Spectral norms of circulant-type matrices involving some well-known numbers

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Abstract: In this paper, we investigate spectral norms for circulant-type matrices, including circulant, skew-circulant and \( g \)-circulant matrices. The entries are product of binomial coefficients with Fibonacci numbers and Lucas numbers, respectively. We obtain identity estimations for these spectral norms. Employing these approaches, we list some numerical tests to verify our results.

KeyWords: Binomial coefficients, Circulant type, Fibonacci numbers, Lucas numbers, Spectral norms

1 Introduction

Circulant, skew-circulant and \( g \)-circulant matrices play important roles in various applications with good foundation. Circulant-type matrix had been applied to the area of discussion about economics, digital image disposal, linear forecast, design of self-regress, etc. For example, the economists can employ spectral norms of those matrices to construct the optimal filter for an economic model, and form the building blocks for most modern circulant-type filters to investigate the rule of certain economics, and so on. The properties of this kind of matrix support lots of benefits for the engineer applications. For the details, please refer to [1, 7, 8, 12, 13, 14, 15, 18, 21, 22, 23, 25], and the references therein. The skew-circulant matrices were collected to construct pre-conditioners for LMF-based ODE codes, Hermitian and skew-Hermitian Toeplitz systems were considered in [3, 10, 11, 17], Lyness employed a skew-circulant matrix to construct an \( s \)-dimensional lattice rules in [16].

Recently, there are lots of research on the spectral distribution and norms of circulant-type matrices. In [5], the authors pointed out the processes based on the eigenvalue of circulant-type matrices with i.i.d. entries, furthermore, they claimed that they converged to a Poisson random measures in vague topology. There were discussions about the convergence in probability and distribution of the spectral norm of circulant-type matrices in [6]. The authors in [4] listed the limiting spectral distribution for a class of circulant-type matrices with heavy tailed input sequence. Eric Ngongiep et al. showed that the singular values of \( g \)-circulants in [19].

Solak established the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries in [20]. İpek investigated an improved estimation for spectral norms in [26]. In this paper, we derive some identity estimates of spectral norms for some circulant-type matrices with product of binomial coefficients with Fibonacci numbers and Lucas numbers, respectively.

The Fibonacci and Lucas sequences \( \{F_k\} \) and \( \{L_k\} \) are defined by the following recursive relations

\[
F_n = F_{n-1} + F_{n-2}
\]

with\( F_0 = 0, F_1 = 1 \), and

\[
L_n = L_{n-1} + L_{n-2}
\]

with\( L_0 = 2, L_1 = 1 \).

The binomial coefficients \( \binom{n}{k} \) are, for all natural numbers \( k \), defined by

\[
(1 + X)^n = \sum_{k=0}^{\infty} \binom{n}{k} X^k.
\]

It is clear that, for \( k > n \),

\[
\binom{n}{k} = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{otherwise} \end{cases}
\]

Let \( \binom{n}{k} \) be the \( k \)-th binomial coefficient of \( n \), \( F_k \) and \( L_k \) denotes the \( k \)-th Fibonacci and Lucas number,
respectively. For anyone \( p \in \mathbb{N} \), we have the following formulas [9]

\[
\begin{align*}
\sum_{i=0}^{n} \binom{n}{i} F_{i+p} &= F_{2n+p}, \\
\sum_{i=0}^{n} \binom{n}{i} L_{i+p} &= L_{2n+p},
\end{align*}
\]

and

\[
\begin{align*}
\sum_{i=0}^{n} \binom{n}{i} 2^i F_{i+p} &= F_{3n+p}, \\
\sum_{i=0}^{n} \binom{n}{i} 2^i L_{i+p} &= L_{3n+p}.
\end{align*}
\]

\[\text{(1)}\]

\[\text{(2)}\]

2 Circulant-type matrices

**Definition 1** [13, 15] A circulant matrix is an \( n \times n \) complex matrix of the form

\[
A_c = \begin{pmatrix}
    a_0 & a_1 & \cdots & a_{n-1} \\
    a_{n-1} & a_0 & \cdots & a_{n-2} \\
    a_{n-2} & a_{n-1} & \cdots & a_{n-3} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_1 & a_2 & \cdots & a_0
\end{pmatrix}_n^n.
\]

The first row of \( A_c \) is \((a_0, a_1, \cdots, a_j, \cdots, a_{n-1})\), its \((j+1)\)-th row is obtained by giving its \( j \)-th row a right circular shift by one position.

Equivalently, a circulant matrix can be described as a polynomial

\[
A_c = f(\eta_c) = \sum_{i=0}^{n-1} a_i \eta_c^i,
\]

where

\[
\eta_c = \begin{pmatrix}
    0 & 1 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 1 \\
    1 & 0 & 0 & \cdots & 0
\end{pmatrix}_n^n.
\]

Obviously, \( \eta_c^n = I_n \).

We are now in a position to discuss the eigenvalues of \( A_c \). Motivated by the relation between matrix and polynomial, we declare that the eigenvalues of \( \eta_c \) are the corresponding eigenvalues of \( A_c \) with the function \( f \) in (4), which is

\[
\lambda(A_c) = f(\lambda(\eta_c)) = \sum_{i=0}^{n-1} a_i \lambda(\eta_c)^i.
\]

Since

\[
\lambda_j(\eta_c) = \omega^j, \quad (j = 0, 1, \cdots, n-1),
\]

where \( \omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \). Thus \( \lambda_j(A_c) \) can be calculated via

\[
\lambda_j(A_c) = \sum_{i=0}^{n-1} a_i (\omega^j)^i,
\]

\[\text{(5)}\]

Similarly, let us recall a skew-circulant matrix.

