Almost Periodic Solution of Predator-Prey System with Beddington-DeAngelis Functional Response on Time Scales

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Abstract: This paper is concerned with a predator-prey system with Beddington-DeAngelis functional response on time scales. Based on the theory of calculus on time scales, by using the properties of almost periodic functions and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the existence of a unique globally attractive positive almost periodic solution of the system are obtained. Finally, an example and numerical simulations are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Permanence; Almost periodic solution; Global attractivity; Time scale.

1 Introduction

In the natural world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can not accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales.

In the past few years, different types of ecosystems with periodic coefficients on time scales have been studied extensively, see, for example, [1]-[6] and the references therein. However, upon considering long-term dynamical behaviors, the periodic parameters often turn out to experience certain perturbations, that is, parameters become periodic up to a small error, then one has to consider the ecosystems to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Therefore, if we consider the effects of the environmental factors (e.g. seasonal effects of weather, food supplies, mating habits, and harvesting), the assumption of almost periodicity is more realistic, more important and more general. Due to these reasons, almost periodicity of continuous or discrete ecosystems received more recently researchers' special attention, see [7-10] and the references therein.

However, to the best of the authors' knowledge, there was few papers published on the existence of almost periodic solution of ecosystems on time scales.

Motivated by the above, in the present paper, we shall study an almost periodic predator-prey system with Beddington-DeAngelis functional response on time scales as follows:

$$
\begin{aligned}
x^{\Delta}(t) &= x(t)[r_1(t) - p_1(t)x(t) - d_1(t)x(\sigma(t))] \\
&\quad - \frac{k_1(t)x(t)y(t)}{a(t)+b(t)x(t)+c(t)y(t)}, \\
y^{\Delta}(t) &= y(t)[-r_2(t) - p_2(t)y(t) - d_2(t)y(\sigma(t))] \\
&\quad + \frac{k_2(t)x(t)y(t)}{a(t)+b(t)x(t)+c(t)y(t)}, \qquad (1)
\end{aligned}
$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale. $x(t)$ denotes the density of prey specie and $y(t)$ denote the density of predator species. All the coefficients $a(t), b(t), c(t), r_i(t), p_i(t), d_i(t), k_i(t) (i = 1, 2)$ are continuous, almost periodic functions.

For convenience, we introduce the notation

$$f^u = \sup_{t\in\mathbb{T}} f(t),\ f^l = \inf_{t\in\mathbb{T}} f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that the coefficients of the almost periodic system (1) satisfy

$$
\begin{aligned}
&\min_{i=1,2}\{a^l, b^l, c^l, r_i^l, p_i^l, d_i^l, k_i^l\} > 0, \\
&\max_{i=1,2}\{a^u, b^u, c^u, r_i^u, p_i^u, d_i^u, k_i^u\} < +\infty.
\end{aligned}
$$

The initial condition of system (1) in the form

$$
\begin{aligned}
&x(t_0) = x_0,\ y(t_0) = y_0,\ t_0 \in \mathbb{T}, \\
&x_0 > 0,\ y_0 > 0. \qquad (2)
\end{aligned}
$$

The aim of this paper is, by using the properties of almost periodic functions and constructing a suitable Lyapunov functional, to obtain sufficient conditions for the existence of a unique globally attractive positive almost periodic solution of the system (1).

For relevant definitions and the properties of almost periodic functions, see [11, 12]. In this paper, for each interval $\mathbb{I}$ of $\mathbb{T}$, we denote by $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$.

2 Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t$.

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R})$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous, if $Y^{\Delta}(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

Since $\mathbb{T}$ is almost periodic, then $\sigma(t)$ is almost periodic. The basic theories of calculus on time scales, one can see [13].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1+\mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p\oplus q = p+q+\mu p\, q,\ \ominus p = -\frac{p}{1+\mu p},\ p\ominus q = p\oplus(\ominus q).$$

Lemma 1. *[13] If $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*

(i) $e_0(t, s) \equiv 1$, $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;

(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;

(v) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;

(vi) $(e_p(t, s))^{\Delta} = p(t)e_p(t, s)$.

