Numerical solution of integro-differential equations of fractional order by Laplace decomposition method

CHANGQING YANG
Huaihai Institute of Technology
Department of Science
Cangwu Road 59, Lianyungang
CHINA
cdruby@.126.com

JIANHUA HOU
Huaihai Institute of Technology
Department of Science
Cangwu Road 59, Lianyungang
CHINA
yangcq@hhit.edu.cn

Abstract: In this paper, Laplace decomposition method is developed to solve linear and nonlinear fractional integro-differential equations. The proposed method is based on the application of Laplace transform to nonlinear fractional integro-differential equation. The nonlinear term can easily be handled with the help of Adomian polynomials. The fractional derivative is described in the Caputo sense. The Laplace decomposition method is found to be fast and accurate. Illustrative examples are included to demonstrate the validity and applicability of presented technique and comparison is made with exacting results.

Key–Words: integro-differential equations, Laplace transform, fractional derivative, Adomian polynomials, Padé approximants

1 Introduction

In this paper, we study the Laplace decomposition method for a special kind of nonlinear fractional integro-differential equation

$$D^{\alpha}y(t) = p(t)y(t) + g(t) + \lambda \int_0^t k(t,\tau)F(y(\tau))d\tau \tag{1}$$

and

$$D^{\alpha}y(t) = p(t)y(t) + g(t) + \lambda \int_0^1 k(t,\tau)F(y(\tau))d\tau \tag{2}$$

for $t \in [0, 1]$, with the initial conditions

$$y^{(i)} = \delta_i, i = 0, 1, 2, \cdots, n-1, n-1 < \alpha \leq n, n \in N,$$

where $g \in L^2([0, 1])$, $p \in L^2([0, 1])$, $k \in L^2([0, 1]^2)$ are known functions, $y(t)$ is the unknown function, D^{α} is the Caputo fractional differential operator of order α.

Such equations arise in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems [1, 2].

In recent years, the analytic results on existence and uniqueness of problems solutions to fractional differential equations have been investigated by many authors [3, 4]. Momani [5] has obtained local and global existence and uniqueness solution of the integro-differential equation. Most of nonlinear fractional integro-differential equations do not have exact analytic solution, so approximation and numerical technique must be used. There are only a few of techniques for the solution of fractional integro-differential equations, since it is relatively a new subject in mathematics.

Recently, several numerical methods to solve fractional differential equations and fractional integro-differential equations have been given. Nawaz [6] employed variational iteration method to solve the problem. Seyed Alizadeh and Domairry [7] presented the homotopy perturbation method for solving integro-differential equations. Also, Momani [8] and Qaralleh [9] applied Adomian polynomials to solve fractional integro-differential equations and systems of fractional integro-differential equations. Zhang and Tang [10] presented homotopy analysis method for higher-order fractional integro-differential equations. Yang [11] applied the hybrid of block-pulse function and Chebyshev polynomials to solve nonlinear Fredholm fractional integro-differential equations. In addition, the applications of collocation method [12, 13, 14], wavelet method [15, 16, 17] and spectral method [18, 19] for solution of fractional integro-differential equations.

The Laplace decomposition method is a numerical algorithm to solve nonlinear ordinary, partial differential equations. Khuri [20] used this method for the approximate solution of a class of nonlinear ordinary differential equations. The numerical technique basically illustrates how the Laplace transform can be used to approximate the solution of the nonlinear differential by manipulating the decomposition method which was first introduced by Adomian. In 2006, Agadjanov [21] developed this method for solution of Diffing equation. The Laplace decomposition method was proved to be compatible with the versatile nature of the physical problems and was applied to a wide class of functional equations [22, 23, 24, 25]. To the best of authors knowledge no attempt have been made to exploit this method to solve nonlinear fractional integro-differential equation. Our aim in this paper is to apply this technique to fractional integro-differential equation.

Here we will investigate the construction of the Padé approximate for the functions studied. The main advantage of Padé approximation over Taylor series approximation is that the Taylor series approximation can exhibit oscillations which may produce an approximation error bound. Moreover, Taylor series approximations can never blow up in a finite region. To overcome these demerits we use the Padé approximate.

The Padé approximate is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function. The $[L/M]$ Padé approximate to a formal power series $y(t) = \sum_{i=0}^{\infty} a_i t^i$ is given by:

$$\left[\frac{L}{M}\right] = \frac{P_L(t)}{Q_M(t)} = \frac{p_0 + p_1 t + \cdots + p_L t^L}{1 + q_1 t + \cdots + q_M t^M}. \quad (3)$$

The two polynomials in the numerator and denominator of (4) have no common factor. This means that the formal power series

$$y(t) = \frac{P_L(t)}{Q_M(t)} + O(t^{L+M+1}).$$

In this case Padé approximate $[L/M]$ is unique determined.

In this paper, we applied Laplace transform and Adomian polynomials to solve nonlinear integro-differential equation of fractional order.

The paper organized as follows: In section 2, we introduce some necessary definitions and properties of the fractional calculus theory and Laplace transform. In section 3, we construct our method to approximate the solution of the fractional integro-differential equation (1) and (2). Numerical examples are given in Section 4.

2 Basic definitions

In this section, we give some definitions and properties of the fractional calculus and Laplace transform.