**Definition 2** [13, 15] A skew-circulant matrix is an \( n \times n \) complex matrix of the following form

\[
A_{sc} = \begin{pmatrix}
    a_0 & a_1 & \cdots & a_{n-1} \\
    -a_{n-1} & a_0 & \cdots & a_{n-2} \\
    -a_{n-2} & -a_{n-1} & \cdots & a_{n-3} \\
    \vdots & \vdots & \ddots & \vdots \\
    -a_1 & -a_2 & \cdots & a_0
\end{pmatrix}_n^n.
\]

Also, a skew-circulant matrix can be described as matrix polynomial

\[
A_{sc} = f(\eta_{sc}) = \sum_{i=0}^{n-1} a_i \eta_{sc}^i,
\]

where

\[
\eta_{sc} = \begin{pmatrix}
    0 & 1 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 1 \\
    -1 & 0 & 0 & \cdots & 0
\end{pmatrix}_n^n.
\]

Obviously, \( \eta_{sc}^n = -I_n \).

To calculate the eigenvalues of \( A_{sc} \), for the same reason, we obtain

\[
\lambda(A_{sc}) = f(\lambda(\eta_{sc})) = \sum_{i=0}^{n-1} a_i \lambda(\eta_{sc})^i.
\]

Since

\[
\lambda_j(\eta_{sc}) = \omega^j \alpha, \quad (j = 0, 1, \cdots, n-1),
\]

where

\[
\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}, \quad \alpha = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n},
\]

so \( \lambda_j(A_{sc}) \) can be computed via

\[
\lambda_j(A_{sc}) = \sum_{i=0}^{n-1} a_i (\omega^j \alpha)^i.
\]
Definition 3 [4, 24] A $g$-circulant matrix is an $n \times n$ complex matrix of the following form

$$A_g = \begin{bmatrix}
    a_0 & a_1 & \cdots & a_{n-1} \\
    a_{n-g} & a_{n-g+1} & \cdots & a_{n-g-1} \\
    a_{n-2g} & a_{n-2g+1} & \cdots & a_{n-2g-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_g & a_{g+1} & \cdots & a_{g-1}
\end{bmatrix}$$

(7)

where $g$ is a nonnegative integer and each of the subscripts is understood to be reduced modulo $n$.

The first row of $A_g$ is $(a_0, a_1, \ldots, a_{n-1})$, its $(j+1)$-th row is obtained by giving its $j$-th row a right circular shift by $g$ positions (equivalently, $g \mod n$ positions). Note that $g = 1$ or $g = n + 1$ yields the standard circulant matrix. If $g = n - 1$, then we obtain the so-called reverse circulant matrix [4].

Definition 4 [2] The spectral norm $\| \cdot \|_2$ of a matrix $A$ with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix $A^*A$:

$$\| A \|_2 = \sqrt{\lambda_{\text{max}}(A^*A)}$$

where $A^*$ denotes the conjugate transpose of $A$. Therefore if $A$ is an $n \times n$ real symmetric matrix or $A$ is a normal matrix, then

$$\| A \|_2 = \max_{1 \leq i \leq n} | \lambda_i |,$$

(8)

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$.

3 Spectral norms of some circulant matrices

We will analyse spectral norms of some given circulant matrices, whose entries are combined binomial coefficients with Fibonacci or Lucas numbers, respectively. For the convenience of the discussion, we set $p = 0$ in (1), and the same conclusions can be deduced for $\forall p \in \mathbb{N}$.

3.1 Spectral norms of some circulant matrices with modified $\binom{n}{i} F_i$ and $\binom{n}{i} L_i$

Definition 5 Some circulant matrices are defined as the following forms:

$$B_1 = \begin{pmatrix}
    \binom{n}{0} F_0 & \cdots & \binom{n}{i} F_i \\
    \binom{n}{m} F_m & \cdots & \binom{n}{m-1} F_{m-1} \\
    \binom{n}{n-1} F_{n-1} & \cdots & \binom{n}{n-3} F_{n-3} \\
    \vdots & \ddots & \vdots \\
    \binom{n}{1} F_1 & \cdots & \binom{n}{0} F_0
\end{pmatrix},$$

$$B_2 = \begin{pmatrix}
    \binom{n}{0} L_0 & \cdots & \binom{n}{i} L_i \\
    \binom{n}{m} L_m & \cdots & \binom{n}{m-1} L_{m-1} \\
    \binom{n}{n-1} L_{n-1} & \cdots & \binom{n}{n-3} L_{n-3} \\
    \vdots & \ddots & \vdots \\
    \binom{n}{1} L_1 & \cdots & \binom{n}{0} L_0
\end{pmatrix},$$

$$B_3 = \begin{pmatrix}
    \binom{m}{0} F_0 & \cdots & -\binom{m}{i} F_i \\
    -\binom{m}{m} F_m & \cdots & \binom{m}{m-1} F_{m-1} \\
    -\binom{m}{m-1} F_{m-1} & \cdots & \binom{m}{m-2} F_{m-2} \\
    \vdots & \ddots & \vdots \\
    -\binom{m}{1} F_1 & \cdots & \binom{m}{0} F_0
\end{pmatrix},$$