Lemma 2. *[14] Assume that $a > 0, b > 0$ and $-a \in \mathcal{R}^+$. Then*

$$y^{\Delta}(t) \geq (\leq) b - ay(t),\ y(t) > 0,\ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \geq (\leq)\frac{b}{a}[1+(\frac{ay(t_0)}{b}-1)e_{(-a)}(t, t_0)],\ t \in [t_0, \infty)_{\mathbb{T}}.$$

Lemma 3. *[14] Assume that $a > 0, b > 0$. Then*

$$y^{\Delta}(t) \leq (\geq) y(t)(b-ay(\sigma(t))),\ y(t) > 0,\ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \leq (\geq)\frac{b}{a}[1+(\frac{b}{ay(t_0)}-1)e_{\ominus b}(t, t_0)],\ t \in [t_0, \infty)_{\mathbb{T}}.$$

Let $\mathbb{T}$ be a time scale with at least two positive points, one of them being always one: $1 \in \mathbb{T}$, there exists at least one point $t \in \mathbb{T}$ such that $0 < t \neq 1$. Define the natural logarithm function on the time scale $\mathbb{T}$ by

$$L_{\mathbb{T}}(t) = \int_1^t \frac{1}{\tau}\Delta\tau,\ t \in \mathbb{T} \cap (0, +\infty).$$

Lemma 4. *[15] Assume that $x : \mathbb{T} \to \mathbb{R}^+$ is strictly increasing and $\widetilde{\mathbb{T}} := x(\mathbb{T})$ is a time scale. If $x^\Delta(t)$ exists for $t \in \mathbb{T}^k$, then*

$$\frac{\Delta}{\Delta t} L_{\mathbb{T}}(x(t)) = \frac{x^\Delta(t)}{x(t)}.$$

Lemma 5. *[13] Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$, then $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$\begin{aligned}(fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)\\ &= f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).\end{aligned}$$

3 Main Results

Assume that the coefficients of (1) satisfy

(H_1) $k_2^u M_1 - a^l r_2^l > 0$;

(H_2) $a^l(r_1^l - p_1^u M_1) - k_1^u M_2 > 0$;

(H_3) $k_2^l m_1 - a^u(r_2^u + p_2^u M_2) > 0$.

Lemma 6. *Let $(x(t), y(t))$ be any positive solution of system (1) with initial condition (2). If (H_1) hold, then system (1) is permanent, that is, any positive solution $(x(t), y(t))$ of system (1) satisfies*

$$m_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_1, \quad (3)$$
$$m_2 \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le M_2, \quad (4)$$

especially if $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, then

$$m_1 \le x(t) \le M_1,\ m_2 \le y(t) \le M_2,\ t \in [t_0, +\infty)_{\mathbb{T}},$$

where

$$\begin{aligned} M_1 &= \frac{r_1^u}{d_1^l},\ M_2 = \frac{k_2^u M_1}{a^l d_2^l} - \frac{r_2^l}{d_2^l},\\ m_1 &= \frac{r_1^l - p_1^u M_1}{d_1^u} - \frac{k_1^u M_2}{a^l d_1^u},\\ m_2 &= \frac{k_2^l m_1}{a^u d_2^u} - \frac{r_2^u + p_2^u M_2}{d_2^u}.\end{aligned}$$

Proof. Assume that $(x(t), y(t))$ be any positive solution of system (1) with initial condition (2). From the first equation of system (1), we have

$$x^\Delta(t) \le x(t)(r_1^u - d_1^l x(\sigma(t))). \quad (5)$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} x(t) \le \frac{r_1^u}{d_1^l} := M_1.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x(t) < M_1 + \varepsilon,\ \forall t \in [T_1, +\infty]_{\mathbb{T}}.$$

From the second equation of system (1), when $t \in [T_1, +\infty)_{\mathbb{T}}$,

$$y^\Delta(t) < y(t)\left[\frac{k_2^u(M_1+\varepsilon)}{a^l} - r_2^l - d_2^l y(\sigma(t))\right].$$