Definition 1 *A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p h_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in space C_μ^n if and only if $f^{(n)} \in C_\mu, n \in N$.*

Definition 2 *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$\begin{aligned} J^\alpha f(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \alpha > 0 \\ J^0 f(t) =& f(t). \end{aligned}$$

Some properties of the operator J^α, are as follows: For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$

$$\begin{aligned} J^\alpha J^\beta f(t) &= J^{\alpha+\beta} f(t), \\ J^\alpha J^\beta f(t) &= J^\beta J^\alpha f(t), \\ J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned}$$

Definition 3 *The fractional derivative D^α of $f(t)$ in the Caputo's sense is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d(\tau), \quad (4)$$

for $n - 1 < \alpha \leq n, n \in N, t > 0, f(t) \in C_{-1}^n$.

Definition 4 *The Laplace transform of a function $f(t), t > 0$ is defined as*

$$\mathscr{L}[f(t)] = F(s) = \int_0^{+\infty} e^{-st} f(t) dt,$$

where s can be either real or complex.

The Laplace transform has several properties, as explained below:

Lemma 5 *Laplace Transform of an Integral: If $F(s) = \mathscr{L}[f(t)]$ then*

$$\mathscr{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}. \quad (5)$$

Definition 6 *Given two functions f and g, we define, for any $t > 0$,*

$$(f * g)(t) = \int_0^t f(x)g(t-x)dx. \quad (6)$$

The function $f * g$ is called the convolution of f and g.

Theorem 7 *The convolution theorem*

$$\mathscr{L}[f * g] = \mathscr{L}[f(t)] \cdot \mathscr{L}[g(t)]. \quad (7)$$

Theorem 8 *The Laplace transform $\mathscr{L}[f(t)]$ of the Caputo derivative is defined as [4]*

$$\mathscr{L}[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad (8)$$

for $n-1 < \alpha \le n$.

3 Analysis of the method

3.1 The method for solution of Volterra integral equation

Firstly, we consider the fractional integro-differential equation of Volterra type. According to Laplace decomposition method we apply Laplace transform first on both sides of (1)

$$\begin{aligned}\mathscr{L}[D^\alpha y(t)] =& \mathscr{L}[p(t)y(t)] + \mathscr{L}[g(t)] \\ &+ \mathscr{L}\left[\lambda \int_0^t k(t,\tau)F(y(\tau))d\tau\right].\end{aligned} \quad (9)$$

Using the differentiation property of Laplace transform (8) we get

$$\begin{aligned}s^\alpha \mathscr{L}[y(t)] - c =& \mathscr{L}[p(t)y(t)] + \mathscr{L}[g(t)] \\ &+ \mathscr{L}\left[\lambda \int_0^t k(t,\tau)F(y(\tau))d\tau\right],\end{aligned} \quad (10)$$

where $c = \sum_{k=0}^{m-1} s^{\alpha-k-1} y^{(k)}(0)$. Thus, the given equation is equivalent to

$$\begin{aligned}\mathscr{L}[y(t)] =& \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathscr{L}[p(t)y(t)] + \frac{1}{s^\alpha}\mathscr{L}[g(t)] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda \int_0^t k(t,\tau)F(y(\tau))d\tau\right].\end{aligned} \quad (11)$$

The second step in Laplace decomposition method is that we represent solution as an infinite series given below

$$y(t) = \sum_{n=0}^{\infty} y_n. \quad (12)$$

The nonlinear operator is decomposed as

$$Ny = F[(y(t)] = \sum_{n=0}^{\infty} A_n(y) \quad (13)$$

where A_n is the Adomian polynomials [26] of $y_0, y_1, y_2, \cdots, y_n, \cdots$ that are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{i=0}^{\infty}\lambda^i y_i)\right]_{\lambda=0}, n = 0, 1, 2, \cdots.$$

For the nonlinear function $Ny = F(y)$ the first Adomian polynomials are given by

$$\begin{aligned}A_0 &= F(y_0), \\ A_1 &= y_1 F^{(1)}(y_0), \\ A_2 &= y_2 F^1(y_0) + \frac{1}{2!} y_1^2 F^{(2)}(y_0), \\ A_3 &= y_3 F^{(1)}(y_0) + y_1 y_2 F^{(2)}(y_0) + \frac{1}{3!} y_1^3 F^{(3)}(y_0), \\ &\vdots \\ A_n &= \sum_{v=1}^{n} c(v,n) F^{(v)}(y_0).\end{aligned}$$

The first index of $c(v,n)$ is the order of derivatives from 1 to n, and the second is the order of the Adomian polynomial. The $c(v,n)$ are products (or sums of products) of v components of f whose subscripts sum to n, divided by the factorial of the number of repeated subscripts.

Substituting (12) and (13) into (11), we will get

$$\begin{aligned}\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] =& \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathscr{L}[g(t)] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[p(t)\sum_{n=0}^{\infty} y_n\right] \\ &+ \frac{\lambda}{s^\alpha}\mathscr{L}\left[\int_0^t k(t,\tau)\sum_{n=0}^{\infty} A_n(y)d\tau\right].\end{aligned} \quad (14)$$

Matching both sides of (14) yields the following iterative algorithm:

$$\mathscr{L}[y_0] = \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathscr{L}[g(t)], \quad (15)$$

$$\begin{aligned}\mathscr{L}[y_1] =& \frac{1}{s^\alpha}\mathscr{L}[p(t)y_0] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda \int_0^t k(t,\tau)A_0(y)d\tau\right],\end{aligned} \quad (16)$$

$$\begin{aligned}\mathscr{L}[y_2] =& \frac{1}{s^\alpha}\mathscr{L}[p(t)y_1] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda \int_0^t k(t,\tau)A_1(y)d\tau\right].\end{aligned} \quad (17)$$