$$B_4 = \begin{pmatrix}
    -\binom{m}{0} F_0 & \cdots & \binom{m}{i} F_i \\
    -\binom{m}{m} F_m & \cdots & -\binom{m}{m-1} F_{m-1} \\
    \binom{m}{m-1} F_{m-1} & \cdots & \binom{m}{m-2} F_{m-2} \\
    \vdots & \ddots & \vdots \\
    \binom{m}{1} F_1 & \cdots & -\binom{m}{0} F_0
\end{pmatrix},$$

$$B_5 = \begin{pmatrix}
    \binom{m}{0} L_0 & \cdots & -\binom{m}{i} L_i \\
    -\binom{m}{m} L_m & \cdots & \binom{m}{m-1} L_{m-1} \\
    \binom{m}{m-1} L_{m-1} & \cdots & \binom{m}{m-2} L_{m-2} \\
    \vdots & \ddots & \vdots \\
    -\binom{m}{1} L_1 & \cdots & \binom{m}{0} L_0
\end{pmatrix},$$

$$B_6 = \begin{pmatrix}
    -\binom{m}{0} L_0 & \cdots & \binom{m}{i} L_i \\
    -\binom{m}{m} L_m & \cdots & -\binom{m}{m-1} L_{m-1} \\
    \binom{m}{m-1} L_{m-1} & \cdots & \binom{m}{m-2} L_{m-2} \\
    \vdots & \ddots & \vdots \\
    \binom{m}{1} L_1 & \cdots & -\binom{m}{0} L_0
\end{pmatrix},$$

where $m, n$ are integers, and $m$ is odd.

Obviously,

$$B_4 = -B_3, \quad B_6 = -B_5.$$

Our main results for those matrices are as follows.

Theorem 1 Let $B_3$ be the matrix defined as in definition 5. Then we have

$$\| B_1 \|_2 = F_{2n}.$$
Proof. Since the circulant matrix $B_1$ is normal (see Definition 4), we claim that the spectral norm of $B_1$ is equal to its spectral radius. Furthermore, applying the irreducible and entrywise nonnegative properties, we claim that $\|B_1\|_2$ (i.e., its spectral norm), is equal to its Perron value. We select an $(n + 1)$-dimensional column vector

$$v = (1, 1, \cdots, 1)^T,$$

then

$$B_1v = \left( \sum_{j=0}^{n} \binom{n}{j} F_j \right) v.$$

Obviously, $\sum_{j=0}^{n} \binom{n}{j} F_j$ is an eigenvalue of $B_1$ associated with $v$, which is necessarily the Perron value of $B_1$. Employing (1), we obtain

$$\|B_1\|_2 = F_{2n}.$$

This completes the proof. □

Using the same approach, we can prove the following result.

**Theorem 2** Let $B_2$ be the matrix defined as in Definition 5. Then we have

$$\|B_2\|_2 = L_{2n}.$$  

**Proof.** Using the same techniques of Theorem 1 and irreducibility and entrywise nonnegativity of the normal matrix $B_2$, we declare that the spectral norm of $B_2$ is the same as its Perron value.

Let

$$v^T = (1, 1, \cdots, 1)^T,$$

Then

$$B_2v = \left( \sum_{i=0}^{n} \binom{n}{i} L_i \right) v.$$

Since $\sum_{i=0}^{n} \binom{n}{i} L_i$ is an eigenvalue of $B_2$ associated with the positive eigenvector $v$, it is equal to Perron value of $B_2$. Combining with the identities, we obtain

$$\|B_2\|_2 = L_{2n},$$

which completes the proof. □

**Corollary 1** Let $A_c$ be defined as (3). For any $p \in \mathbb{N}$, the following statements are true:

1. If $\binom{m}{0} F_p, \binom{m}{1} F_{1+p}, \binom{m}{2} F_{2+p}, \cdots, \binom{m}{n} F_{n+p}$ is the first row of $A_c$, then

$$\|A_c\|_2 = F_{2n+p}.$$  

2. If $\binom{n}{0} L_p, \binom{n}{1} L_{1+p}, \binom{n}{2} L_{2+p}, \cdots, \binom{n}{m} L_{m+p}$ is the first row of $A_c$, then

$$\|A_c\|_2 = L_{2n+p}.$$  

**Theorem 3** Let $B_3$ and $B_4$ be defined as in Definition 5, respectively, and $m$ be odd. Then

$$\|B_3\|_2 = F_{2m}, \quad \|B_4\|_2 = F_{2m}.$$  

**Proof.** Noticing (5) and (8), it is clear that the spectral norm of $B_3$ can be calculated by

$$\|B_3\|_2 = \max_{0 \leq j \leq m} |\lambda_j(B_3)| = \max_{0 \leq j \leq m} \left| \sum_{i=0}^{m} a_i(\omega^j)^i \right| = \sum_{i=0}^{m} |a_i|,$$

where $a_i = (-1)^i \binom{m}{i} F_i$.

Note that, if $m$ is odd, the $m + 1$ is even, then

$$\lambda_j(\eta_c) = \omega^m = -1$$

is an eigenvalue of $\eta_c$, so the identity holds, i.e.,

$$\|B_3\|_2 = \sum_{i=0}^{m} |a_i|. \quad (10)$$

Combining (1) and (10) yields $\|B_3\|_2 = F_{2m}$.

