Spectral norms of circulant-type matrices involving some well-known numbers

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Abstract: In this paper, we investigate spectral norms for circulant-type matrices, including circulant, skew-circulant and g-circulant matrices. The entries are product of binomial coefficients with Fibonacci numbers and Lucas numbers, respectively. We obtain identity estimations for these spectral norms. Employing these approaches, we list some numerical tests to verify our results.

KeyWords: Binomial coefficients, Circulant type, Fibonacci numbers, Lucas numbers, Spectral norms

1 Introduction

Circulant, skew-circulant and g-circulant matrices play important roles in various applications with good foundation. Circulant-type matrix had been applied to the area of discussion about economics, digital image disposal, linear forecast, design of self-regress, etc. For example, the economists can employ spectral norms of those matrices to construct the optimal filter for an economic model, and form the building blocks for most modern circulant-type filters to investigate the rule of certain economics, and so on. The properties of this kind of matrix support lots of benefits for the engineer applications. For the details, please refer to [1, 7, 8, 12, 13, 14, 15, 18, 21, 22, 23, 25], and the references therein. The skew-circulant matrices were collected to construct pre-conditioners for LMF-based ODE codes, Hermitian and skew-Hermitian Toeplitz systems were considered in[3, 10, 11, 17], Lyness employed a skew-circulant matrix to construct an s-dimensional lattice rules in[16].

Recently, there are lots of research on the spectral distribution and norms of circulant-type matrices. In [5], the authors pointed out the processes based on the eigenvalue of circulant-type matrices with i.i.d. entries, furthermore, they claimed that they converged to a Poisson random measures in vague topology. There were discussions about the convergence in probability and distribution of the spectral norm of circulant-type matrices in [6]. The authors in [4] listed the limiting spectral distribution for a class of circulant-type matrices with heavy tailed input sequence. Eric Ngongiep $et\ al.$ showed that the singular values of g-circulants in [19].

Solak established the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries in [20]. İpek investigated an improved estimation for spectral norms in [26]. In this paper, we derive some identity estimates of spectral norms for some circulant-type matrices with product of binomial coefficients with Fibonacci numbers and Lucas numbers, respectively.

The Fibonacci and Lucas sequences $\{F_k\}$ and $\{L_k\}$ are defined by the following recursive relations

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0, F_1 = 1$, and

$$L_n = L_{n-1} + L_{n-2}$$

with $L_0 = 2$, $L_1 = 1$.

The binomial coefficients $\binom{n}{k}$ are, for all natural numbers k, defined by

$$(1+X)^n = \sum_{k\geq 0} \binom{n}{k} X^k.$$

It is clear that, for $k > n$,

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{k} = 0.$$

Let $\binom{n}{k}$ be the k-th binomial coefficient of n, F_k and L_k denotes the k-th Fibonacci and Lucas number, respectively. For anyone $p \in \mathbb{N}$, we have the following formulas [9]

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}F_{i+p}&=F_{2n+p},\\ \sum_{i=0}^{n}\binom{n}{i}L_{i+p}&=L_{2n+p},\end{aligned}\tag{1}$$

and

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}2^iF_{i+p}&=F_{3n+p},\\ \sum_{i=0}^{n}\binom{n}{i}2^iL_{i+p}&=L_{3n+p}.\end{aligned}\tag{2}$$

2 Circulant-type matrices

Definition 1 *[13, 15] A circulant matrix is an $n\times n$ complex matrix of the form*

$$A_c=\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1}\\ a_{n-1} & a_0 & \dots & a_{n-2}\\ a_{n-2} & a_{n-1} & \dots & a_{n-3}\\ \vdots & \vdots & \ddots & \vdots\\ a_1 & a_2 & \dots & a_0\end{pmatrix}_{n\times n}. \tag{3}$$

The first row of A_c is $(a_0, a_1, \cdots, a_j, \cdots, a_{n-1})$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by one positions.

Equivalently, a circulant matrix can be described as a polynomial

$$A_c=f(\eta_c)=\sum_{i=0}^{n-1}a_i\eta_c^i, \tag{4}$$

where

$$\eta_c=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}_{n\times n}.$$

Obviously, $\eta_c^n = I_n$.

We are now in a position to discuss the eigenvalues of A_c. Motivated by the relation between matrix and polynomial, we declare that the eigenvalues of η_c are the corresponding eigenvalues of A_c with the function f in (4), which is

$$\lambda(A_c)=f(\lambda(\eta_c))=\sum_{i=0}^{n-1}a_i\lambda(\eta_c)^i.$$

Since

$$\lambda_j(\eta_c)=\omega^j,\ (j=0,1,\cdots,n-1),$$

where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Thus $\lambda_j(A_c)$ can be calculated via

$$\lambda_j(A_c)=\sum_{i=0}^{n-1}a_i(\omega^j)^i, \tag{5}$$

Similarly, let us recall a skew-circulant matrix.

Definition 2 *[13, 15] A skew-circulant matrix is an $n\times n$ complex matrix of the following form*

$$A_{sc}=\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1}\\ -a_{n-1} & a_0 & \dots & a_{n-2}\\ -a_{n-2} & -a_{n-1} & \dots & a_{n-3}\\ \vdots & \vdots & \ddots & \vdots\\ -a_1 & -a_2 & \dots & a_0\end{pmatrix}. \tag{6}$$

Also, a skew-circulant matrix can be described as matrix polynomial

$$A_{sc}=f(\eta_{sc})=\sum_{i=0}^{n-1}a_i\eta_{sc}^i,$$

where

$$\eta_{sc}=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1\\ -1 & 0 & 0 & \dots & 0\end{pmatrix}_{n\times n}.$$

Obviously, $\eta_{sc}^n = -I_n$.