In general, the recursive relation is given by

$$\begin{aligned}\mathscr{L}[y_{n+1}] =& \frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_n\right] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^t k(t,\tau)A_n(y)d\tau\right].\end{aligned} \tag{18}$$

Applying inverse Laplace transform to (15-18), so our required recursive relation is given below

$$y_0(t) = H(t), \tag{19}$$

and

$$\begin{aligned}y_{n+1}(t) =& \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_n\right]\right] + \\ &\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^t k(t,\tau)A_n(y)d\tau\right]\right],\end{aligned} \tag{20}$$

where $H(t)$ is a function that arises from the source term and the prescribed initial conditions. The initial solution is important, the choice of (19) as the initial solution always leads to noise oscillation during the iteration procedure. The modified Laplace decomposition method [27] suggests that the function $H(t)$ defined above in (19) be decomposed into two parts:

$$H(t) = H_1(t) + H_2(t).$$

Instead of iteration procedure (19) and (20), we suggest the following modification

$$y_0(t) = H_1(t), \tag{21}$$

$$\begin{aligned}y_1(t) =& H_2(t) + \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_0\right]\right] + \\ &\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^t k(t,\tau)A_0(y)d\tau\right]\right],\end{aligned} \tag{22}$$

$$\begin{aligned}y_{n+1}(t) =& \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_n\right]\right] + \\ &\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^t k(t,\tau)A_n(y)d\tau\right]\right].\end{aligned} \tag{23}$$

The solution through the modified Laplace decomposition method high depend on the choice of $H_1(t)$ and $H_2(t)$. We will show how to suitably choose $H_1(t)$ and $H_2(t)$ by examples.

3.2 The method for solution of Fredholm integral equation

Secondly, we consider the nonlinear Fredholm integro-differential equation of fractional order. We apply the Laplace transform to both sides of (2)

$$\begin{aligned}&\mathscr{L}[D^\alpha y(t)] = \mathscr{L}[p(t)y(t)] + \mathscr{L}[g(t)] \\ &+ \mathscr{L}\left[\lambda\int_0^1 k(t,\tau)F(y(\tau))d\tau\right].\end{aligned} \tag{24}$$

Using the differentiation property of Laplace transform we can get

$$\begin{aligned}&s^\alpha\mathscr{L}[y(t)] - c = \mathscr{L}[p(t)y(t)] + \mathscr{L}[g(t)] \\ &+ \mathscr{L}\left[\lambda\int_0^1 k(t,\tau)F(y(\tau))d\tau\right],\end{aligned} \tag{25}$$

where $c = \sum_{k=0}^{m-1} s^{\alpha-k-1}y^{(k)}(0)$, and

$$\begin{aligned}\mathscr{L}[y(t)] =& \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathscr{L}[p(t)y(t)] + \frac{1}{s^\alpha}\mathscr{L}[g(t)] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)F(y(\tau))d\tau\right].\end{aligned} \tag{26}$$

In the same way, we represent solution as an infinite series given below

$$y(t) = \sum_{n=0}^{\infty} y_n. \tag{27}$$

The nonlinear operator is decomposed as

$$Ny = F[(y(t)] = \sum_{n=0}^{\infty} A_n(y) \tag{28}$$

Substituting (27)and (28) into (26), we can obtain

$$\begin{aligned}\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] =& \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathscr{L}[g(t)] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[p(t)\sum_{n=0}^{\infty} y_n\right] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)\sum_{n=0}^{\infty} A_n(y)d\tau\right].\end{aligned} \tag{29}$$

Matching both sides of (29) yields the following iterative algorithm:

$$\mathscr{L}[y_0] = \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathscr{L}[g(t)], \tag{30}$$

$$\begin{aligned}\mathscr{L}[y_1] =& \frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_0\right] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)A_0(y)d\tau\right],\end{aligned} \tag{31}$$

$$\begin{aligned}\mathscr{L}[y_2] =& \frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_1\right] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)A_1(y)d\tau\right].\end{aligned} \tag{32}$$

In general, the recursive relation is given by

$$\begin{aligned}\mathscr{L}[y_{n+1}] =& \frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_n\right] \\ &+ \frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^t k(t,\tau)A_n(y)d\tau\right]. \end{aligned} \quad (33)$$

Applying inverse Laplace transform to (30-33), so our required recursive relation is given below

$$y_0(t) = H(t), \quad (34)$$

$$y_{n+1}(t) = \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_n\right]\right] + \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)A_n(y)d\tau\right]\right]. \quad (35)$$

The function $H(t)$ defined in (34) can be decomposed into two parts

$$H(t) = H_1(t) + H_2(t).$$

So, the modified recursion relation is obtained

$$\begin{aligned} y_0(t) &= H_1(t), \\ y_1(t) &= H_2(t) + \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_0\right]\right] + \\ &\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)A_0(y)d\tau\right]\right], \\ y_{n+1}(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[p(t)y_n\right]\right] + \\ &\mathscr{L}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}\left[\lambda\int_0^1 k(t,\tau)A_n(y)d\tau\right]\right]. \end{aligned}$$

4 Numerical examples

In order to show the effectiveness of the Laplace decomposition method for solving integro-differential equations of fractional order, we present some examples. All the results are calculated by using the symbolic calculus software Mathematica.

Example 1 Consider the following linear fractional Volterra integro-differential equation [12]:

$$\begin{aligned} D^{3/4}y(t) =& \frac{6t^{9/4}}{\Gamma(13/4)} + \left(\frac{-t^2e^t}{5}\right)y(t) \\ &+ \int_0^t e^t\tau y(\tau)d\tau \end{aligned} \quad (36)$$

with the initial condition

$$y(0) = 0 \quad (37)$$

and the the exact solution is $y(t) = t^3$. First, we apply the Laplace transform to both sides of (36)

$$\begin{aligned}\mathscr{L}[D^{3/4}y(t)] =& \mathscr{L}\left[\frac{6t^{9/4}}{\Gamma(13/4)}\right] + \mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)y(t)\right] \\ &+ \mathscr{L}\left[\int_0^t e^t\tau y(\tau)d\tau\right]\end{aligned}$$

Using the property of Laplace transform and the initial conditions (37), we get

$$\begin{aligned} s^{\frac{3}{4}}\mathscr{L}[y(t)] =& \mathscr{L}\left[\frac{6t^{9/4}}{\Gamma(13/4)}\right] + \mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)y(t)\right] \\ &+ \mathscr{L}\left[\int_0^t e^t\tau y(\tau)d\tau\right]\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}[y(t)] =& \frac{1}{s^{3/4}}\Bigg\{\mathscr{L}\left[\frac{6t^{9/4}}{\Gamma(13/4)}\right] + \mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)y(t)\right] \\ &+ \mathscr{L}\left[\int_0^t e^t\tau y(\tau)d\tau\right]\Bigg\}.\end{aligned}$$

Substituting (12) and (13) into above equation, we have

$$\begin{aligned}\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] =& \frac{1}{s^{3/4}}\Bigg\{\mathscr{L}\left[\frac{6t^{9/4}}{\Gamma(13/4)}\right] \\ &+ \mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)\sum_{n=0}^{\infty} y_n\right] \\ &+ \mathscr{L}\left[\int_0^t e^t\tau\sum_{n=0}^{\infty} y_n(\tau)d\tau\right]\Bigg\}.\end{aligned} \quad (38)$$

Match both side of (38), we have the following relation:

$$\begin{aligned}\mathscr{L}[y_0] &= \frac{1}{s^{3/4}}\mathscr{L}\left[\frac{6t^{9/4}}{\Gamma(13/4)}\right], \\ \mathscr{L}[y_1] &= \frac{1}{s^{3/4}}\mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)y_0\right] \\ &+ \frac{1}{s^{3/4}}\mathscr{L}\left[\int_0^t e^t\tau y_0(\tau)d\tau\right], \\ \mathscr{L}[y_{n+1}] &= \frac{1}{s^{3/4}}\mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)y_n\right] \\ &+ \frac{1}{s^{3/4}}\mathscr{L}\left[\int_0^t e^t\tau y_n(\tau)d\tau\right].\end{aligned}$$

Applying inverse Laplace transform to above equations we get

$$\begin{aligned}
&y_0 = t^3,\\
&y_1 = \mathscr{L}^{-1}\left[\frac{1}{s^{3/4}}\mathscr{L}\left[\left(\frac{-t^2e^t}{5}\right)t^3\right]\right]\\
&\quad + \mathscr{L}^{-1}\left[\frac{1}{s^{3/4}}\mathscr{L}\left[\int_0^t e^t\tau\cdot\tau^3 d\tau\right]\right] = 0,\\
&y_{n+1} = 0.
\end{aligned}$$

Therefore, the solution is obtained to be

$$y(t) = \sum_{n=0}^{\infty} y_n = t^3.$$

The results are better than the results of [12].

Example 2 Consider the following nonlinear Volterra integro-differential equation with a difference kernel [15]

$$D^{\frac{6}{5}}y(t) = \frac{5}{2\Gamma(4/5)}t^{\frac{4}{5}} - \frac{t^9}{252} + \int_0^t (t-\tau)^2[y(\tau)]^3 d\tau, \tag{39}$$

for $0 \le t < 1$ and with the initial condition

$$y(0) = y'(0) = 0 \tag{40}$$

Applying the Laplace transform to both sides of (39) and using the initial conditions we obtain

$$\begin{aligned}
s^{\frac{6}{5}}\mathscr{L}[y(t)] =& \mathscr{L}\left[\frac{5}{2\Gamma(4/5)}t^{\frac{4}{5}} - \frac{t^9}{252}\right]\\
&+ \mathscr{L}\left[\int_0^t (t-\tau)^2[y(\tau)]^3 d\tau\right].
\end{aligned}$$

Applying convolution theorem (8), we arrive at

$$\begin{aligned}
s^{\frac{6}{5}}\mathscr{L}[y(t)] =& \mathscr{L}\left[\frac{5}{2\Gamma(4/5)}t^{\frac{4}{5}} - \frac{t^9}{252}\right]\\
&+ \mathscr{L}[t^2]\cdot\mathscr{L}\left[y(t)^3\right].
\end{aligned}$$

Hence

$$\mathscr{L}[y(t)] = \frac{2}{s^3} - 1440\frac{1}{s^{56/5}} + \frac{2}{s^{21/5}}\cdot\mathscr{L}\left[y(t)^3\right]. \tag{41}$$

Substituting (12) and (13) into (41) leads to

$$\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] = \frac{2}{s^3} - 1440\frac{1}{s^{56/5}} + \frac{2}{s^{21/5}}\cdot\mathscr{L}\left[\sum_{n=0}^{\infty} A_n\right].$$

So we have following relation:

$$\mathscr{L}[y_0] = \frac{2}{s^3}, \tag{42}$$

$$\mathscr{L}[y_1] = 1440\frac{-1}{s^{56/5}} + \frac{2}{s^{21/5}}\cdot\mathscr{L}\left[\sum_{n=0}^{\infty} A_0\right], \tag{43}$$

$$\mathscr{L}[y_{n+1}] = \frac{2}{s^{21/5}}\cdot\mathscr{L}\left[\sum_{n=0}^{\infty} A_n\right],\ n \ge 1. \tag{44}$$

Taking the inverse Laplace transform of both sides of (42,43), and using the recursive relation (44) gives

$$\begin{aligned}
&y_0 = t^2,\\
&y_1 = 0,\\
&\vdots\\
&y_{n+1} = 0.
\end{aligned}$$

Therefore, the solution is obtained to be

$$y(t) = \sum_{n=0}^{\infty} y_n = t^2,$$

which is the exact solution.

Example 3 Consider the following Volterra integro-differential equation of fractional order

$$D^\alpha y(t) = 1 + \int_0^t y'(\tau)y(\tau)d\tau, \tag{45}$$

for $0 \le t < 1$, $0 < \alpha \le 1$ and with the initial condition $y(0) = 0$. Applying the Laplace transform to both sides of (45) gives

$$\mathscr{L}[D^\alpha y(t)] = \mathscr{L}[1] + \mathscr{L}\left[\int_0^t y'(\tau)y(\tau)d\tau\right],$$

so that

$$s^\alpha\mathscr{L}[y(t)] = \frac{1}{s} + \mathscr{L}\left[\int_0^t y'(\tau)y(\tau)d\tau\right],$$

or equivalently

$$\mathscr{L}[y(t)] = \frac{1}{s^{\alpha+1}} + \frac{1}{s^\alpha}\mathscr{L}\left[\int_0^t y'(\tau)y(\tau)d\tau\right].$$

Substituting the series assumption for $y(t)$ and the Adomian polynomials for $y'y$ as given above in (12) and (13) respectively, we obtain

$$\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] = \frac{1}{s^{\alpha+1}} + \frac{1}{s^\alpha}\mathscr{L}\left[\int_0^t \sum_{n=0}^{\infty} A_n d\tau\right].$$

So we can get the following relation:

$$\mathscr{L}[y_0] = \frac{1}{s^{\alpha+1}}, \quad (46)$$

$$\mathscr{L}[y_{n+1}] = \frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^t A_n d\tau\right], n \geq 0. \quad (47)$$

Taking the inverse Laplace transform of both sides of (46) and (47) gives

$$y_0 = \frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$y_n = \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^t A_{n-1}d\tau\right]\right], n \geq 1$$

The general form of the approximation $y(t)$ is given by

$$y(t) = \sum_{k=0}^{n} C_k t^{(2k+1)\alpha}, \quad (48)$$

where the coefficients are given by

$$C_0 = \frac{1}{\Gamma(1+\alpha)},$$

$$C_1 = C_0C_0\alpha\frac{\Gamma(2\alpha)}{\Gamma(1+3\alpha)},$$

$$C_2 = (C_0C_1 3\alpha + C_0C_1\alpha)\frac{\Gamma(4\alpha)}{\Gamma(1+5\alpha)},$$

$$C_3 = (C_0C_2 5\alpha + C_1C_1 3\alpha + C_2C_0\alpha)\frac{\Gamma(6\alpha)}{\Gamma(1+7\alpha)},$$

$$\vdots$$

$$C_n = (C_0C_n(2n-1)\alpha + C_1C_{n-1}(2n-2)\alpha + \cdots + C_0C_n\alpha)\frac{\Gamma(2n\alpha)}{\Gamma(1+(2n+1)\alpha)}.$$

To consider the behavior of solution of solution for different value of α, we will take advantage of the formula (48) available for $0 < \alpha \leq 1$, and consider the following two special cases: First order case: Setting $\alpha = 1$ in (48), we obtain the approximate solution in a series form as

$$y(t) \approx t + \frac{1}{6}t^3 + \frac{1}{30}t^5 + \frac{17}{2520}t^7 + \frac{29}{22680}t^9 + \frac{431}{2494800}t^{11}.$$

The [5/5] Padé approximate gives

$$y(t) \approx \left[\frac{5}{5}\right] = \frac{-139/3780t^5 + 19/18t^3 + t}{1 + 8/9t^2 - 55/252t^4}.$$

A comparison between the exact and the approximate solutions at 10 points is demonstrated for $n = 5$ in Table1. From Table1, it can be found that the obtained approximate solutions are very close to the exact solution.

Fractional order case: In this case we will examine the equation (46). Setting $\alpha = 1/2$ and $n = 5$

$$y(t) \approx 1.1284t^{1/2} + 0.9578t^{3/2} + 0.6504t^{5/2} + 0.6151t^{7/2} + 0.6039t^{9/2} + 0.8494t^{11/2}.$$

For simplicity, let $t^{1/2} = x$ then,

$$y(x) = 1.1284x + 0.9578x^3 + 0.6504x^5 + 0.6151x^7 + 0.6039x^9 + 0.8494x^{11}.$$

Calculating the [5/5] Padé approximate and recalling that $x = t^{1/2}$, we get

$$y(t) \approx \frac{-0.2798t^{5/2} - 0.2529t^{3/2} + 1.1283t^{1/2}}{1 - 1.0729t + 0.0863t^2}$$

Similarly, we can get the results for $\alpha = 1/3, 1/4$. The obtained numerical results for $\alpha = 1/2, 1/3, 1/4$ and $\alpha = 1$ are summarized in Figure1. The comparisons how that as $\alpha \to 1$, the approximate solutions tend to $y(t) = \sqrt{2}tan(\sqrt{2}t/2)$, which is the exact solution of the equation in the case of $\alpha = 1$.

Table 1: The exact and approximate solutions of Example 3($\alpha = 1$)

t	Exact solution	Numerical solution	Absolute error
0	0	0	0
0.1	1.001670e-001	1.001670e-001	8.7624e-014
0.2	2.013440e-001	2.013440e-001	4.4005e-011
0.3	3.045825e-001	3.045825e-001	1.6426e-009
0.4	4.110194e-001	4.110194e-001	2.1086e-008
0.5	5.219305e-001	5.219303e-001	1.5085e-007
0.6	6.387957e-001	6.387949e-001	7.4741e-007
0.7	7.633858e-001	7.633829e-001	2.8841e-006
0.8	8.978815e-001	8.978722e-001	9.3090e-006
0.9	1.045043e+000	1.045016e+000	2.6341e-005

Example 4 Consider the nonlinear Fredholm fractional integro-differential equation [16, 28]

$$D^{\frac{5}{3}}y(t) = g(t) + \int_0^1 (x+t)^2[y(x)]^3dx, \quad (49)$$

where $g(t) = \frac{6}{\Gamma(1/3)}\sqrt[3]{t} - t^2/7 - t/4 - 1/9$, with the initial condition $y'(0) = y(0) = 0$. First, we apply

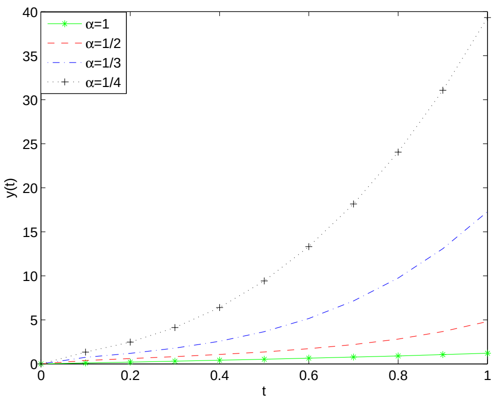

Figure 1: Comparison of approximate solution for $\alpha = 1/4, 1/3, 1/2, 1$.

Laplace transform and its properties to both sides of (49), we have

$$\mathscr{L}[D^{\frac{5}{3}}y(t)] = \mathscr{L}\left[g(t)\right]+\mathscr{L}\left[\int_0^1 (x+t)^2[y(x)]^3dx\right].$$

The initial conditions give

$$s^{\frac{5}{3}}\mathscr{L}[y(t)] = \mathscr{L}\left[g(t)\right]+\mathscr{L}\left[\int_0^1 (x+t)^2[y(x)]^3dx\right].$$

Assuming an infinite series solution of the form (27), we have

$$\begin{aligned}\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] =& \frac{2}{s^3} - \frac{1}{7s^{143}} - \frac{1}{4s^{\frac{11}{3}}} - \frac{1}{9s^{\frac{8}{3}}} \\ &+ \frac{1}{s^{\frac{5}{3}}}\mathscr{L}\left[\int_0^1 (x+t)^2\sum_{n=0}^{\infty} A_n dx\right],\end{aligned} \tag{50}$$

where the nonlinear operator $F(y) = y^3$ is decomposed as in terms of the Adomian polynomials. The first few Adomian polynomials are

$$\begin{aligned}A_0 &= y_0^3,\\ A_1 &= 3y_0^2y_1,\\ A_2 &= 3y_0^2y_2 + 3y_0y_1^2,\\ A_3 &= 2y_0^2y_3 + 6y_0y_1y_2 + y_3^3,\\ &\cdots.\end{aligned}$$

Matching both sides of (50), the components of $y(t)$ can be defined as follows:

$$\mathscr{L}[y_0] = \frac{2}{s^3}, \tag{51}$$

$$\begin{aligned}\mathscr{L}[y_1] =& -\frac{1}{7s^{\frac{14}{3}}} - \frac{1}{4s^{\frac{11}{3}}} - \frac{1}{9s^{\frac{8}{3}}} \\ &+ \frac{1}{s^{\frac{5}{3}}}\mathscr{L}\left[\int_0^1 (x+t)^2\sum_{n=0}^{\infty} A_0 dx\right],\end{aligned} \tag{52}$$

$$\vdots$$

$$\mathscr{L}[y_n] = \frac{1}{s^{\frac{5}{3}}}\mathscr{L}\left[\int_0^1 (x+t)^2\sum_{n=0}^{\infty} A_{n-1} dx\right]. \tag{53}$$

Taking the inverse Laplace transform of (51) gives

$$y_0 = t^2.$$

Applying that $A_0 = y_0^3$, we obtain

$$\begin{aligned}\mathscr{L}[y_1] =& -\frac{1}{7s^{\frac{14}{3}}} - \frac{1}{4s^{\frac{11}{3}}} - \frac{1}{9s^{\frac{8}{3}}} \\ &+ \frac{1}{s^{\frac{5}{3}}}\mathscr{L}\left[\int_0^1 (x+t)^2\cdot x^6 dx\right].\end{aligned}$$

Taking the inverse Laplace transform of both sides of the above equation gives

$$y_1 = 0.$$

So we can get

$$y_n = 0, \quad n > 1.$$

Therefore, the solution is obtained to be

$$y(t) = \sum_{n=0}^{\infty} y_n = t^2,$$

which is the exact solution.

Example 5 Consider the nonlinear Fredholm fractional integro-differential equation[11, 16, 28]

$$D^{\alpha}y(t) = 1 - \frac{1}{4}t + \int_0^1 xt[y(x)]^2dx, \tag{54}$$

for $0 < \alpha \leq 1$ and with the initial condition $y(0) = 0$. Applying the Laplace transform to both sides of (54) gives

$$\mathscr{L}[D^{\alpha}y(t)] = \mathscr{L}\left[1 - \frac{1}{4}t\right] + \mathscr{L}\left[\int_0^1 xt[y(x)]^2dx\right],$$

so that

$$\mathscr{L}[y(t)] = \frac{1}{s^{\alpha+1}} - \frac{1}{4s^{\alpha+2}} + \frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^1 xt[y(x)]^2dx\right].$$

Assuming an infinite series solution of the form (27), we have

$$\mathscr{L}\left[\sum_{n=0}^{\infty} y_n\right] = \frac{1}{s^{\alpha+1}} - \frac{1}{4s^{\alpha+2}} + \frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^1 xt\sum_{n=0}^{\infty} A_n dx\right],$$

where the nonlinear operator $F(y) = y^2$ is decomposed as in terms of the Adomian polynomials. The first few Adomian polynomials are

$$\begin{aligned}
A_0 &= y_0^2,\\
A_1 &= 2y_0y_1,\\
A_2 &= 2y_0y_2 + y_1^2,\\
A_3 &= 2y_0y_3 + 2y_1y_2,\\
A_4 &= 2y_0y_4 + y_2^2 + 2y_1y_3,\\
A_5 &= 2y_0y_5 + 2y_1y_4 + 2y_2^3,\\
A_6 &= 2y_0y_6 + 2y_1y_5 + 2y_2y_4 + y_3^2,\\
&\cdots
\end{aligned}$$

So we can get the following relation:

$$\mathscr{L}[y_0] = \frac{1}{s^{\alpha+1}}, \tag{55}$$

$$\mathscr{L}[y_1] = -\frac{1}{4s^{\alpha+2}} + \frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^1 xt\sum_{n=0}^{\infty} A_1 dx\right], \tag{56}$$

$$\mathscr{L}[y_n] = \frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^1 xt\sum_{n=0}^{\infty} A_n dx\right], n \geq 2. \tag{57}$$

The inverse Laplace transform applied to (55-57) results

$$\begin{aligned}
y_0 &= \frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
y_1 &= -\frac{t^{1+\alpha}}{4\Gamma(2+\alpha)} + \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^1 xtA_0 dx\right]\right],\\
y_n &= \mathscr{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}\left[\int_0^1 xtA_{n-1} dx\right]\right], n \geq 2.
\end{aligned}$$

When $\alpha = 1$ in (54) we can get

$$\begin{aligned}
y_0 &= t,\\
y_1 &= -\frac{t^2}{4\Gamma(3)} + \mathscr{L}^{-1}\left[\frac{1}{s}\mathscr{L}\left[\int_0^1 t\cdot x^3 dx\right]\right] = 0.
\end{aligned}$$

Similarly, we can obtain

$$y_n = 0, \ \ n > 1.$$

So the solution is $y(t) = \sum_{n=0}^{\infty} y_n = t$ which is the exact solution.

When $\alpha = 1/2$, we can obtain

$$\begin{aligned}
y_0 &= 1.1284t^{1/2},\\
y_1 &= 0.1312t^{3/2},\\
y_2 &= 0.0557t^{3/2},\\
y_3 &= 0.0262t^{3/2},\\
&\vdots
\end{aligned}$$

Similarly, we can get the results for $\alpha = 1/3, 1/4$. The obtained numerical results for $\alpha = 1/2, 1/3, 1/4$ and $\alpha = 1$ are summarized in Figure 2. From Figure 2, we can see the numerical solution is in very good agreement with the wavelet methods in [11, 20].

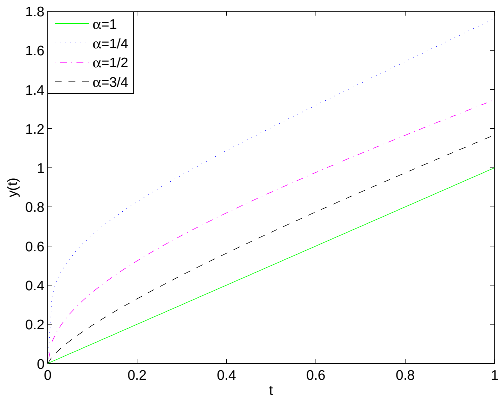

Figure 2: The approximate solution of Example4 for $\alpha = 1/4, 1/3, 1/2, 1$.