In the same manner, we can show $\|B_4\|_2 = F_{2m}$. This completes the proof. □

**Corollary 2** Let $A_c$ be defined as (3) and $m$ is odd. Then we have

1. If $\binom{m}{0} F_p, \binom{m}{1} F_{1+p}, \cdots, \binom{m}{n} F_{n+p}$ is the first row of $A_c$, then

$$\|A_c\|_2 = F_{2m+p},$$

where $\forall p \in \mathbb{N}$.

2. If $\binom{m}{0} F_p, \binom{m}{1} F_{1+p}, \cdots, \binom{m}{m} F_{m+p}$ is the first row of $A_c$, then

$$\|A_c\|_2 = F_{2m+p},$$

where $\forall p \in \mathbb{N}$.

**Theorem 4** Let $B_5$ and $B_6$ be defined as in Definition 5, respectively, and $m$ be odd. Then

$$\|B_5\|_2 = L_{2m}, \quad \|B_6\|_2 = L_{2m}. \quad (11)$$  

**Proof.** From proof of Theorem 3, we see that (11) holds. □
Corollary 3 Let $A_c$ be defined as (3) and $m$ be odd. Then we have
1. If $\binom{m}{0}L_p - \binom{m}{1}L_{1+p}, \ldots, -\binom{m}{m}L_{m+p}$ is the first row of $A_c$, then
   \[ \|A_c\|_2 = L_{2m+p}, \]
   where $\forall p \in \mathbb{N}$.
2. If $-\binom{m}{0}L_p, \binom{m}{1}L_{1+p}, \ldots, \binom{m}{m}L_{m+p}$ is the first row of $A_c$, then
   \[ \|A_c\|_2 = L_{2m+p}, \]
   where $\forall p \in \mathbb{N}$.

3.2 Spectral norms of some circulant matrices with modified $\binom{n}{i}2^i F_i$ and $\binom{n}{i}2^i L_i$

Definition 6 The circulant matrices are defined as the following forms

\[ \begin{pmatrix} \binom{n}{0}2^0 F_0 & \cdots & \binom{n}{n}2^n F_n \\ \binom{n}{1}2^1 L_1 & \cdots & \binom{n}{n-1}2^{n-1} L_n \\ \vdots & \ddots & \vdots \\ \binom{n}{n}2^n F_0 & \cdots & \binom{n}{0}2^0 L_0 \end{pmatrix}, \]

\[ \begin{pmatrix} \binom{n}{0}2^0 L_0 & \cdots & \binom{n}{n}2^n L_n \\ \binom{n}{1}2^1 L_1 & \cdots & \binom{n}{n-1}2^{n-1} L_n \\ \vdots & \ddots & \vdots \\ \binom{n}{n}2^n L_0 & \cdots & \binom{n}{0}2^0 L_0 \end{pmatrix}, \]

\[ \begin{pmatrix} \binom{m}{0}2^0 F_0 & \cdots & -\binom{m}{m}2^m F_m \\ -\binom{m}{1}2^1 F_1 & \cdots & \binom{m}{m-1}2^{m-1} F_{m-1} \\ \vdots & \ddots & \vdots \\ -\binom{m}{m}2^m F_0 & \cdots & \binom{m}{0}2^0 F_0 \end{pmatrix}, \]

Employing the same approaches as in the above subsection, we list the main results for those matrices.

Theorem 5 Let $\tilde{B}_1$ be defined as in Definition 6. Then
   \[ \|\tilde{B}_1\|_2 = F_{3n}. \]

Theorem 6 Let $\tilde{B}_2$ be defined as in Definition 6. The we have
   \[ \|\tilde{B}_2\|_2 = L_{3n}. \]

Corollary 4 Let $A_c$ be defined by (3). For any $p \in \mathbb{N}$, we have
1. $\binom{n}{0}F_p, \binom{n}{1}2F_{1+p}, \binom{n}{2}2^2 F_{2+p}, \ldots, \binom{n}{n}2^n F_{n+p}$ is the first row of $A_c$, then
   \[ \|A_c\|_2 = F_{3n+p}. \]
2. $\binom{n}{0}L_p, \binom{n}{1}2L_{1+p}, \binom{n}{2}2^2 L_{2+p}, \ldots, \binom{n}{n}2^n L_{n+p}$ is the first row of $A_c$, then
   \[ \|A_c\|_2 = L_{3n+p}. \]

Theorem 7 Let $\tilde{B}_3$ and $\tilde{B}_4$ be defined as in Definition 6, respectively, and $m$ be odd. Then
   \[ \|\tilde{B}_3\|_2 = F_{3m}, \|\tilde{B}_4\|_2 = F_{3m}. \]

Corollary 5 Let $A_c$ be defined as in (3) and $m$ be odd. Then
1. $\binom{m}{0}2^0 F_p, -\binom{m}{1}2^1 F_{1+p}, \ldots, -\binom{m}{m}2^m F_{m+p}$ is the first row of $A_c$, then
   \[ \|A_c\|_2 = F_{3m+p}, \]
   where $\forall p \in \mathbb{N}$.
2. $\binom{m}{0}2^0 F_p, \binom{m}{1}2^1 F_{1+p}, \ldots, \binom{m}{m}2^m F_{m+p}$ is the first row of $A_c$, then
   \[ \|A_c\|_2 = F_{3m+p}, \]
   where $\forall p \in \mathbb{N}$.