To calculate the eigenvalues of A_{sc}, for the same reason, we obtain

$$\lambda(A_{sc})=f(\lambda(\eta_{sc}))=\sum_{i=0}^{n-1}a_i\lambda(\eta_{sc})^i.$$

Since

$$\lambda_j(\eta_{sc})=\omega^j\alpha,\ (j=0,1,\cdots,n-1),$$

where

$$\omega=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},\ \ \alpha=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n},$$

so $\lambda_j(A_{sc})$ can be computed via

$$\lambda_j(A_{sc})=\sum_{i=0}^{n-1}a_i(\omega^j\alpha)^i.$$

Definition 3 *[4, 24] A g-circulant matrix is an $n \times n$ complex matrix of the following form*

$$A_g = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-g} & a_{n-g+1} & \cdots & a_{n-g-1} \\ a_{n-2g} & a_{n-2g+1} & \cdots & a_{n-2g-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_g & a_{g+1} & \cdots & a_{g-1} \end{pmatrix}, \quad (7)$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n.

The first row of A_g is $(a_0, a_1, \cdots, a_{n-1})$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by g positions(equivalently, g mod n positions). Note that $g = 1$ or $g = n + 1$ yields the standard $circulant\ matrix$. If $g = n - 1$, then we obtain the so called reverse circulant matrix [4].

Definition 4 *[2] The spectral norm $\|\cdot\|_2$ of a matrix A with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix A^*A:*

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}.$$

where A^* denotes the conjugate transpose of A. Therefore if A is an $n \times n$ real symmetric matrix or A is a normal matrix, then

$$\|A\|_2 = \max_{1 \le i \le n} |\lambda_i|, \quad (8)$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A.

3 Spectral norms of some circulant matrices

We will analyse spectral norms of some given circulant matrices, whose entries are combined binomial coefficients with Fibonacci or Lucas numbers, respectively. For the convenience of the discussion, we set $p = 0$ in (1), and the same conclusions can be deduced for $\forall p \in \mathbb{N}$.

3.1 Spectral norms of some circulant matrices with modified $\binom{n}{i}F_i$ and $\binom{n}{i}L_i$

Definition 5 *Some circulant matrices are defined as the following forms:*

$$B_1 = \begin{pmatrix} \binom{n}{0}F_0 & \cdots & \binom{n}{n}F_n \\ \binom{n}{n}F_n & \cdots & \binom{n}{n-1}F_{n-1} \\ \binom{n}{n-1}F_{n-1} & \cdots & \binom{n}{n-3}F_{n-3} \\ \vdots & \ddots & \vdots \\ \binom{n}{1}F_1 & \cdots & \binom{n}{0}F_0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \binom{n}{0}L_0 & \cdots & \binom{n}{n}L_n \\ \binom{n}{n}L_n & \cdots & \binom{n}{n-1}L_{n-1} \\ \binom{n}{n-1}L_{n-1} & \cdots & \binom{n}{n-3}L_{n-3} \\ \vdots & \ddots & \vdots \\ \binom{n}{1}L_1 & \cdots & \binom{n}{0}L_0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \binom{m}{0}F_0 & \cdots & -\binom{m}{m}F_m \\ -\binom{m}{m}F_m & \cdots & \binom{m}{m-1}F_{m-1} \\ \binom{m}{m-1}F_{m-1} & \cdots & -\binom{m}{m-2}F_{m-2} \\ \vdots & \ddots & \vdots \\ -\binom{m}{1}F_1 & \cdots & \binom{m}{0}F_0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} -\binom{m}{0}F_0 & \cdots & \binom{m}{m}F_m \\ \binom{m}{m}F_m & \cdots & -\binom{m}{m-1}F_{m-1} \\ -\binom{m}{m-1}F_{m-1} & \cdots & \binom{m}{m-2}F_{m-2} \\ \vdots & \ddots & \vdots \\ \binom{m}{1}F_1 & \cdots & -\binom{m}{0}F_0 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} \binom{m}{0}L_0 & \cdots & -\binom{m}{m}L_m \\ -\binom{m}{m}L_m & \cdots & \binom{m}{m-1}L_{m-1} \\ \binom{m}{m-1}L_{m-1} & \cdots & -\binom{m}{m-2}L_{m-2} \\ \vdots & \ddots & \vdots \\ -\binom{m}{1}L_1 & \cdots & \binom{m}{0}L_0 \end{pmatrix},$$

$$B_6 = \begin{pmatrix} -\binom{m}{0}L_0 & \cdots & \binom{m}{m}L_m \\ \binom{m}{m}L_m & \cdots & -\binom{m}{m-1}L_{m-1} \\ -\binom{m}{m-1}L_{m-1} & \cdots & \binom{m}{m-2}L_{m-2} \\ \vdots & \ddots & \vdots \\ \binom{m}{1}L_1 & \cdots & -\binom{m}{0}L_0 \end{pmatrix},$$

where $m,\ n$ are integers, and m is odd.

Obviously,

$$B_4 = -B_3, \quad B_6 = -B_5.$$

Our main results for those matrices are as follows.