Let $\varepsilon \to 0$, then

$$y^\Delta(t) \le y(t)\left[\frac{k_2^u M_1}{a^l} - r_2^l - d_2^l y(\sigma(t))\right]. \quad (6)$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} y(t) = \frac{k_2^u M_1}{a^l d_2^l} - \frac{r_2^l}{d_2^l} := M_2.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$y(t) < M_2 + \varepsilon,\ \forall t \in [T_2, +\infty]_{\mathbb{T}}.$$

On the other hand, from the first equation of system (1), when $t \in [T_2, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} x^\Delta(t) > x(t)\Big[& r_1^l - p_1^u(M_1+\varepsilon) - \frac{k_1^u(M_2+\varepsilon)}{a^l}\\ & - d_1^u x(\sigma(t))\Big].\end{aligned}$$

Let $\varepsilon \to 0$, then

$$x^\Delta(t) \ge x(t)\left[r_1^l - p_1^u M_1 - \frac{k_1^u M_2}{a^l} - d_1^u x(\sigma(t))\right]. \quad (7)$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} x(t) = \frac{r_1^l - p_1^u M_1}{d_1^u} - \frac{k_1^u M_2}{a^l d_1^u} := m_1.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$x(t) > m_1 - \varepsilon,\ \forall t \in [T_3, +\infty]_{\mathbb{T}}.$$

From the second equation of system (1), when $t \in [T_3, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} y^\Delta(t) > y(t)\Big[& \frac{k_2^l(m_1-\varepsilon)}{a^u} - r_2^u - p_2^u(M_2+\varepsilon)\\ & - d_2^u y(\sigma(t))\Big].\end{aligned}$$

Let $\varepsilon \to 0$, then

$$y^{\Delta}(t) \geq y(t)\left[\frac{k_2^l m_1}{a^u} - r_2^u - p_2^u M_2 - d_2^u y(\sigma(t))\right]. \quad (8)$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} y(t) = \frac{k_2^l m_1}{a^u d_2^u} - \frac{r_2^u + p_2^u M_2}{d_2^u} := m_2.$$

Then, for arbitrary small positive constant $\varepsilon > 0$, there exists a $T_4 > T_3$ such that

$$y(t) > m_2 - \varepsilon, \ \forall t \in [T_4, +\infty]_{\mathbb{T}}.$$

In special case, if $m_1 \leq x_0 \leq M_1$, $m_2 \leq y_0 \leq M_2$, by Lemma 2, it follows from (5)-(8) that

$$m_1 \leq x(t) \leq M_1, \ m_2 \leq y(t) \leq M_2, \ t \in [t_0, +\infty)_{\mathbb{T}},$$

This completes the proof. □

Let $S(\mathbb{T})$ be the set of all solutions $(x(t), y(t))$ of system (1) satisfying $m_1 \leq x(t) \leq M_1$, $m_2 \leq y(t) \leq M_2$ for all $t \in \mathbb{T}$.

Lemma 7. $S(\mathbb{T}) \neq \emptyset$.

Proof. By Lemma 6, we see that for any $t_0 \in \mathbb{T}$ with $m_1 \leq x_0 \leq M_1$, $m_2 \leq y_0 \leq M_2$, system (1) has a solution $(x(t), y(t))$ satisfying $m_1 \leq x(t) \leq M_1$, $m_2 \leq y(t) \leq M_2, t \in [t_0, +\infty)_{\mathbb{T}}$. Since $a(t)$, $b(t)$, $c(t)$, $r_i(t)$, $p_i(t)$, $d_i(t)$, $k_i(t)$, $\sigma(t), i = 1, 2$ are almost periodic, there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $a(t + t_n) \to a(t), b(t+t_n) \to b(t), c(t+t_n) \to c(t), r_i(t+t_n) \to r(t), p_i(t + t_n) \to p_i(t), d_i(t + t_n) \to d_i(t), k_i(t + t_n) \to k_i(t), \sigma(t + t_n) \to \sigma(t), i = 1, 2$ as $n \to +\infty$ uniformly on $\mathbb{T}$.