Theorem 8 Let $\tilde{B}_5$ and $\tilde{B}_6$ be defined as in Definition 6, respectively, and $m$ be odd. Then
   \[ \|\tilde{B}_5\|_2 = L_{3m}, \|\tilde{B}_6\|_2 = L_{3m}. \]
Corollary 6 Let $A_c$ be as (3) and $m$ be odd. Then we have

(1) if $\left(\begin{array}{c} m \\ 0 \end{array}\right) 2^0 L_0, \left(\begin{array}{c} m \\ 1 \end{array}\right) 2^1 L_{1+p}, \ldots, \left(\begin{array}{c} m \\ m \end{array}\right) 2^m L_{m+p}$

is the first row of $A_c$, then

$$\|A_c\|_2 = L_{3m+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(\begin{array}{c} m \\ 0 \end{array}\right) 2^0 L_0, \left(\begin{array}{c} m \\ 1 \end{array}\right) 2^1 L_{1+p}, \ldots, \left(\begin{array}{c} m \\ m \end{array}\right) 2^m L_{m+p}$

is the first row of $A_c$, then

$$\|A_c\|_2 = L_{3m+p},$$

where $\forall p \in \mathbb{N}$.

4 Spectral norms of skew-circulant matrices

4.1 Spectral norms of skew-circulant matrices with modified $\left(\begin{array}{c} n \\ i \end{array}\right) F_i$ and $\left(\begin{array}{c} n \\ j \end{array}\right) L_i$

An odd-order alternative skew-circulant matrix is defined as follows, where $s$ is even.

$$B_7 = \begin{pmatrix} 
\left(\begin{array}{c} n \\ 0 \end{array}\right) F_0 & \cdots & \left(\begin{array}{c} n \\ s \end{array}\right) F_s \\
-\left(\begin{array}{c} n \\ s \end{array}\right) F_s & \cdots & -\left(\begin{array}{c} n \\ s \end{array}\right) F_{s-1} \\
\left(\begin{array}{c} s \\ s-1 \end{array}\right) F_{s-1} & \cdots & \left(\begin{array}{c} s \\ s-2 \end{array}\right) F_{s-2} \\
\vdots & \ddots & \vdots \\
\left(\begin{array}{c} 1 \\ 0 \end{array}\right) F_1 & \cdots & \left(\begin{array}{c} 1 \\ 0 \end{array}\right) F_0 
\end{pmatrix}, \quad (14a)$$

$$B_8 = \begin{pmatrix} 
-\left(\begin{array}{c} n \\ 0 \end{array}\right) F_0 & \cdots & -\left(\begin{array}{c} n \\ s \end{array}\right) F_s \\
\left(\begin{array}{c} n \\ s \end{array}\right) F_s & \cdots & \left(\begin{array}{c} n \\ s \end{array}\right) F_{s-1} \\
-\left(\begin{array}{c} s \\ s-1 \end{array}\right) F_{s-1} & \cdots & -\left(\begin{array}{c} s \\ s-2 \end{array}\right) F_{s-2} \\
\vdots & \ddots & \vdots \\
-\left(\begin{array}{c} 1 \\ 0 \end{array}\right) F_1 & \cdots & -\left(\begin{array}{c} 1 \\ 0 \end{array}\right) F_0 
\end{pmatrix}, \quad (14b)$$

$$B_9 = \begin{pmatrix} 
\left(\begin{array}{c} n \\ 0 \end{array}\right) L_0 & \cdots & \left(\begin{array}{c} n \\ s \end{array}\right) L_s \\
-\left(\begin{array}{c} n \\ s \end{array}\right) L_s & \cdots & -\left(\begin{array}{c} n \\ s \end{array}\right) L_{s-1} \\
\left(\begin{array}{c} s \\ s-1 \end{array}\right) L_{s-1} & \cdots & \left(\begin{array}{c} s \\ s-2 \end{array}\right) L_{s-2} \\
\vdots & \ddots & \vdots \\
\left(\begin{array}{c} 1 \\ 0 \end{array}\right) L_1 & \cdots & \left(\begin{array}{c} 1 \\ 0 \end{array}\right) L_0 
\end{pmatrix}, \quad (14c)$$

$$B_{10} = \begin{pmatrix} 
-\left(\begin{array}{c} n \\ 0 \end{array}\right) L_0 & \cdots & -\left(\begin{array}{c} n \\ s \end{array}\right) L_s \\
\left(\begin{array}{c} n \\ s \end{array}\right) L_s & \cdots & \left(\begin{array}{c} n \\ s \end{array}\right) L_{s-1} \\
-\left(\begin{array}{c} s \\ s-1 \end{array}\right) L_{s-1} & \cdots & -\left(\begin{array}{c} s \\ s-2 \end{array}\right) L_{s-2} \\
\vdots & \ddots & \vdots \\
-\left(\begin{array}{c} 1 \\ 0 \end{array}\right) L_1 & \cdots & -\left(\begin{array}{c} 1 \\ 0 \end{array}\right) L_0 
\end{pmatrix}, \quad (14d)$$

Obviously,

$$B_8 = -B_7, \quad B_{10} = -B_9.$$
Theorem 10 Let $B_9$ be the matrix defined as in (14d), and let $s$ be even. Then
\[ \|B_9\|_2 = L_{2s}, \]
Moreover,
\[ \|B_{10}\|_2 = L_{2s}. \]

Proof. Replacing $B_7$ by $B_9$ in (15) yields
\[ |\lambda_j(B_9)| \leq \sum_{i=0}^{s} \binom{s}{i} L_i, \quad (j = 0, 1, \ldots, s - 1). \]

Note that $\lambda_{ac} = -1$ is an eigenvalue of $\eta_{sc}$ ($s + 1$ is odd), we obtain the corresponding eigenvalue of $B_9$
\[ \lambda_j(B_9) = \sum_{i=0}^{s} a_i \lambda_{ac}^i = \sum_{i=0}^{s} a_i (-1)^i \]
\[ = -\sum_{i=0}^{s} \binom{s}{i} L_i, \]
where $a_i = (-1)^{i+1} \binom{s}{i} L_i$ in $B_9$.