Theorem 1 *Let B_1 be the matrix defined as in definition 5. Then we have*

$$\|B_1\|_2 = F_{2n}.$$

Proof. Since the circulant matrix B_1 is normal (see Definition 4), we claim that the spectral norm of B_1 is equal to its spectral radius. Furthermore, applying the irreducible and entrywise nonnegative properties, we claim that $\|B_1\|_2$ ($i.e.$, its spectral norm), is equal to its Perron value. We select an $(n+1)$-dimensional column vector

$$v = (1,\ 1,\ \cdots,\ 1)^T,$$

then

$$B_1 v = \left(\sum_{j=0}^{n} \binom{n}{j} F_j\right) v.$$

Obviously, $\sum_{j=0}^{n} \binom{n}{j} F_j$ is an eigenvalue of B_1 associated with v, which is necessarily the Perron value of B_1. Employing (1), we obtain

$$\|B_1\|_2 = F_{2n}.$$

This completes the proof. □

Using the same approach, we can prove the following result.

Theorem 2 *Let B_2 be the matrix defined as in Definition 5. Then we have*

$$\|B_2\|_2 = L_{2n}.$$

Proof. Using the same techniques of Theorem 1 and irreducibility and entrywise nonnegativity of the normal matrix B_2, we declare that the spectral norm of B_2 is the same as its Perron value.

Let

$$v^T = \underbrace{(1,\ 1,\ \cdots,1)}_{n+1}.$$

Then

$$B_2 v = (\sum_{i=0}^{n} \binom{n}{i} L_i) v.$$

Since $\sum_{i=0}^{n} \binom{n}{i} L_i$ is an eigenvalue of B_2 associated with the positive eigenvector v, it is equal to Perron value of B_2. Combining with the identities, we obtain

$$\|B_2\|_2 = L_{2n},$$

which completes the proof. □

Corollary 1 *Let A_c be defined as (3). For any $p \in \mathbb{N}$, the following statements are true:*

(1) If $\left(\binom{n}{0}F_p, \binom{n}{1}F_{1+p}, \binom{n}{2}F_{2+p}, \cdots, \binom{n}{n}F_{n+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = F_{2n+p}.$$

(2) If $\left(\binom{n}{0}L_p, \binom{n}{1}L_{1+p}, \binom{n}{2}L_{2+p}, \cdots, \binom{n}{n}L_{n+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = L_{2n+p}.$$

Theorem 3 *Let B_3 and B_4 be defined as in Definition 5, respectively, and m be odd. Then*

$$\|B_3\|_2 = F_{2m},\ \ \|B_4\|_2 = F_{2m}.$$

Proof. Noticing (5) and (8), it is clear that the spectral norm of B_3 can be calculated by

$$\begin{aligned}\|B_3\|_2 &= \max_{0\le j\le m} |\lambda_j(B_3)| = \max_{0\le j\le m}\left|\sum_{i=0}^{m} a_i(\omega^j)^i\right| \\ &\le \max_{0\le j\le m}\left\{\sum_{i=0}^{m} |a_i|\cdot|(\omega^j)^i|\right\} = \sum_{i=0}^{m}|a_i|,\end{aligned}$$

where $a_i = (-1)^i\binom{m}{i}F_i$.

Note that, if m is odd, the $m+1$ is even, then

$$\lambda_{j_0}(\eta_c) = \omega^{j_0} = -1$$

is an eigenvalue of η_c, so the identity holds, $i.e.$,

$$\|B_3\|_2 = \sum_{i=0}^{m} |a_i|. \tag{10}$$

Combining (1) and (10) yields $\|B_3\|_2 = F_{2m}$.

In the same manner, we can show $\|B_4\|_2 = F_{2m}$. This completes the proof. □

Corollary 2 *Let A_c be defined as (3) and m is odd. Then we have*

(1) If $\left(\binom{m}{0}F_p, -\binom{m}{1}F_{1+p}, \ldots, -\binom{m}{m}F_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = F_{2m+p},$$

where $\forall p \in \mathbb{N}$.

(2) If $\left(-\binom{m}{0}F_p, \binom{m}{1}F_{1+p}, \ldots, \binom{m}{m}F_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = F_{2m+p},$$

where $\forall p \in \mathbb{N}$.

Theorem 4 *Let B_5 and B_6 be defined as in Definition 5, respectively, and m be odd. Then*

$$\|B_5\|_2 = L_{2m},\ \|B_6\|_2 = L_{2m}. \tag{11}$$

Proof. From proof of Theorem 3 , we see that (11) holds. □

Corollary 3 *Let A_c be defined as (3) and m be odd. Then we have*

(1) If $\left(\binom{m}{0}L_p, -\binom{m}{1}L_{1+p}, \ldots, -\binom{m}{m}L_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = L_{2m+p},$$

where $\forall p \in \mathbb{N}$.

(2) If $\left(-\binom{m}{0}L_p, \binom{m}{1}L_{1+p}, \ldots, \binom{m}{m}L_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = L_{2m+p},$$

where $\forall p \in \mathbb{N}$.