We claim that $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are uniformly bounded and equi-continuous on any bounded interval in $\mathbb{T}$.

In fact, for any bounded interval $[\alpha, \beta]_{\mathbb{T}} \subset \mathbb{T}$, when n is large enough, $\alpha + t_n > t_0$, then $t + t_n > t_0, \forall t \in [\alpha, \beta]_{\mathbb{T}}$. So, $m_1 \leq x(t + t_n) \leq M_1$, $m_2 \leq y(t+t_n) \leq M_2$ for any $t \in [\alpha, \beta]_{\mathbb{T}}$, that is, $\{x(t+t_n)\}$ and $\{y(t + t_n)\}$ are uniformly bounded. On the other hand, $\forall t_1, t_2 \in [\alpha, \beta]_{\mathbb{T}}$, from the mean value theorem of differential calculus on time scales, we have

$$\begin{aligned}
&|x(t_1 + t_n) - x(t_2 + t_n)| \\
&\leq M_1\left[r_1^u + (p_1^u + d_1^u)M_1 + \frac{k_1^u M_2}{a^l + b^l m_1 + c^l m_2}\right] \\
&\times |t_1 - t_2|, \qquad (9) \\
&|y(t_1 + t_n) - y(t_2 + t_n)| \\
&\leq M_2\left[r_2^u + (p_2^u + d_2^u)M_2 + \frac{k_2^u M_1}{a^l + b^l m_1 + c^l m_2}\right] \\
&\times |t_1 - t_2|. \qquad (10)
\end{aligned}$$

The inequalities (9) and (10) show that $\{x(t + t_n)\}$ and $\{y(t + t_n)\}$ are equi-continuous on $[\alpha, \beta]_{\mathbb{T}}$. By the arbitrary of $[\alpha, \beta]_{\mathbb{T}}$, the conclusion is valid.

By Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, we still denote it as $\{t_n\}$, such that

$$x(t + t_n) \to u(t), y(t + t_n) \to v(t),$$

as $n \to +\infty$ uniformly in t on any bounded interval in $\mathbb{T}$. For any $\theta \in \mathbb{T}$, we can assume that $t_n + \theta \geq t_0$ for all n, and let $t \geq 0$, integrate both equations of system (1) from $t_n + \theta$ to $t + t_n + \theta$, we have

$$\begin{aligned}
&x(t + t_n + \theta) - x(t_n + \theta) \\
&= \int_{t_n+\theta}^{t+t_n+\theta} x(s)[r_1(s) - p_1(s)x(s) \\
&- d_1(s)x(\sigma(s))] - \frac{k_1(s)x(s)y(s)}{a(s) + b(s)x(s) + c(s)y(s)}\Delta s \\
&= \int_{\theta}^{t+\theta} x(s + t_n)[r_1(s + t_n) \\
&- p_1(s + t_n)x(s + t_n) - d_1(s + t_n)x(\sigma(s + t_n))] \\
&- \frac{k_1(s + t_n)x(s + t_n)y(s + t_n)}{\Phi(s + t_n)}\Delta s,
\end{aligned}$$

and

$$\begin{aligned}
&y(t + t_n + \theta) - y(t_n + \theta) \\
&= \int_{t_n+\theta}^{t+t_n+\theta} y(s)[-r_2(s) - p_2(s)y(s) \\
&- d_2(s)y(\sigma(s))] + \frac{k_2(s)x(s)y(s)}{a(s) + b(s)x(s) + c(s)y(s)}\Delta s \\
&= \int_{\theta}^{t+\theta} y(s + t_n)[-r_2(s + t_n) \\
&- p_2(s + t_n)y(s + t_n) - d_2(s + t_n)y(\sigma(s + t_n))] \\
&+ \frac{k_2(s + t_n)x(s + t_n)y(s + t_n)}{\Phi(s + t_n)}\Delta s.
\end{aligned}$$