Obviously,
\[ |\lambda_j(B_9)| = \sum_{i=0}^{s} \binom{s}{i} L_i = \max_{0 \leq j \leq s} |\lambda_j(B_9)|. \quad (16) \]

Since the skew-circulant matrix $B_9$ is normal, combining (1), (8) and (16) yields
\[ \|B_9\|_2 = \max_{0 \leq j \leq s} |\lambda_j(B_9)| = \sum_{i=0}^{s} \binom{s}{i} L_i \]
\[ = L_{2s}. \]

Similarly, we can calculate the identity for $B_{10}$. Then we complete the proof.

Corollary 8 Let $A_{sc}$ be as (6) and $s$ be even. Then
(1) if \((\binom{s}{0} L_p, -\binom{s}{1} L_{1+p}, \ldots, \binom{s}{s} L_{s+p})\) is the first row of $A_{sc}$, then
\[ \|A_{sc}\|_2 = L_{2s+p}, \]
where $\forall p \in \mathbb{N}$.

(2) if \((-\binom{s}{0} L_p, \binom{s}{1} L_{1+p}, \ldots, -\binom{s}{s} L_{s+p})\) is the first row of $A_{sc}$, then
\[ \|A_{sc}\|_2 = L_{2s+p}, \]
where $p \in \mathbb{N}$.

4.2 Spectral norms of skew-circulant matrices with modified \((\binom{s}{0}) L, \binom{s}{1} L_{1+p}, \ldots, \binom{s}{s} L_{s+p}\)

Similarly, set $s$ is even, then we list some odd-order alternative skew-circulant matrices as follows.
\[ \tilde{B}_7 = \begin{pmatrix} \binom{s}{0} L_0 & \ldots & \binom{s}{s} L_s \\ -\binom{s}{0} L_1 & \ldots & \binom{s}{s-1} L_{s-1} \\ \vdots & \ddots & \vdots \\ -\binom{s}{s} L_s & \ldots & \binom{s}{s} L_0 \end{pmatrix}, \quad (17a) \]
\[ \tilde{B}_8 = \begin{pmatrix} \binom{s}{0} L_0 & \ldots & \binom{s}{s} L_s \\ -\binom{s}{s-1} L_{s-1} & \ldots & \binom{s}{s} L_{s-1} \\ \vdots & \ddots & \vdots \\ -\binom{s}{s} L_s & \ldots & \binom{s}{s} L_0 \end{pmatrix}, \quad (17b) \]
\[ \tilde{B}_9 = \begin{pmatrix} \binom{s}{0} L_0 & \ldots & \binom{s}{s} L_s \\ -\binom{s}{s-1} L_{s-1} & \ldots & \binom{s}{s} L_{s-1} \\ \vdots & \ddots & \vdots \\ -\binom{s}{s} L_s & \ldots & \binom{s}{s} L_0 \end{pmatrix}, \quad (17c) \]
\[ \tilde{B}_{10} = \begin{pmatrix} \binom{s}{0} L_0 & \ldots & \binom{s}{s} L_s \\ -\binom{s}{s-1} L_{s-1} & \ldots & \binom{s}{s} L_{s-1} \\ \vdots & \ddots & \vdots \\ -\binom{s}{s} L_s & \ldots & \binom{s}{s} L_0 \end{pmatrix}, \quad (17d) \]

Obviously, \(\tilde{B}_8 = -\tilde{B}_7, \tilde{B}_{10} = -\tilde{B}_9\).

Theorem 11 Let $\tilde{B}_7$ be defined as before, and $s$ be even. Then
\[ \|\tilde{B}_7\|_2 = F_{3s}, \]
and
\[ \|\tilde{B}_8\|_2 = F_{3s}. \]

Corollary 9 Let $A_{sc}$ be defined by (6) and $s$ be even. Then we have
(1) if \((\binom{s}{0} L_p, -\binom{s}{1} L_{1+p}, \ldots, \binom{s}{s} L_{s+p})\) is the first row of $A_{sc}$, then
\[ \|A_{sc}\|_2 = F_{3s+p}, \]
where $\forall p \in \mathbb{N}$.
Theorem 12. Let $B_9$ and $B_{10}$ be the matrix defined by (17c) and (17d), respectively, and let $s$ be even. Then

$$\|B_9\|_2 = L_{3s}, \quad \|B_{10}\|_2 = L_{3s}.$$ 

Corollary 10. Let $A_{sc}$ be defined as in (6) and let $s$ be even. Then

1. If \((-1)^2 L_p, (\cdot)^2_1 L_{1+p}, \ldots, (\cdot)^2 L_{s+p}\) is the first row of $A_{sc}$, then

$$\|A_{sc}\|_2 = L_{3s+p},$$

where $\forall p \in \mathbb{N}$.