3.2 Spectral norms of some circulant matrices with modified $\binom{n}{i}2^iF_i$ and $\binom{n}{i}2^iL_i$

Definition 6 *The circulant matrices are defined as the following forms*

$$\tilde{B}_1 = \begin{pmatrix} \binom{n}{0}2^0F_0 & \ldots & \binom{n}{n}2^nF_n \\ \binom{n}{n}2^nF_n & \ldots & \binom{n}{n-1}2^{n-1}F_{n-1} \\ \binom{n}{n-1}2^{n-1}F_{n-1} & \ldots & \binom{n}{n-3}2^{n-3}F_{n-3} \\ \vdots & \ddots & \vdots \\ \binom{n}{1}2^1F_1 & \ldots & \binom{n}{0}2^0F_0 \end{pmatrix},$$

$$\tilde{B}_2 = \begin{pmatrix} \binom{n}{0}2^0L_0 & \ldots & \binom{n}{n}2^nL_n \\ \binom{n}{n}2^nL_n & \ldots & \binom{n}{n-1}2^{n-1}L_{n-1} \\ \binom{n}{n-1}2^{n-1}L_{n-1} & \ldots & \binom{n}{n-3}2^{n-3}L_{n-3} \\ \vdots & \ddots & \vdots \\ \binom{n}{1}2^1L_1 & \ldots & \binom{n}{0}2^0L_0 \end{pmatrix},$$

$$\tilde{B}_3 = \begin{pmatrix} \binom{m}{0}2^0F_0 & \ldots & -\binom{m}{m}2^mF_m \\ -\binom{m}{m}2^mF_m & \ldots & \binom{m}{m-1}2^{m-1}F_{m-1} \\ \binom{m}{m-1}2^{m-1}F_{m-1} & \ldots & -\binom{m}{m-2}2^{m-2}F_{m-2} \\ \vdots & \ddots & \vdots \\ -\binom{m}{1}2^1F_1 & \ldots & \binom{m}{0}2^0F_0 \end{pmatrix},$$

$$\tilde{B}_4 = \begin{pmatrix} -\binom{m}{0}2^0F_0 & \ldots & \binom{m}{m}2^mF_m \\ \binom{m}{m}2^mF_m & \ldots & -\binom{m}{m-1}2^{m-1}F_{m-1} \\ -\binom{m}{m-1}2^{m-1}F_{m-1} & \ldots & \binom{m}{m-2}2^{m-2}F_{m-2} \\ \vdots & \ddots & \vdots \\ \binom{m}{1}2^1F_1 & \ldots & -\binom{m}{0}2^0F_0 \end{pmatrix},$$

$$\tilde{B}_5 = \begin{pmatrix} \binom{m}{0}2^0L_0 & \ldots & -\binom{m}{m}2^mL_m \\ -\binom{m}{m}2^mL_m & \ldots & \binom{m}{m-1}2^{m-1}L_{m-1} \\ \binom{m}{m-1}2^{m-1}L_{m-1} & \ldots & -\binom{m}{m-2}2^{m-2}L_{m-2} \\ \vdots & \ddots & \vdots \\ -\binom{m}{1}2^1L_1 & \ldots & \binom{m}{0}2^0L_0 \end{pmatrix},$$

$$\tilde{B}_6 = \begin{pmatrix} -\binom{m}{0}2^0L_0 & \ldots & \binom{m}{m}2^mL_m \\ \binom{m}{m}2^mL_m & \ldots & -\binom{m}{m-1}2^{m-1}L_{m-1} \\ -\binom{m}{m-1}2^{m-1}L_{m-1} & \ldots & \binom{m}{m-2}2^{m-2}L_{m-2} \\ \vdots & \ddots & \vdots \\ \binom{m}{1}2^1L_1 & \ldots & -\binom{m}{0}2^0L_0 \end{pmatrix}$$

where m, n are integers, and m is odd. Obviously, $\tilde{B}_4 = -\tilde{B}_3$, $\tilde{B}_6 = -\tilde{B}_5$.

Employing the same approaches as in the above subsection, we list the main results for those matrices.

Theorem 5 *Let $\tilde{B}_1$ be defined as in Definition 6. Then*

$$\|\tilde{B}_1\|_2 = F_{3n}.$$

Theorem 6 *Let $\tilde{B}_2$ be defined as in Definition 6. The we have*

$$\|\tilde{B}_2\|_2 = L_{3n}.$$

Corollary 4 *Let A_c be defined by (3). For any $p \in \mathbb{N}$, we have*

(1) if $\left(\binom{n}{0}F_p, \binom{n}{1}2F_{1+p}, \binom{n}{2}2^2F_{2+p}, \cdots, \binom{n}{n}2^nF_{n+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = F_{3n+p}.$$

(2) if $\left(\binom{n}{0}L_p, \binom{n}{1}2L_{1+p}, \binom{n}{2}2^2L_{2+p}, \cdots, \binom{n}{n}2^nL_{n+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = L_{3n+p}.$$

Theorem 7 *Let $\tilde{B}_3$ and $\tilde{B}_4$ be defined as in Definition 6 , respectively, and m be odd. Then*

$$\|\tilde{B}_3\|_2 = F_{3m}, \ \ \|\tilde{B}_4\|_2 = F_{3m}.$$

Corollary 5 *Let A_c be defined as in (3) and m be odd. Then*

(1) if $\left(\binom{m}{0}2^0F_p, -\binom{m}{1}2^1F_{1+p}, \ldots, -\binom{m}{m}2^mF_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = F_{3m+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(-\binom{m}{0}2^0F_p, \binom{m}{1}2^1F_{1+p}, \ldots, \binom{m}{m}2^mF_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = F_{3m+p},$$

where $\forall p \in \mathbb{N}$.