5 Conclusion

In this paper, Laplace decomposition method has been successfully applied to finding the approximate solution of nonlinear fractional integro-differential equation. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear fractional integro-differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems. Finally, the behavior of the solution can be formally determined by using the Padé approximate.

The proposed method can be applied for other nonlinear fractional differential equations, systems of differential and integral equation.

Acknowledgements: The authors are grateful to Huaihai institute of technology which provided support through Grant No. Z2011151, Z2012021.

References:

[1] M. Caputo, Linear models of dissipation whose Q is almost frequency in dependent-II, *Geophys. J. Roy. Astron. Soc.* 13, 1967. pp. 529-539.

[2] W. E. Olmstead and R. A. Handelsman, Diffusion in a semi-infinite region with nonlinear surface dissipation, *SIAM Rev.* 18, 1976, pp. 275-291.

[3] K. Diethem and N. J. Ford, Analysis of fractional differential equation, *J. Math. Anal. Appl.* 256, 2002, pp. 229-248.

[4] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[5] S. M. Momani, Local and global existence theorems on fractional integro-differential equations, *J. Fract. Calc.* 18, 2000, pp. 81-86.

[6] Y. Nawaz, Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations, *Comput. Math. Appl.* 61, 2011, pp. 2330-2341.

[7] S. R. Seyed Alizadeh, G. G. Domairry and S. Karimpour, An approximation of the analytical solution of the linear and nonlinear integro-differential equations by homotopy perturbation method, *Acta. Appl. Math.* 104, 2008, pp. 355-366.

[8] S. Momani and M. A. Noor, Numerical methods for four-order fractional integro-differential equations, *Appl. Math. Comput.* 182, 2006, pp. 754-760.

[9] S. Momani and R. Qaralleh, An efficient method for solving systems of fractional integro-differential equations, *Comput. Math. Appl.* 52, 2006, pp. 459-470.

[10] X. Zhang, B. Tang and Y. He, Homotopy analysis method for higher-order fractional integro-differential equations, *Comput. math. appl.* 62, 2011, pp. 3194-3203.

[11] C. Yang, Numerical solution of nonlinear Fredholm integrodifferential equations of fractional order by using hybrid of block-pulse functions and Chebyshev polynomials, *Math. Probl. Eng.* 2011, 2011, pp. 1-11.

[12] E. A. Rawashdeh, Numerical solution of fractional integro-differential equations by collocation method, *Appl. math. comput* 176, 2006, pp. 1-6.

[13] A. Saadatmandi and M. Dehghan, A Legendre collocation method for fractional integro-differential equations, *J. Vib. Control.* 17, 2011, pp. 2050-2058.

[14] X. Ma and C. Huang, Numerical solution of fractional integro-differential equations by a hybrid collocation methodq, *Appl. Math. Comput.* 219, 2013, pp. 6750-6760.

[15] H. Saeedi and M. Mohseni Moghadam, Numerical solution of nonlinear Volterra integro-differential equations of arbitrary order by CAS wavelets, *Commun. Nonlinear. Sci. Numer. Simulat.* 16, 2011 pp. 1216-1226.

[16] L. Zhu and Q. Fan, Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet, *Commun. Nonlinear. Sci. Numer. Simulat.* 17, 2012, pp. 2333-2341.

[17] L. Zhu and Q. Fan, Numerical solution of nonlinear fractional-order Volterra integro-differential equations by SCW, *Commun. Nonlinear. Sci. Numer. Simulat.* 18, 2013, pp. 1203-1213.

[18] P. Mokhtary and F. Ghoreishi, The L^2-convergence of the Legendre spectral Tau matrix formulation for nonlinear fractional integro differential equations, *Numer. Algor.* 58, 2011, pp. 475-496.

[19] M. M. Khader and N. H. Sweilam, On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method, *Appl. Math. Model.* 37, 2013, pp. 9819-9828.

[20] S. A. Khuri, A Laplace decomposition algorithm applied to a class of nonlinear differential equations, *J. Math. Appl.* 1, 2001, pp. 141-155.

[21] E. Yusufoglu, Numerical solution of Duffing equation by the Laplace decomposition algorithm, *Appl. Math. Comput.* 177, 2006, pp. 572-580.

[22] Y. Khan and N. Faraz, A new approach to differential difference equations, *J. Adv. Res. Differ. Equ.* 2(1), 2010, pp. 1-12.

[23] S. Islam, Y. Khan, N. Faraz and F. Austin, Numerical solution of logistic differential equations by using the Laplace decomposition method, *World Applied Sciences J.* 8(9), 2010, pp. 1100-1105.

[24] A. M. Wazwaz, The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations, *Appl. Math. Comput.* 216, 2010, pp. 1304-1309.

[25] M. Khan and M. Hussain, Application of Laplace decomposition method on semi-infinite domain, *Numer. Algor.* 56, 2011, pp. 211-288.

[26] A. M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, *Appl. Math. Comput.* 111, 2002, pp. 33-51.

[27] Y. Khan, A effective modification of the Laplace decomposition method for nonlinear equations, *Int. J. Nonlin. Sci. Num. Simul.* 10, 2009, pp. 1373-1376.

[28] H. Saeedi, M. Mohseni Moghadamb, N. Mollahasani and G. N. Chuev, A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order, *Commun. Nonlinear. Sci. Numer. Simulat.* 16, 2011, pp. 1154-1163.