2. If \((-1)^2 L_p, (\cdot)^2_1 L_{1+p}, \ldots, (\cdot)^2 L_{s+p}\) is the first row of $A_{sc}$, then

$$\|A_{sc}\|_2 = L_{3s+p},$$

where $\forall p \in \mathbb{N}$.

5 Spectral norms of $g$-circulant matrices

Inspired by the above propositions, we will analyse spectral norms of some given $g$-circulant matrices.

Lemma 1. [24] The $(n+1) \times (n+1)$ matrix $Q_g$ is unitary if and only if

$$(n+1, g) = 1,$$  \hspace{1cm} (18)

where $Q_g$ is a $g$-circulant matrix with first row $e^* = [1, 0, \cdots, 0]$.

Lemma 2. [24] A is a $g$-circulant matrix with first row $[a_0, a_1, \cdots, a_n]$ if and only if

$$A = Q_g C,$$

where

$$C = \text{circ}(a_0, a_1, \cdots, a_n).$$

In the following, we assume that $(n+1, g) = 1$.

5.1 Spectral norms of $g$-circulant matrices with modified \((n) F_i\) and \((n) L_i\)

We list two $(n+1) \times (n+1)$ $g$-circulant matrices as following.

$$B_{11} = \begin{pmatrix} (n)^0 F_1 & \cdots & (n)^n F_n \\ (n)^0 L_0 & \cdots & (n)^n L_n \\ \vdots & \ddots & \vdots \\ (n)^0 F_{n-1} & \cdots & (n)^n L_{n-1} \\ (n)^0 F_{n+1} & \cdots & (n)^n L_{n+1} \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} (n)^0 L_0 & \cdots & (n)^n L_n \\ (n)^0 F_0 & \cdots & (n)^n F_n \\ \vdots & \ddots & \vdots \\ (n)^0 F_{n-1} & \cdots & (n)^n L_{n-1} \\ (n)^0 F_{n+1} & \cdots & (n)^n L_{n+1} \end{pmatrix}.$$  \hspace{1cm} (20a)

Theorem 13. Let $B_{11}$ and $B_{12}$ be defined as the matrix (20a) and (20b), respectively. Then

$$\|B_{11}\|_2 = F_{2n}, \quad \|B_{12}\|_2 = L_{2n}.$$  \hspace{1cm} (20b)

Proof. According to Lemma 1 Lemma 2, the $g$-circulant matrix $B_{11}$ is normal, we claim that the spectral norm of $B_{11}$ is equal to its spectral radius. Applying the irreducible and entrywise nonnegative properties, we claim that $\|B_{11}\|_2$ (i.e., its spectral norm), is equal to its Perron value. We select a $(n+1)$-dimensional column vector $v = (1, 1, \cdots, 1)^T$, then

$$B_{11}v = \sum_{i=0}^{n} \binom{n}{i} F_i v.$$  \hspace{1cm} (21)

Obviously, $\sum_{i=0}^{n} \binom{n}{i} F_i$ is an eigenvalue of $B_{11}$ associated with $v$, which is necessarily the Perron value of $B_{11}$. Employing (1), we obtain

$$\|B_{11}\|_2 = F_{2n}.$$  \hspace{1cm} (22)

Employing the same techniques, we can obtain the equality for $B_{12}$. This completes the proof.  \hspace{1cm} $\square$

Corollary 11. Let $A_g$ be as (7) and $(n+1, g) = 1$. Then

1. If \((-1)^2 L_p, (\cdot)^2_1 F_{1+p}, \cdots, (\cdot)^2 L_{n+p}\) is the first row of $A_g$, then

$$\|A_g\|_2 = F_{2n+p},$$

2. If \((-1)^2 L_p, (\cdot)^2_1 L_{1+p}, \cdots, (\cdot)^2 L_{n+p}\) is the first row of $A_g$, then

$$\|A_g\|_2 = L_{2n+p},$$
where \( \forall p \in \mathbb{N} \).

(2) if \( \binom{n}{p} L_{ap}, \binom{n}{1} L_{1+p}, \ldots, \binom{n}{p} L_{n+p} \) is the first row of \( A_g \), then
\[
\|A_g\|_2 = L_{2n+p},
\]
where \( \forall p \in \mathbb{N} \).

5.2 Spectral norms of \( g \)-circulant matrices with modified \( \binom{n}{i} 2^i F_i \) and \( \binom{n}{i} 2^i L_i \)

We list two \( (n+1) \times (n+1) \) \( g \)-circulant matrices with \( \binom{n}{i} 2^i F_i \) and \( \binom{n}{i} 2^i L_i \). Following the same techniques, we can prove these theorems.

\[
\tilde{B}_{11} = \begin{pmatrix}
\binom{n}{0} 2^0 F_0 & \cdots & \binom{n}{n} 2^n F_n \\
\binom{n}{n-1} 2^{n-1} F_{n-1} & \cdots & \binom{n-1}{n-1} 2^{n-1} F_{n-1} \\
\vdots & \ddots & \vdots \\
\binom{n}{1} 2^1 F_1 & \cdots & \binom{n}{0} 2^0 F_0
\end{pmatrix}
\]