where $\Phi(s+t_n) = a(s+t_n) + b(s+t_n)x(s+t_n) + c(s + t_n)y(s + t_n)$. Using the Lebesgues dominated convergence theorem, we have

$$\begin{aligned}
&u(t + \theta) - u(\theta) \\
&= \int_{\theta}^{t+\theta} x(s)[r_1(s) - p_1(s)x(s) - d_1(s)x(\sigma(s))] \\
&\quad - \frac{k_1(s)x(s)y(s)}{a(s) + b(s)x(s) + c(s)y(s)}\Delta s, \\
&v(t + \theta) - v(\theta) \\
&= \int_{\theta}^{t+\theta} y(s)[-r_2(s) - p_2(s)y(s) - d_2(s)y(\sigma(s))] \\
&\quad + \frac{k_2(s)x(s)y(s)}{a(s) + b(s)x(s) + c(s)y(s)}\Delta s.
\end{aligned}$$

This means that $(u(t), v(t))$ is a solution of system (1), and by the arbitrary of θ, $(u(t), v(t))$ is a solution of system (1) on $\mathbb{T}$. It is clear that

$$m_1 \leq u(t) \leq M_1,\ m_2 \leq v(t) \leq M_2,\ \forall t \in \mathbb{T}.$$

This completes the proof. □

Lemma 8. *In addition to the conditions $(H_1)-(H_3)$, assume further that the coefficients of system (1) satisfy the following conditions:*

(H_4) $p_1^l + \frac{k_2^l b^l m_1}{(a^u+b^u M_1+c^u M_2)^2} - \frac{k_1^u b^u M_2}{(a^l)^2} - \frac{k_2^u}{a^l} > 0$;

(H_5) $p_2^l + \frac{k_1^l}{a^u+b^u M_1+c^u M_2} + \frac{k_2^l c^l m_1}{(a^u+b^u M_1+c^u M_2)^2} - \frac{k_1^u c^u M_2}{(a^l)^2} > 0.$

Then system (1) is globally attractive.

Proof. Let $z_1(t) = (x_1(t), y_1(t))$ and $z_2(t) = (x_2(t), y_2(t))$ be any two positive solutions of system (1). By (H_4) and (H_5), there exists a sufficient small positive constant ε_0 $(0 < \varepsilon_0 < \min\{m_1, m_2\})$ such that

$$\Gamma_1 =: p_1^l + \frac{k_2^l b^l (m_1 - \varepsilon_0)}{(a^u + b^u(M_1+\varepsilon_0) + c^u(M_2+\varepsilon_0))^2} - \frac{k_1^u b^u (M_2+\varepsilon_0)}{(a^l)^2} - \frac{k_2^u}{a^l} > 0, \quad (11)$$

$$\Gamma_2 =: p_2^l + \frac{k_1^l}{a^u + b^u(M_1+\varepsilon_0) + c^u(M_2+\varepsilon_0)} + \frac{k_2^l c^l (m_1 - \varepsilon_0)}{(a^u + b^u(M_1+\varepsilon_0) + c^u(M_2+\varepsilon_0))^2} - \frac{k_1^u c^u (M_2+\varepsilon_0)}{(a^l)^2} > 0. \quad (12)$$

It follows from (3)-(4) that for the above ε_0, there exists a $T > 0$ such that

$$m_1 - \varepsilon_0 < x_i(t) < M_1 + \varepsilon_0,$$
$$m_2 - \varepsilon_0 < y_i(t) < M_2 + \varepsilon_0,$$

for $t \in [T, +\infty)_{\mathbb{T}},\ i = 1, 2$.

Since $x_i(t), y_i(t), i = 1, 2$ are positive, bounded and differentiable functions on $\mathbb{T}$, then there exists two positive, bounded and differentiable functions $f(t), g(t), t \in \mathbb{T}$, such that $x_i(t)(1+f(t)), y_i(t)(1+g(t)), i = 1, 2$ are strictly increasing on $\mathbb{T}$, respectively. By Lemma 3 and Lemma 4, we have

$$\frac{\Delta}{\Delta t} L_{\mathbb{T}}(x_i(t)[1+f(t)]) = \frac{x_i^{\Delta}(t)}{x_i(t)} + \frac{x_i(\sigma(t)) f^{\Delta}(t)}{x_i(t)[1+f(t)]},$$
$$\frac{\Delta}{\Delta t} L_{\mathbb{T}}(y_i(t)[1+g(t)]) = \frac{y_i^{\Delta}(t)}{y_i(t)} + \frac{y_i(\sigma(t)) g^{\Delta}(t)}{y_i(t)[1+g(t)]}.$$

Here, we can choose two functions $f(t)$ and $g(t)$ such that $\frac{|f^{\Delta}(t)|}{1+f(t)}$ and $\frac{|g^{\Delta}(t)|}{1+g(t)}$ be two bounded functions on $\mathbb{T}$, that is,

$$0 < \zeta_1 < \frac{|f^{\Delta}(t)|}{1+f(t)} < \xi_1,\ 0 < \zeta_2 < \frac{|g^{\Delta}(t)|}{1+g(t)} < \xi_2,$$

for all $t \in \mathbb{T}$, where $\zeta_i, \xi_i, i = 1, 2$ are positive constants.

Set

$$V(t) = e_{-\delta}(t,T)|L_{\mathbb{T}}(x_1(t)(1+f)) - L_{\mathbb{T}}(x_2(t)(1+f))| + e_{-\delta}(t,T)|L_{\mathbb{T}}(y_1(t)(1+g)) - L_{\mathbb{T}}(y_2(t)(1+g))|,$$

where $\delta \geq 0$ is a constant (if $\mu(t) = 0$, then $\delta = 0$; if $\mu(t) > 0$, then $\delta > 0$). It follows from the mean value theorem of differential calculus on time scales for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned}
&\frac{1}{M_1+\varepsilon_0}|x_1(t) - x_2(t)| \\
&\leq |L_{\mathbb{T}}(x_1(t)(1+f(t))) - L_{\mathbb{T}}(x_2(t)(1+f(t)))| \\
&\leq \frac{1}{m_1-\varepsilon_0}|x_1(t) - x_2(t)|, \qquad (13)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{M_2+\varepsilon_0}|y_1(t) - y_2(t)| \\
&\leq |L_{\mathbb{T}}(y_1(t)(1+g(t))) - L_{\mathbb{T}}(y_2(t)(1+g(t)))| \\
&\leq \frac{1}{m_2-\varepsilon_0}|y_1(t) - y_2(t)|. \qquad (14)
\end{aligned}$$

Let $\gamma = \min\{\Gamma_1(m_1 - \varepsilon_0), \Gamma_2(m_2 - \varepsilon_0)\}$. We divide the proof into two cases.

Case I. If $\mu(t) > 0$, set $\delta > \max\{\frac{(M_1+\varepsilon_0)\xi_1}{m_1-\varepsilon_0}, \frac{(M_2+\varepsilon_0)\xi_2}{m_2-\varepsilon_0}, \gamma\}$ and $1 - \mu(t)\delta < 0$. Calculating the upper right derivatives of $V(t)$ along the solution of system (1), it follows from (11)-(14), (H_4) and (H_5) that for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned}
&D^+V(t) \\
&= e_{-\delta}(t,T)\mathrm{sgn}(x_1(t) - x_2(t))\Big[\frac{x_1^{\Delta}(t)}{x_1(t)} - \frac{x_2^{\Delta}(t)}{x_2(t)} \\
&\quad + \frac{f^{\Delta}(t)}{1+f(t)}\Big(\frac{x_1(\sigma(t))}{x_1(t)} - \frac{x_2(\sigma(t))}{x_2(t)}\Big)\Big] \\
&\quad - \delta e_{-\delta}(t,T)|L_{\mathbb{T}}(x_1(\sigma(t))(1+f(\sigma(t)))) \\
&\quad - L_{\mathbb{T}}(x_2(\sigma(t))(1+f(\sigma(t))))| \\
&\quad + e_{-\delta}(t,T)\mathrm{sgn}(y_1(t) - y_2(t))\Big[\frac{y_1^{\Delta}(t)}{y_1(t)} - \frac{y_2^{\Delta}(t)}{y_2(t)} \\
&\quad + \frac{g^{\Delta}(t)}{1+g(t)}\Big(\frac{y_1(\sigma(t))}{y_1(t)} - \frac{y_2(\sigma(t))}{y_2(t)}\Big)\Big] \\
&\quad - \delta e_{-\delta}(t,T)|L_{\mathbb{T}}(y_1(\sigma(t))(1+g(\sigma(t)))) \\
&\quad - L_{\mathbb{T}}(y_2(\sigma(t))(1+g(\sigma(t))))|
\end{aligned}$$

$$
\begin{aligned}
&= e_{-\delta}(t,T)\mathrm{sgn}(x_1(t)-x_2(t))\Big[-p_1(t)(x_1(t)\\
&\quad -x_2(t))-d_1(t)(x_1(\sigma(t))-x_2(\sigma(t)))\\
&\quad -k_1(t)\Big(\frac{y_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{y_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\Big)\\
&\quad +\frac{f^{\Delta}(t)}{1+f(t)}\frac{x_1(\sigma(t))x_2(t)-x_1(t)x_2(\sigma(t))}{x_1(t)x_2(t)}\Big]\\
&\quad -\delta e_{-\delta}(t,T)|L_{\mathbb{T}}(x_1(\sigma(t))(1+f(\sigma(t))))\\
&\quad -L_{\mathbb{T}}(x_2(\sigma(t))(1+f(\sigma(t))))|\\
&\quad +e_{-\delta}(t,T)\mathrm{sgn}(y_1(t)-y_2(t))\Big[-p_2(t)(y_1(t)\\
&\quad -y_2(t))-d_2(t)(y_1(\sigma(t))-y_2(\sigma(t)))\\
&\quad +k_2(t)\Big(\frac{x_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{x_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\Big)\\
&\quad +\frac{g^{\Delta}(t)}{1+g(t)}\frac{y_1(\sigma(t))y_2(t)-y_1(t)y_2(\sigma(t))}{y_1(t)y_2(t)}\Big]\\
&\quad -\delta e_{-\delta}(t,T)|L_{\mathbb{T}}(y_1(\sigma(t))(1+g(\sigma(t))))\\
&\quad -L_{\mathbb{T}}(y_2(\sigma(t))(1+g(\sigma(t))))|\\
&\le e_{-\delta}(t,T)\mathrm{sgn}(x_1(t)-x_2(t))\Big[-p_1(t)(x_1(t)\\
&\quad -x_2(t))-d_1(t)(x_1(\sigma(t))-x_2(\sigma(t)))\\
&\quad -k_1(t)\Big(\frac{y_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{y_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad +\frac{y_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{y_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\Big)\\
&\quad +\frac{f^{\Delta}(t)}{1+f(t)}\Big(\frac{x_1(\sigma(t))(x_2(t)-x_1(t))}{x_1(t)x_2(t)}\\
&\quad +\frac{x_1(t)(x_1(\sigma(t))-x_2(\sigma(t)))}{x_1(t)x_2(t)}\Big)\Big]\\
&\quad -e_{-\delta}(t,T)\frac{\delta}{M_1+\varepsilon_0}|x_1(\sigma(t))-x_2(\sigma(t))|\\
&\quad +e_{-\delta}(t,T)\mathrm{sgn}(y_1(t)-y_2(t))\Big[-p_2(t)(y_1(t)\\
&\quad -y_2(t))-d_2(t)(y_1(\sigma(t))-y_2(\sigma(t)))\\
&\quad +k_2(t)\Big(\frac{x_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{x_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad +\frac{x_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{x_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\Big)\\
&\quad +\frac{g^{\Delta}(t)}{1+g(t)}\Big(\frac{y_1(\sigma(t))(y_2(t)-y_1(t))}{y_1(t)y_2(t)}\\
&\quad +\frac{y_1(t)(y_1(\sigma(t))-y_2(\sigma(t)))}{y_1(t)y_2(t)}\Big)\Big]\\
&\quad -e_{-\delta}(t,T)\frac{\delta}{M_2+\varepsilon_0}|y_1(\sigma(t))-y_2(\sigma(t))|\\
&\le -e_{-\delta}(t,T)\Big(p_1(t)-\frac{k_1(t)b(t)y_2(t)}{a^2(t)}\\
&\quad -\frac{k_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}+\frac{k_2(t)b(t)x_2(t)}{\Psi(t)}\\
&\quad +\frac{f^{\Delta}(t)}{1+f(t)}\frac{x_1(\sigma(t))}{x_1(t)x_2(t)}\Big)|x_1(t)-x_2(t)|\\
&\quad -e_{-\delta}(t,T)\Big(p_2(t)+\frac{k_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)}\\
&\quad -\frac{k_1(t)c(t)y_2(t)}{a^2(t)}+\frac{k_2(t)c(t)x_2(t)}{\Psi(t)}\\
&\quad +\frac{g^{\Delta}(t)}{1+g(t)}\frac{y_1(\sigma(t))}{y_1(t)y_2(t)}\Big)|y_1(t)-y_2(t)|\\
&\quad -e_{-\delta}(t,T)\Big(\frac{\delta}{M_1+\varepsilon_0}+d_1(t)\\
&\quad -\frac{|f^{\Delta}(t)|}{1+f(t)}\frac{1}{x_2(t)}\Big)|x_1(\sigma(t))-x_2(\sigma(t))|\\
&\quad -e_{-\delta}(t,T)\Big(\frac{\delta}{M_2+\varepsilon_0}+d_2(t)\\
&\quad -\frac{|g^{\Delta}(t)|}{1+g(t)}\frac{1}{y_2(t)}\Big)|y_1(\sigma(t))-y_2(\sigma(t))|\\
&\le -\Gamma_1(m_1-\varepsilon_0)e_{-\delta}(t,T)|L_{\mathbb{T}}(x_1(t)(1+f(t)))\\
&\quad -L_{\mathbb{T}}(x_2(t)(1+f(t)))|\\
&\quad -\Gamma_2(m_2-\varepsilon_0)e_{-\delta}(t,T)|L_{\mathbb{T}}(y_1(t)(1+g(t)))\\
&\quad -L_{\mathbb{T}}(y_2(t)(1+g(t)))|\\
&= -\gamma V(t), \qquad (15)
\end{aligned}
$$