Theorem 8 *Let $\tilde{B}_5$ and $\tilde{B}_6$ be defined as in Definition 6, respectively, and m be odd. Then*

$$\|\tilde{B}_5\|_2 = L_{3m}, \ \|\tilde{B}_6\|_2 = L_{3m}. \tag{13}$$

Corollary 6 *Let A_c be as (3) and m be odd. Then we have*

(1) if $\left(\binom{m}{0}2^0L_p, -\binom{m}{1}2^1L_{1+p}, \dots, -\binom{m}{m}2^mL_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = L_{3m+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(-\binom{m}{0}2^0L_p, \binom{m}{1}2^1L_{1+p}, \dots, \binom{m}{m}2^mL_{m+p}\right)$ is the first row of A_c, then

$$\|A_c\|_2 = L_{3m+p},$$

where $\forall p \in \mathbb{N}$.

4 Spectral norms of skew-circulant matrices

4.1 Spectral norms of skew-circulant matrices with modified $\binom{n}{i}F_i$ and $\binom{n}{i}L_i$

An odd-order alternative skew-circulant matrix is defined as follows, where s is even.

$$B_7 = \begin{pmatrix} \binom{s}{0}F_0 & \cdots & \binom{s}{s}F_s \\ -\binom{s}{s}F_s & \cdots & -\binom{s}{s-1}F_{s-1} \\ \binom{s}{s-1}F_{s-1} & \cdots & \binom{s}{s-2}F_{s-2} \\ \vdots & \ddots & \vdots \\ \binom{s}{1}F_1 & \cdots & \binom{s}{0}F_0 \end{pmatrix}, \quad (14a)$$

$$B_8 = \begin{pmatrix} -\binom{s}{0}F_0 & \cdots & -\binom{s}{s}F_s \\ \binom{s}{s}F_s & \cdots & \binom{s}{s-1}F_{s-1} \\ -\binom{s}{s-1}F_{s-1} & \cdots & -\binom{s}{s-2}F_{s-2} \\ \vdots & \ddots & \vdots \\ -\binom{s}{1}F_1 & \cdots & -\binom{s}{0}F_0 \end{pmatrix}, \quad (14b)$$

$$B_9 = \begin{pmatrix} \binom{s}{0}L_0 & \cdots & \binom{s}{s}L_s \\ -\binom{s}{s}L_s & \cdots & -\binom{s}{s-1}L_{s-1} \\ \binom{s}{s-1}L_{s-1} & \cdots & \binom{s}{s-2}L_{s-2} \\ \vdots & \ddots & \vdots \\ \binom{s}{1}L_1 & \cdots & \binom{s}{0}L_0 \end{pmatrix}, \quad (14c)$$

$$B_{10} = \begin{pmatrix} -\binom{s}{0}L_0 & \cdots & -\binom{s}{s}L_s \\ \binom{s}{s}L_s & \cdots & \binom{s}{s-1}L_{s-1} \\ -\binom{s}{s-1}L_{s-1} & \cdots & -\binom{s}{s-2}L_{s-2} \\ \vdots & \ddots & \vdots \\ -\binom{s}{1}L_1 & \cdots & -\binom{s}{0}L_0 \end{pmatrix}. \quad (14d)$$

Obviously,

$$B_8 = -B_7, \quad B_{10} = -B_9.$$

Theorem 9 *Let B_7 and B_8 be defined as in (14a) and (14b), respectively, and s be even. Then*

$$\|B_7\|_2 = F_{2s}, \quad \|B_8\|_2 = F_{2s}.$$

Proof. We use (2) and (8) to calculate the spectral norm of B_7. For all $j = 0, 1, \cdots, s$,

$$\begin{aligned} |\lambda_j(B_7)| &= \left|\sum_{i=0}^{s} a_i(\omega^j\alpha)^i\right| \\ &\le \sum_{i=0}^{s} |a_i| \cdot |(\omega^j\alpha)^i| \quad (15) \\ &= \sum_{i=0}^{s} |a_i| = \sum_{i=0}^{s} \binom{s}{i} F_i, \end{aligned}$$

where $a_i = (-1)^i\binom{s}{i}F_i$.

Since all skew-circulant matrices are normal, we deduce that

$$\|B_7\|_2 = \max_{0\le j\le s} |\lambda_j(B_7)|.$$

If s is even, then $s+1$ is odd. We assert that $\lambda_{sc} = -1$ is an eigenvalue of η_{sc}. We calculate the corresponding eigenvalue of B_7 as follows

$$\begin{aligned} \lambda_{\hat{j}}(B_7) &= \sum_{i=0}^{s} a_i\lambda_{sc}^i = \sum_{i=0}^{s} a_i(-1)^i \\ &= \sum_{i=0}^{s} \binom{s}{i} F_i, \end{aligned}$$

where we have used (2).

Noticing that (15), we claim that $\lambda_{\hat{j}}(B_7)$ is the maximum eigenvalue of B_7), which means

$$\|B_7\|_2 = \sum_{i=0}^{s} \binom{s}{i} F_i.$$

Thus, from (1) we obtain $\|B_7\|_2 = F_{2s}$.

Following the same techniques for B_8, we complete the proof. □

Corollary 7 *Let A_{sc} be defined as (6) and let s be even. Then we have*

(1) if $\left(\binom{s}{0}F_p, -\binom{s}{1}F_{1+p}, \dots, \binom{s}{s}F_{s+p}\right)$ is the first row of A_{sc}, then

$$\|A_{sc}\|_2 = F_{2s+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(-\binom{s}{0}F_p, \binom{s}{1}F_{1+p}, \dots, -\binom{s}{s}F_{s+p}\right)$ is the first row of A_{sc}, then

$$\|A_{sc}\|_2 = F_{2s+p},$$

where $p \in \mathbb{N}$.