(21a)

\[
\tilde{B}_{12} = \begin{pmatrix}
\binom{n}{0} 2^0 L_0 & \cdots & \binom{n}{n} 2^n L_n \\
\binom{n}{n-1} 2^{n-1} L_{n-1} & \cdots & \binom{n-1}{n-1} 2^{n-1} L_{n-1} \\
\vdots & \ddots & \vdots \\
\binom{n}{1} 2^1 L_1 & \cdots & \binom{n}{0} 2^0 L_0
\end{pmatrix}
\]

(21b)

**Theorem 14** Let \( \tilde{B}_{11} \) and \( \tilde{B}_{12} \) be defined as in (21a) and (21b), respectively. Then
\[
\|\tilde{B}_{11}\|_2 = F_{3n}, \quad \|\tilde{B}_{12}\|_2 = L_{3n}.
\]

**Corollary 12** Let \( A_g \) be as in (7) and \( (n+1, g) = 1 \).

The we have

(1) if \( \binom{n}{0} 2^0 F_0, \binom{n}{1} 2^1 F_{1+p}, \ldots, \binom{n}{p} 2^n F_{n+p} \) is the first row of \( A_g \), then
\[
\|A_g\|_2 = F_{3n+p},
\]
where \( \forall p \in \mathbb{N} \).

(2) if \( \binom{n}{0} 2^0 L_0, \binom{n}{1} 2^1 L_{1+p}, \ldots, \binom{n}{p} 2^n L_{n+p} \) is the first row of \( A_g \), then
\[
\|A_g\|_2 = L_{3n+p},
\]
where \( p \in \mathbb{N} \).

Here, we give a proposition without proof.

**Proposition 1** Let \( A_{g_i} \) \( (i = 1, 2) \) be a \( g_i \)-circulant matrix as in (7), respectively. Then
\[
\|A_{g_1}\|_2 = \|A_{g_2}\|_2,
\]
where \( (n+1, g_1) = 1 \), \( (n+1, g_2) = 1 \) and \( g_1 \neq g_2 \).

6 Numerical examples

**Example 1.** In this example, we give the numerical results for \( B_1 \) and \( B_2 \) in Table 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( |B_1|_2 )</td>
<td>3</td>
<td>8</td>
<td>21</td>
<td>55</td>
<td>144</td>
<td>377</td>
<td>987</td>
</tr>
<tr>
<td>( |B_2|_2 )</td>
<td>7</td>
<td>18</td>
<td>47</td>
<td>123</td>
<td>322</td>
<td>843</td>
<td>2207</td>
</tr>
</tbody>
</table>

**Example 2.** In this example, we list the numerical results for alternative circulant matrices \( B_i \) \( (i = 3, 4, 5, 6) \) in Table 2.

Table 2: Spectral norms of alternative circulant matrices

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( |B_3|_2 )</td>
<td>1</td>
<td>8</td>
<td>55</td>
<td>377</td>
<td>2584</td>
</tr>
<tr>
<td>( |B_4|_2 )</td>
<td>1</td>
<td>8</td>
<td>55</td>
<td>377</td>
<td>2584</td>
</tr>
<tr>
<td>( |B_5|_2 )</td>
<td>3</td>
<td>18</td>
<td>123</td>
<td>843</td>
<td>5778</td>
</tr>
<tr>
<td>( |B_6|_2 )</td>
<td>3</td>
<td>18</td>
<td>123</td>
<td>843</td>
<td>5778</td>
</tr>
</tbody>
</table>

**Example 3.** In this example, we reveal the numerical results for alternative skew-circulant matrices \( B_i \) \( (i = 7, \cdots, 10) \) in Table 3.

Table 3: Spectral norms of alternative skew-circulant matrices

<table>
<thead>
<tr>
<th>( s )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( |B_7|_2 )</td>
<td>4</td>
<td>21</td>
<td>144</td>
<td>987</td>
</tr>
<tr>
<td>( |B_8|_2 )</td>
<td>4</td>
<td>21</td>
<td>144</td>
<td>987</td>
</tr>
<tr>
<td>( |B_9|_2 )</td>
<td>7</td>
<td>47</td>
<td>322</td>
<td>2207</td>
</tr>
<tr>
<td>( |B_{10}|_2 )</td>
<td>7</td>
<td>47</td>
<td>322</td>
<td>2207</td>
</tr>
</tbody>
</table>

**Example 4.** In this example, we show the numerical results for \( B_{11} \) and \( B_{12} \) in Table 4.

Table 4: Spectral norms of \( B_{11} \) and \( B_{12} \)

<table>
<thead>
<tr>
<th>( n+1 )</th>
<th>5</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( |B_{11}|_2 )</td>
<td>21</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>( |B_{12}|_2 )</td>
<td>47</td>
<td>47</td>
<td>322</td>
</tr>
</tbody>
</table>
The above results demonstrate that the identities of spectral norms for the given matrices hold.

7 Conclusion

This paper had discussed the explicit formulations for identity estimations of spectral norms for circulant, skew-circulant matrices and \( g \)-circulant matrices, whose entries are binomial coefficients combined with Fibonacci and Lucas numbers, respectively. Furthermore, if \( a_j \) take other values, we can obtain more interesting identities. The same approaches can be used to verify those identities. By setting different \( p \in \mathbb{N} \), we can obtain much more results. It is an open problem to investigate the properties of \( B_{ii}, (i = 1, 2, \cdots , 12) \), such as the explicit formulations for determinants and inverses, only using the entries in the first row. The economists can use them to construct the optimal filter for some economic model and design the most modern circulant-type filters, investigate the rules of some given model in economics.

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References:


