}$$

Clearly u is a mild solution of the nonlocal problem (1) if and only if u is a fixed point of the map G. Write $Y_r = \{u \in C([0, b]; X) : \|u(t)\| \leq r \text{ for } t \in [0, b]\}$.

Here, we list the following hypotheses:

(HA) $A : D(A) \subseteq X \to X$, generates a compact strongly continuous operator semigroup $\{T(t) : t \geq 0\}$, that is, $T(t)$ is compact for $t > 0$. Moreover, there exists a positive constant $M > 0$ such that $M = \sup_{0\leq t\leq b} \|T(t)\|$ (see[23]).

(b1) $g : C([0, b]; X) \to X$ is continuous.

(b2) For any $r > 0$, the set $g(\overline{conv}GY_r)$ is precompact, where $\overline{conv}B$ denotes the convex closed hull of set $B \subseteq C([0, b]; X)$.

Remark 5 Clearly the condition (b2) is weaker than the compactness and convexity of g. The same hypothesis can be seen in Xue[5], where the author considered the existence of mild solutions for semilinear differential equations with nonlocal conditions when A is a linear, densely defined operator on X which generates a C_0-semigroup. After the proof of our main result, we will give some special types of nonlocal item g which are neither Lipschitz nor compact, but satisfy the condition (b2) in the next corollaries.

Theorem 6 *Assume that the hypotheses* (HF), (HA), (b1), (b2) *are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0, b]$ *provided that*

$$\begin{aligned} & M[\sup_{u\in Y_r} \|g(u)\| + b\sup\{\|f(t)\| : \\ & \quad t \in [0, b], f \in S_F(u), u \in Y_r\}\ ] \leq r. \quad (5) \end{aligned}$$

To prove the above theorem, we need some lemmas.

Lemma 7 *Suppose that conditions* (HF) *and* (HA) *are satisfied. Then the map*

$$Q : Y_r \to P(C([0, b]; X))$$

defined by

$$\begin{aligned} (Qu)(t) \ = \ & \Big\{v \in C([0, b]; X) : \\ & v(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s, \ f \in S_F(u)\Big\} \end{aligned}$$

is compact.

Proof: It is enough to show that QY_r is relatively compact in $C([0,b];X)$.

Firstly, we show that, for each $t \in [0,b]$, $u \in Y_r$ the set $(Qu)(t)$ is relatively compact in X. In fact, if $t = 0$, then $(Qu)(0) = 0$. For given $t \in (0,b]$ and $\varepsilon \in (0,t)$,

$$\begin{aligned}(Q^\varepsilon u)(t) &:= \left\{ \int_0^{t-\varepsilon} T(t-s)f(s)\,\mathrm{d}s : \ f \in S_F(u) \right\} \\ &= \left\{ T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)f(s)\,\mathrm{d}s : \right. \\ &\qquad \left. f \in S_F(u) \right\}\end{aligned}$$

is relatively compact in X since $T(\varepsilon)$ is compact for $\varepsilon > 0$. Then, as

$$(Q^\varepsilon u)(t) \to (Qu)(t), \qquad \text{as } \varepsilon \to 0,$$

we conclude that, for each $t \in [0,b]$, $(Qu)(t)$ is relatively compact in X by using the total boundedness.

Next, we prove the equicontinuity of QY_r. We let $0 \le t_1 < t_2 \le b$, $f \in S_F(u)$, $u \in Y_r$ and obtain

$$\begin{aligned} &\left\| \int_0^{t_2} T(t_2-s)f(s)\mathrm{d}s - \int_0^{t_1} T(t_1-s)f(s)\mathrm{d}s \right\| \\ = &\left\| \int_0^{t_1} [T(t_2-s)-T(t_1-s)]f(s)\,\mathrm{d}s \right. \\ &\left. + \int_{t_1}^{t_2} T(t_2-s)f(s)\,\mathrm{d}s \right\| \\ \le &\int_0^{t_1} \left\| T(t_2-s)-T(t_1-s) \right\|_{L(X)} \|f(s)\|\mathrm{d}s \\ &+ M\int_{t_1}^{t_2} \|f(s)\|\,\mathrm{d}s. \end{aligned} \tag{6}$$