Theorem 10 *Let B_9 be the matrix de fined as in (14d), and let s be even. Then*

$$\|B_9\|_2 = L_{2s},$$

Moreover,

$$\|B_{10}\|_2 = L_{2s}.$$

Proof. Replacing B_7 by B_9 in (15) yields

$$|\lambda_j(B_9)| \le \sum_{i=0}^{s} \binom{s}{i} L_i, \quad (j = 0, 1, \cdots, s-1).$$

Note that $\lambda_{sc} = -1$ is an eigenvalue of η_{sc} ($s+1$ is odd), we obtain the corresponding eigenvalue of B_9

$$\lambda_{\bar{j}}(B_9) = \sum_{i=0}^{s} a_i \lambda_{sc}^i = \sum_{i=0}^{s} a_i (-1)^i$$
$$= -\sum_{i=0}^{s} \binom{s}{i} L_i,$$

where $a_i = (-1)^{i+1}\binom{s}{i} L_i$ in B_9.

Obviously,

$$|\lambda_{\bar{j}}(B_9)| = \sum_{i=0}^{s} \binom{s}{i} L_i = \max_{0\le j\le s} |\lambda_j(B_9)|. \quad (16)$$

Since the skew-circulant matrix B_9 is normal, combining (1), (8) and (16) yields

$$\|B_9\|_2 = \max_{0\le j\le s} |\lambda_j(B_9)| = \sum_{i=0}^{s} \binom{s}{i} L_i$$
$$= L_{2s}.$$

Similarly, we can calculate the identity for B_{10}. Then we complete the proof. □

Corollary 8 *Let A_{sc} be as (6) and s be even. Then*

(1) if $\left(\binom{s}{0}L_p, -\binom{s}{1}L_{1+p}, \ldots, \binom{s}{s}L_{s+p}\right)$ is the first row of A_{sc}, then

$$\|A_{sc}\|_2 = L_{2s+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(-\binom{s}{0}L_p, \binom{s}{1}L_{1+p}, \ldots, -\binom{s}{s}L_{s+p}\right)$ is the first row of A_{sc}, then

$$\|A_{sc}\|_2 = L_{2s+p},$$

where $p \in \mathbb{N}$.

4.2 Spectral norms of skew-circulant matrices with modified $\binom{n}{i}2^iF_i$ and $\binom{n}{i}2^iL_i$

Similarly, set s is even, then we list some odd-order alternative skew-circulant matrices as follows.

$$\tilde{B}_7 = \begin{pmatrix} \binom{s}{0}2^0F_0 & \cdots & \binom{s}{s}2^sF_s \\ -\binom{s}{s}2^sF_s & \cdots & -\binom{s}{s-1}2^{s-1}F_{s-1} \\ \binom{s}{s-1}2^{s-1}F_{s-1} & \cdots & \binom{s}{s-2}2^{s-2}F_{s-2} \\ \vdots & \ddots & \vdots \\ \binom{s}{1}2^1F_1 & \cdots & \binom{s}{0}2^0F_0 \end{pmatrix}, \quad (17a)$$

$$\tilde{B}_8 = \begin{pmatrix} -\binom{s}{0}2^0F_0 & \cdots & -\binom{s}{s}2^sF_s \\ \binom{s}{s}2^sF_s & \cdots & \binom{s}{s-1}2^{s-1}F_{s-1} \\ -\binom{s}{s-1}2^{s-1}F_{s-1} & \cdots & -\binom{s}{s-2}2^{s-2}F_{s-2} \\ \vdots & \ddots & \vdots \\ -\binom{s}{1}2^1F_1 & \cdots & -\binom{s}{0}2^0F_0 \end{pmatrix}, \quad (17b)$$

$$\tilde{B}_9 = \begin{pmatrix} \binom{s}{0}2^0L_0 & \cdots & \binom{s}{s}2^sL_s \\ -\binom{s}{s}2^sL_s & \cdots & -\binom{s}{s-1}2^{s-1}L_{s-1} \\ \binom{s}{s-1}2^{s-1}L_{s-1} & \cdots & \binom{s}{s-2}2^{s-2}L_{s-2} \\ \vdots & \ddots & \vdots \\ \binom{s}{1}2^1L_1 & \cdots & \binom{s}{0}2^0L_0 \end{pmatrix}, \quad (17c)$$

$$\tilde{B}_{10} = \begin{pmatrix} -\binom{s}{0}2^0L_0 & \cdots & -\binom{s}{s}2^sL_s \\ \binom{s}{s}2^sL_s & \cdots & \binom{s}{s-1}2^{s-1}L_{s-1} \\ -\binom{s}{s-1}2^{s-1}L_{s-1} & \cdots & -\binom{s}{s-2}2^{s-2}L_{s-2} \\ \vdots & \ddots & \vdots \\ -\binom{s}{1}2^1L_1 & \cdots & -\binom{s}{0}2^0L_0 \end{pmatrix}. \quad (17d)$$

Obviously, $\tilde{B}_8 = -\tilde{B}_7$, $\tilde{B}_{10} = -\tilde{B}_9$.

Theorem 11 *Let $\tilde{B}_7$ be defined as before, and s be even. Then*

$$\|\tilde{B}_7\|_2 = F_{3s},$$

and

$$\|\tilde{B}_8\|_2 = F_{3s}.$$

Corollary 9 *Let A_{sc} be defined by (6) and s be even. Then we have*

(1) if $\left(\binom{s}{0}2^0F_p, -\binom{s}{1}2^1F_{1+p}, \ldots, \binom{s}{s}2^sF_{s+p}\right)$ is the first row of A_{sc}, then

$$\|A_{sc}\|_2 = F_{3s+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(-\binom{s}{0}2^0F_p, \binom{s}{1}2^1F_{1+p}, \ldots, -\binom{s}{s}2^sF_{s+p}\right)$ *is the first row of* A_{sc}*, then*

$$\|A_{sc}\|_2 = F_{3s+p},$$

where $p \in \mathbb{N}$.

Theorem 12 *Let* $\tilde{B}_9$ *and* $\tilde{B}_{10}$ *be the matrix defined by (17c) and (17d), respectively, and let s be even. Then*

$$\|\tilde{B}_9\|_2 = L_{3s}, \quad \|\tilde{B}_{10}\|_2 = L_{3s}.$$

Corollary 10 *Let* A_{sc} *be defined as in (6) and let s be even. Then*

(1) if $\left(\binom{s}{0}2^0L_p, -\binom{s}{1}2^1L_{1+p}, \ldots, \binom{s}{s}2^sL_{s+p}\right)$ *is the first row of* A_{sc}*, then*

$$\|A_{sc}\|_2 = L_{3s+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(-\binom{s}{0}2^0L_p, \binom{s}{1}2^1L_{1+p}, \ldots, -\binom{s}{s}2^sL_{s+p}\right)$ *is the first row of* A_{sc}*, then*

$$\|A_{sc}\|_2 = L_{3s+p},$$

where $p \in \mathbb{N}$.

5 Spectral norms of g-circulant matrices

Inspired by the above propositions, we will analyse spectral norms of some given g-circulant matrices.

Lemma 1 *[24] The* $(n+1)\times(n+1)$ *matrix* Q_g *is unitary if and only if*

$$(n+1, g) = 1, \tag{18}$$

where Q_g *is a g-circulant matrix with first row* $e^* = [1, 0, \cdots, 0]$.

Lemma 2 *[24] A is a g-circulant matrix with first row* $[a_0, a_1, \cdots, a_n]$ *if and only if*

$$A = Q_gC, \tag{19}$$

where

$$C = circ(a_0, a_1, \cdots, a_n).$$

In the following, we assume that $(n+1, g) = 1$.

5.1 Spectral norms of g-circulant matrices with modified $\binom{n}{i}F_i$ and $\binom{n}{i}L_i$

We list two $(n+1)\times(n+1)$ g-circulant matrices as following.

$$B_{11} = \begin{pmatrix} \binom{n}{0}F_0 & \cdots & \binom{n}{n}F_n \\ \binom{n}{n-g+1}F_{n-g+1} & \cdots & \binom{n}{n-g}F_{n-g} \\ \binom{n}{n-2g+1}F_{n-2g+1} & \cdots & \binom{n}{n-2g}F_{n-2g} \\ \vdots & \ddots & \vdots \\ \binom{n}{g}F_g & \cdots & \binom{n}{g-1}F_{g-1} \end{pmatrix}, \tag{20a}$$

$$B_{12} = \begin{pmatrix} \binom{n}{0}L_0 & \cdots & \binom{n}{n}L_n \\ \binom{n}{n-g+1}L_{n-g+1} & \cdots & \binom{n}{n-g}L_{n-g} \\ \binom{n}{n-2g+1}L_{n-2g+1} & \cdots & \binom{n}{n-2g}L_{n-2g} \\ \vdots & \ddots & \vdots \\ \binom{n}{g}L_g & \cdots & \binom{n}{g-1}L_{g-1} \end{pmatrix}. \tag{20b}$$

Theorem 13 *Let* B_{11} *and* B_{12} *be defined as the matrix (20a) and (20b), respectively. Then*

$$\|B_{11}\|_2 = F_{2n}, \quad \|B_{12}\|_2 = L_{2n}.$$

Proof. According to Lemma 1 Lemma 2, the g-circulant matrix B_{11} is normal, we claim that the spectral norm of B_{11} is equal to its spectral radius. Applying the irreducible and entrywise nonnegative properties, we claim that $\|B_{11}\|_2$ (i.e., its spectral norm), is equal to its Perron value. We select a $(n+1)$-dimensional column vector $v = (1, 1, \cdots, 1)^T$, then

$$B_{11}v = \left(\sum_{i=0}^{n}\binom{n}{i}F_i\right)v.$$

Obviously, $\sum_{i=0}^{n}\binom{n}{i}F_i$ is an eigenvalue of B_{11} associated with v, which is necessarily the Perron value of B_{11}. Employing (1), we obtain

$$\|B_{11}\|_2 = F_{2n}.$$

Employing the same techniques, we can obtain the equality for B_{12}. This completes the proof. □

Corollary 11 *Let* A_g *be as (7) and* $(n+1, g) = 1$*. Then*

(1) if $\left(\binom{n}{0}F_p, \binom{n}{1}F_{1+p}, \ldots, \binom{n}{n}F_{n+p}\right)$ *is the first row of* A_g*, then*

$$\|A_g\|_2 = F_{2n+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(\binom{n}{0}L_p, \binom{n}{1}L_{1+p}, \ldots, \binom{n}{n}L_{n+p}\right)$ *is the first row of* A_g*, then*

$$\|A_g\|_2 = L_{2n+p},$$

where $\forall p \in \mathbb{N}$.

5.2 Spectral norms of g-circulant matrices with modified $\binom{n}{i}2^iF_i$ and $\binom{n}{i}2^iL_i$

We list two $(n+1) \times (n+1)$ g-circulant matrices with $\binom{n}{i}2^iF_i$ and $\binom{n}{i}2^iL_i$. Following the same techniques, we can prove these theorems.

$$\tilde{B}_{11} = \begin{pmatrix} \binom{n}{0}2^0F_0 & \cdots & \binom{n}{n}2^nF_n \\ \binom{n}{n}2^nF_n & \cdots & \binom{n}{n-1}2^{n-1}F_{n-1} \\ \binom{n}{n-1}2^{n-1}F_{n-1} & \cdots & \binom{n}{n-2}2^{n-2}F_{n-2} \\ \vdots & \ddots & \vdots \\ \binom{n}{1}2^1F_1 & \cdots & \binom{n}{0}2^0F_0 \end{pmatrix}, \tag{21a}$$

$$\tilde{B}_{12} = \begin{pmatrix} \binom{n}{0}2^0L_0 & \cdots & \binom{n}{n}2^nL_n \\ \binom{n}{n}2^nL_n & \cdots & \binom{n}{n-1}2^{n-1}L_{n-1} \\ \binom{n}{n-1}2^{n-1}L_{n-1} & \cdots & \binom{n}{n-2}2^{n-2}L_{n-2} \\ \vdots & \ddots & \vdots \\ \binom{n}{1}2^1L_1 & \cdots & \binom{n}{0}2^0L_0 \end{pmatrix}. \tag{21b}$$

Theorem 14 *Let* $\tilde{B}_{11}$ *and* $\tilde{B}_{12}$ *be defined as in (21a) and (21b), respectively. Then*

$$\|\tilde{B}_{11}\|_2 = F_{3n}, \ \ \|\tilde{B}_{12}\|_2 = L_{3n}.$$

Corollary 12 *Let* A_g *be as in (7) and* $(n+1, g) = 1$. *The we have*

(1) if $\left(\binom{n}{0}2^0F_p, \binom{n}{1}2^1F_{1+p}, \ldots, \binom{n}{n}2^nF_{n+p}\right)$ *is the first row of* A_g*, then*

$$\|A_g\|_2 = F_{3n+p},$$

where $\forall p \in \mathbb{N}$.

(2) if $\left(\binom{n}{0}2^0L_p, \binom{n}{1}2^1L_{1+p}, \ldots, \binom{n}{n}2^nL_{s+p}\right)$ *is the first row of* A_g*, then*

$$\|A_g\|_2 = L_{3n+p},$$

where $p \in \mathbb{N}$.

Here, we give a proposition without proof.

Proposition 1 *Let* A_{g_i} $(i = 1, 2)$ *be a* g_i*-circulant matrix as in (7), respectively. Then*

$$\|A_{g_1}\|_2 = \|A_{g_2}\|_2,$$

where $(n+1, g_1) = 1, \ (n+1, g_2) = 1$ *and* $g_1 \neq g_2$.

6 Numerical examples

Example 1. In this example, we give the numerical results for B_1 and B_2 in Table 1.

Table 1: Spectral norms of circulant matrices B_1 and B_2

n	2	3	4	5	6	7	8
$\|B_1\|_2$	3	8	21	55	144	377	987
$\|B_2\|_2$	7	18	47	123	322	843	2207

Example 2. In this example, we list the numerical results for alternative circulant matrices B_i $(i = 3, 4, 5, 6)$ in Table 2.

Table 2: Spectral norms of alternative circulant matrices

m	1	3	5	7	9
$\|B_3\|_2$	1	8	55	377	2584
$\|B_4\|_2$	1	8	55	377	2584
$\|B_5\|_2$	3	18	123	843	5778
$\|B_6\|_2$	3	18	123	843	5778

Example 3. In this example, we reveal the numerical results for alternative skew-circulant matrices B_i $(i = 7, \cdots, 10)$ in Table 3.

Table 3: Spectral norms of alternative skew-circulant matrices

s	2	4	6	8
$\|B_7\|_2$	4	21	144	987
$\|B_8\|_2$	4	21	144	987
$\|B_9\|_2$	7	47	322	2207
$\|B_{10}\|_2$	7	47	322	2207

Example 4. In this example, we show the numerical results for B_{11} and B_{12} in Table 4.

Table 4: Spectral norms of B_{11} and B_{12}

$n+1$	5			7		8	
g	2	3	4	5	6	3	5
$\|B_{11}\|_2$	21	21	21	144	144	377	377
$\|B_{12}\|_2$	47	47	47	322	322	843	843

The above results demonstrate that the identities of spectral norms for the given matrices hold.

7 Conclusion

This paper had discussed the explicit formulations for identity estimations of spectral norms for circulant, skew-circulant matrices and g-circulant matrices, whose entries are binomial coefficients combined with Fibonacci and Lucas numbers, respectively. Furthermore, if a_j take other values, we can obtain more interesting identities. The same approaches can be used to verify those identities. By setting different $p \in \mathbb{N}$, we can obtain much more results. It is an open problem to investigate the properties of $B_i, (i = 1, 2, \cdots, 12)$, such as the explicit formulations for determinants and inverses, only using the entries in the first row. The economists can use them to construct the optimal filter for some economic model and design the most modern circulant-type filters, investigate the rules of some given model in economics.

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