If $t_1 = 0$, then the right hand of (6) can be made small when t_2 is small independent of $u \in Y_r$. If $t_1 > 0$, then we can find a small number $\varepsilon > 0$ with $t_1 - \varepsilon > 0$, then it follows from (6) that

$$\begin{aligned} &\int_0^{t_1} \left\| T(t_2-s)-T(t_1-s) \right\|_{L(X)} \|f(s)\|\,\mathrm{d}s \\ &+ M\int_{t_1}^{t_2} \|f(s)\|\,\mathrm{d}s \\ &\le \int_0^{t_1-\varepsilon} \left\| T(t_2-s)-T(t_1-s) \right\|_{L(X)} \|f(s)\|\,\mathrm{d}s \\ &\quad + 2M\varepsilon \cdot \max\{\|f(s)\| : f \in S_F(u), u \in Y_r\} \\ &\quad + M\int_{t_1}^{t_2} \|f(s)\|\,\mathrm{d}s. \end{aligned} \tag{7}$$

Here, as $T(t)$ is compact for $t > 0$. Thus $T(t)$ is operator norm continuous for $t > 0$. Therefore, we have

$$\int_0^{t_1-\varepsilon} \left\| T(t_2-s)-T(t_1-s) \right\|_{L(X)} \|f(s)\|\,\mathrm{d}s \to 0,$$

as $t_1 \to t_2$, uniformly for all $f \in S_F(u)$ and $u \in Y_r$. Then, from (7), we see that

$$\{(Qu)(\cdot) : u \in Y_r\}$$

is equicontinuous. By Ascoli-Arzela theorem, we know that QY_r is relatively compact in $C([0,b];X)$. This completes the proof. □

Proof of Theorem 6. We will prove G has a fixed point by using Lemma 3. Firstly we claim that G maps Y_r into itself. For $t \in [0,b]$, $u \in Y_r$, $f \in S_F(u)$, from (5), we have

$$\begin{aligned} \|(Gu)(t)\| &\le \|T(t)g(u)\| + \int_0^t \|T(t-s)f(s)\|\mathrm{d}s \\ &\le M[\sup_{u \in Y_r} \|g(u)\| + b\sup\{\|f(t)\| : \\ &\qquad t \in [0,b], f \in S_F(u), u \in Y_r\}\ ] \\ &\le r. \end{aligned}$$

The key of the following proof is that we construct a set $Y \subseteq Y_r$ such that $G : Y \to Y$ is a compact map with nonempty closed, convex values. Indeed, it is easy to check that G has convex values. From the proof of Theorem 4, we have got that G has a closed graph on $C([0,b];X)$. Then G has closed values on Y_r. Now, it remains to prove that G is compact.

For each $t \in (0,b]$, the set $\{T(t)g(u) : u \in Y_r\}$ is relatively compact in X since $T(t)$ is compact for $t > 0$. Next, we prove that G_1Y_r is equicontinuous on $[\eta,b]$ for any small positive number η. As $T(t)$ is operator norm continuous for $t > 0$. Then for $u \in Y_r$ and $\eta \le t_1 < t_2 \le b$, we have

$$\begin{aligned} &\|T(t_2)g(u) - T(t_1)g(u)\| \\ = &\|[T(t_2) - T(t_1)]g(u)\| \to 0, \end{aligned}$$

as $t_1 \to t_2$, uniformly for all $u \in Y_r$.

Moreover, from Lemma 7, we know that for each $t \in [0,b]$, the set $(G_2Y_r)(t)$ is relatively compact in X and G_2Y_r is equicontinuous on $[0,b]$. Thus, for

$$G = G_1 + G_2,$$

we have proved that $(GY_r)(t)$ is relatively compact for each $t \in (0,b]$ and GY_r is equicontinuous on $[\eta,b]$ for any small positive number η.

Let

$$Y = \overline{conv}GY_r.$$

We get that Y is a bounded closed and convex subset of $C([0,b];X)$, $Y \subset Y_r$ and $GY \subset Y$. It is easy to see that $GY(t)$ is relatively compact in X for every $t \in (0,b]$ and GY is equicontinuous on $[\eta,b]$ for any small positive number η. From hypothesis (b2) we know that $g(Y) = g(\overline{conv}GY_r)$ is pre-compact.

Now we claim that $G : Y \to Y$ is a compact map. In fact, $(G_1Y)(t)$ is relatively compact in X for every $t \geq 0$ since

$$g(Y) = g(\overline{conv}GY_r)$$

is pre-compact by hypothesis (b2). It remains to prove that G_1Y is equicontinuous on $[0, b]$. For that, let $u \in Y$, and $0 \leq t_1 < t_2 \leq b$, we have

$$\|(G_1u)(t_1) - (G_1u)(t_2)\| \leq \|[T(t_1) - T(t_2)]g(u)\|.$$

In view of the compactness of $g(Y)$ and the strong continuity of $T(t)$ on $[0, b]$, we obtain the equicontinuity of G_1Y on $[0, b]$. Thus,

$$G_1 : Y \to C([0, b]; X)$$

is a compact map by Ascoli-Arzela theorem and hence $G = G_1 + G_2$ is also compact from Lemma 7. Since G has a closed graph, then we also have that G is u.s.c.

Finally, due to Lemma 3, G has at least one fixed point $u \in Gu$ and u is a mild solution of the nonlocal problem (1). This completes the proof. □

Next, we will give some special types of nonlocal item g which is neither Lipschitz nor compact but satisfies the condition (b2). We give the following assumptions.

(c1) $g : C([0, b]; X) \to X$ is a continuous map, which maps Y_r into a bounded set, and there is a $\delta = \delta(r) \in (0, b)$ such that $g(u) = g(v)$ for any $u, v \in Y_r$, with $u(s) = v(s)$, $s \in [\delta, b]$.

(c2) $g : (C([0, b]; X), \|\cdot\|_{L^1}) \to X$ is continuous.

Corollary 8 *Assume that the hypotheses* (HF)*,* (HA)*,* (c1) *are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0, b]$ *provided that*

$$M[\sup_{u\in Y_r} \|g(u)\| + b \sup\{\|f(t)\| : t \in [0, b], f \in S_F(u), u \in Y_r\}\ ] \leq r.$$

Proof: Let

$$(GY_r)_\delta = \{u \in C([0, b]; X) : u(t) = v(t) \text{ for } t \in [\delta, b],\ u(t) = v(\delta) \text{ for } t \in [0, \delta),\ \text{where } v \in GY_r\}$$

From the proof of Theorem 6, we know that $(GY_r)_\delta$ is precompact in $C([0, b]; X)$. Moreover, by conditions (c1),

$$g(\overline{conv}GY_r) = g(\overline{conv}(GY_r)_\delta)$$

is also precompact in $C([0, b]; X)$. Thus all the conditions in Theorem 6 are satisfied. Therefore there exists at least one mild solution of nonlocal problem (1). □

Remark 9 In Zhu and Li [17], the authors get the existence of nonlocal problem (1) under condition (c1) by using the method of approximation solutions. Now we can obtain that result as a special case of Theorem 6. In many studies of nonlocal Cauchy problems, for example[1, 3], the map g is given by

$$g(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p)) = \sum_{i=1}^{p} c_i u(t_i),$$

for some given constants c_i. In these cases,

$$g(t_1, \cdots, t_p, u(t_1), \cdots, u(t_p))$$

allows the measurements at $t = t_1, \cdots, t_p$, rather than just at $t = 0$. It is easy to see that g satisfies condition (c1).

Corollary 10 *Assume that the hypotheses* (HF)*,* (HA)*,* (c2) *are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0, b]$ *provided that*

$$M[\sup_{u\in Y_r} \|g(u)\| + b \sup\{\|f(t)\| : t \in [0, b], f \in S_F(u), u \in Y_r\}\ ] \leq r.$$

Proof: According to Theorem 6, we should only prove that the hypothesis (b2) is satisfied.

For arbitrary $\varepsilon > 0$, there exists $0 < \delta < b$ such that

$$\int_0^\delta \|u(s)\| \, \mathrm{d}s < \varepsilon$$

for $u \in GY_r$. Let $(GY_r)_\delta = \{u \in C([0, b]; X) : u(t) = v(t) \text{ for } t \in [\delta, b],\ u(t) = v(\delta) \text{ for } t \in [0, \delta),\ \text{where } v \in GY_r\}$. From the proof of Theorem 6, we know that $(GY_r)_\delta$ is precompact in $C([0, b]; X)$ which implies that $(GY_r)_\delta$ is precompact in $L^1([0, b]; X)$. Thus GY_r is pre-compact in $L^1([0, b]; X)$ as it has a ε-net $(GY_r)_\delta$.

By condition $g : (C([0, b]; X), \|\cdot\|_{L^1}) \to X$ is continuous and

$$\overline{conv}GY_r \subseteq (L)\overline{conv}GY_r,$$

it follows that condition (b2) is satisfied, where $(L)\overline{conv}B$ denotes the convex and closed hull of B in $L^1([0, b]; X)$. Therefore, the nonlocal problem (1) has at least one mild solution on $[0, b]$. □

Remark 11 In [11], Aizicovici and McKibben suppose that

$$g : L^1([0, b]; X) \to X$$

is continuous with linear growth condition with respect to the norm of $L^1([0, b]; X)$. By using the fixed point theorem on $L^1([0,b];X)$, they obtain the existence of integral solutions for nonlinear nonlocal problems with multivalued perturbations. In this paper, by using the fixed point theorem on $C([0,b];X)$ rather than on $L^1([0,b];X)$, we also get the mild solutions for the nonlocal problem (1).

Corollary 12 *Consider the nonlocal Cauchy problem* (1). *Let the hypotheses* (HA), (HF), (b1), (b2) *hold true. Suppose that there exist* $\alpha \in L^1([0,b];R^+)$ *and a non-decreasing function*

$$K : R^+ \to R^+$$

such that

$$\begin{aligned} &\|F(t,x)\| := \sup\{\|y\| : y \in F(t,x)\} \\ \leq\ & \alpha(t)K(\|x\|), \end{aligned} \tag{8}$$

and

$$\|g(u)\| \leq k_1\|u\|_C^\beta + k_2,\ \ k_1, k_2 \in R^+,\ 0 \leq \beta < 1 \tag{9}$$

then the nonlocal problem (1) *has at least one mild solution on* $[0,b]$*, provided that*

$$\limsup_{r\to\infty} [\frac{K(r)}{r}\int_0^b \alpha(t)\,\mathrm{d}t] < \frac{1}{M}. \tag{10}$$

Proof: From (10), we know that

$$\limsup_{r\to\infty} \frac{M[k_1 r^\beta + k_2 + K(r)\int_0^b \alpha(t)\,\mathrm{d}t]}{r} < 1,$$

which implies there exists some $r > 0$ such that

$$M[k_1 r^\beta + k_2 + K(r)\int_0^b \alpha(t)\,\mathrm{d}t \leq r.$$

Then we can get that the condition (5) is true. Thus, all the conditions in Theorem 6 are satisfied and the nonlocal problem (1) has at least one mild solution on $[0,b]$. This completes the proof. □

5 g is completely continuous

In this section, we give the existence result when the nonlocal item g is completely continuous, i.e., g is continuous and maps a bounded set into a relatively compact set. Here the hypothesis (b2) in Section 4 is not needed. We list the following hypotheses:

(d1) $g : C([0,b];X) \to X$ is a completely continuous operator.

(d2) There exists a constant $r > 0$ such that

$$\begin{aligned} M[&\sup_{u\in Y_r}\|g(u)\| + b\sup\{\|f(t)\| : \\ &t \in [0,b], f \in S_F(u), u \in Y_r\}\ ] \leq r, \end{aligned}$$

where $Y_r := \{u \in C([0,b];X) : \|u(t)\| \leq r \text{ for } t \in [0,b]\}$.

Theorem 13 *Assume that the hypotheses (HF), (HA), (d1),(d2) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0,b]$.

Proof. For any $u \in Y_r$, we define an operator G on $C([0,b];X)$ by

$$\begin{aligned} (Gu)(t) &= \Big\{v \in C([0,b];X) : v(t) = T(t)g(u) \\ &\quad + \int_0^t T(t-s)f(s)\,\mathrm{d}s,\ f \in S_F(u)\Big\}, \end{aligned}$$

with

$$(G_1u)(t) = T(t)g(u),$$

$$\begin{aligned} (G_2u)(t) &= \Big\{v \in C([0,b];X) : v(t) \\ &= \int_0^t T(t-s)f(s)\,\mathrm{d}s, f \in S_F(u)\Big\}. \end{aligned}$$

Firstly, we claim that G maps Y_r into itself. In fact, for $t \in [0,b]$, $u \in Y_r$, $f \in S_F(u)$, from hypothesis (d2), we have

$$\begin{aligned} \|(Gu)(t)\| &\leq \|T(t)g(u)\| \\ &\quad + \int_0^t \|T(t-s)f(s)\|\,\mathrm{d}s \\ &\leq M[\sup_{u\in Y_r}\|g(u)\| + b\sup\{\|f(t)\| : \\ &\qquad t \in [0,b], f \in S_F(u), u \in Y_r\}\ ] \\ &\leq r. \end{aligned}$$

Subsequently, we will prove that G has a fixed point by using Lemma 3.

We show that G is a compact map with nonempty compact, closed and convex values. Let $v_1, v_2 \in Gu$. Then there exist $f_1, f_2 \in S_F(u)$ such that

$$v_1(t) = T(t)g(u) + \int_0^t T(t-s)f_1(s)\,\mathrm{d}s,$$

$$v_2(t) = T(t)g(u) + \int_0^t T(t-s)f_2(s)\,\mathrm{d}s.$$

For any given $\lambda \in [0,1]$, we have

$$\begin{aligned} &\lambda v_1(t) + (1-\lambda)v_2(t) = T(t)g(u) \\ &+ \int_0^t T(t-s)[\lambda f_1(s) + (1-\lambda)f_2(s)]\,\mathrm{d}s. \end{aligned}$$

As the set $S_F(u)$ is convex in $L^1([0,b];X)$, we get that

$$\lambda f_1 + (1-\lambda)f_2 \in L^1([0,b];X),$$

and

$$\lambda f_1 + (1-\lambda)f_2 \in S_F(u).$$

So we have that

$$\lambda v_1 + (1-\lambda)v_2 \in Gu,$$

i.e., G has convex values.

Now we show that G has closed values. Let $(u_m)_{m\in\mathrm{N}}, (v_m)_{m\in\mathrm{N}} \subset C([0,b];X)$, satisfying

$$u_m \to u,\ v_m \in G(u_m),\ v_m \to v,$$

in $C([0,b];X)$. Then there exists a sequence $\{f_m\}_{m=1}^{\infty} \subset L^1([0,b];X)$, $f_m \in S_F(u_m)$ for $m \geq 1$, such that

$$v_m(t) = T(t)g(u_m) + \int_0^t T(t-s)f_m(s)\,\mathrm{d}s,$$

for all $t \in [0,b]$. Consider the linear operator

$$\Gamma : L^1([0,b];X) \to C([0,b];X)$$

defined as

$$(\Gamma f)(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

Obviously, Γ is linear and continuous. Then from Lemma 2, we get that $\Gamma \circ S_F(\cdot)$ is a closed graph operator. Moreover, we have

$$v_m(\cdot) - T(\cdot)g(u_m) \in \Gamma \circ S_F(u_m).$$

Since $u_m \to u$ and $v_m \to v$, we obtain that

$$v(\cdot) - T(\cdot)g(u) \in \Gamma \circ S_F(u),$$

that is,

$$v(t) - T(t)g(u) = \int_0^t T(t-s)f(s)\,\mathrm{d}s,$$

for some $f \in S_F(u)$. Therefore, G has a closed graph. Hence G has closed values on $C([0,b];X)$.

Now, it remains to prove that G is compact, or G_1, G_2 are compact.

For each $t \in [0,b]$, the set $\{T(t)g(u) : u \in Y_r\}$ is relatively compact in X as g is compact and $T(t)$ is strongly continuous. Then for $0 \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned} &\|T(t_2)g(u) - T(t_1)g(u)\| \\ &= \|[T(t_2) - T(t_1)]g(u)\| \to 0, \quad \text{as } t_1 \to t_2, \end{aligned}$$

uniformly for all $u \in Y_r$, due to the compactness of g and the strong continuity of $T(t)$. Now an application of Ascoli-Arzela theorem justifies G_1 is a compact operator. Moreover, from Lemma 7, we know that G_2 is a compact operator. Thus $G = G_1 + G_2$ is compact on Y_r. Since G has a closed graph. Thus we also have that G is upper semi-continuous.

Finally, due to Lemma 3, G has at least one fixed point $u \in G(u)$, and u is a mild solution of the problem (1). This completes the proof. $\square$

6 An example

As an application of our abstract results, we give the following partial differential system with nonlocal conditions:

$$\begin{cases} \frac{\partial}{\partial t}\omega(t,x) \in \frac{\partial^2}{\partial x^2}\omega(t,x) + F(t,\omega(t,x)), \\ \quad 0 \leq t \leq b,\ \ 0 \leq x \leq \pi, \\ \omega(t,0) = \omega(t,\pi) = 0, \\ \omega(0,x) = g(\omega(t,x)), \end{cases} \tag{11}$$

where $X = L^2([0,\pi])$.

We consider the operator

$$A : D(A) \subseteq X \to X$$

defined by

$$Az = z'',$$

with

$$D(A) = \{z \in X : z, z' \text{ are absolutely continuous},$$

$$z'' \in X, z(0) = z(\pi) = 0\}.$$

From Pazy ([23]), we know that A generates a compact C_0-semigroup $T(t)$. This implies that A satisfies the condition (HA).

We assume that the following conditions hold:

(1) $F : [0,b] \times X \to P(X)$ is a multivalued map defined by

$$F(t,z)(x) = F(t,z(x)),\ \ 0 \leq t \leq b,\ 0 \leq x \leq \pi,$$

and assumptions (HF) and (8) hold.

(2) $g : C([0,b];X) \to X$ is a continuous function defined by

$$g(\omega(t,\xi)) = \sum_{j=1}^{q} c_j \omega(s_j,\xi),\ \ 0 \leq \xi \leq \pi,$$

$\omega \in C([0,b];X),\ 0 < s_1 < \cdots < s_q < b.$

(3) $g : C([0,b];X) \to X$ is a continuous function defined by

$$g(\omega(t,\xi)) = \sum_{j=1}^{q} c_j \sqrt[3]{\omega(s_j,\xi)}, \quad 0 \leq \xi \leq \pi,$$

$\omega \in C([0,b];X),\ 0 < s_1 < \cdots < s_q < b.$

(4) $g : C([0,b];X) \to X$ is a continuous function defined by

$$g(\omega(t,\xi)) = \int_0^b h(s)\lg(1+|\omega(s,\xi)|)\,\mathrm{d}s,\ \ 0 \leq \xi \leq \pi,$$

$\omega \in C([0,b];X), h \in L^1([0,b];R).$

From the above assumptions, the partial differential system (11) can be reformulated as the abstract problem (1). Then we have the following results:

(i) Under the conditions (1)+(2), the assumptions in Theorem 4 are satisfied. Therefore, the corresponding system (1) has at least a mild solution.

(ii) Under the conditions (1)+(3), the assumptions in Theorem 6 are satisfied for large $r > 0$. Therefore, the corresponding system (1) has at least a mild solution.

(iii) Under the conditions (1)+(4), the assumptions in Theorem 13 are satisfied for large $r > 0$. Therefore, the corresponding system (1) has at least a mild solution.

7 Conclusion Remark

The existence of semilinear nonlocal differential inclusions is discussed in this paper. We introduce some techniques to study the cases that the nonlocal function g is Lipschitz continuous or compact, is not Lipschitz continuous and not compact, respectively. We give some general assumptions on the multivalued function F and nonlocal function g, which covers and extends some results in this area.

We also remark that the method here can be used to study impulsive differential inclusions with nonlocal conditions. The impulsive differential systems can be used to model processes which are subjected to short perturbations whose duration can be negligible in comparison with the duration of the process. For more details on this theory and its applications, we refer to the monographs of Lakshmikantham et al. citelak3 and the papers of [9, 25]. We can study the following impulsive differential inclusions

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)), \ \ t \in [0, b], \\ \Delta u(t_i) = I_i(u(t_i)), \ \ i = 1, 2, \cdots, p, \\ \ 0 < t_1 < t_2 < \cdots < t_p < b, \\ u(0) = g(u), \end{cases}$$

where $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, $u(t_i^+)$, $u(t_i^-)$, denote the right and the left limit of u at t_i. Then we can also release some conditions on the impulsive functions I_i and nonlocal function g and get some new results.

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