where $\Psi(t)=(a(t)+b(t)x_1(t)+c(t)y_1(t))(a(t)+b(t)x_2(t)+c(t)y_2(t))$.

By the comparison theorem and (15), we have

$$
\begin{aligned}
V(t) &\le e_{-\gamma}(t,T)V(T)\\
&< 2\Big(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+\frac{M_2+\varepsilon_0}{m_2-\varepsilon_0}\Big)e_{-\gamma}(t,T),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&e_{-\delta}(t,T)(|L_{\mathbb{T}}(x_1(t)(1+f(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+f(t)))|\\
&+|L_{\mathbb{T}}(y_1(t)(1+g(t)))-L_{\mathbb{T}}(y_2(t)(1+g(t)))|)
\end{aligned}
$$

$$< 2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+\frac{M_2+\varepsilon_0}{m_2-\varepsilon_0}\bigg)e_{-\gamma}(t,T),$$

then

$$\frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+\frac{1}{M_2+\varepsilon_0}|y_1(t)-y_2(t)| < 2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+\frac{M_2+\varepsilon_0}{m_2-\varepsilon_0}\bigg)e_{(-\gamma)\ominus(-\delta)}(t,T). \quad (16)$$

Since $1-\mu(t)\delta<0$ and $0<\gamma<\delta$, then $(-\gamma)\ominus(-\delta)<0$. It follows from (16) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

Case II. If $\mu(t)=0$, set $\delta=0$, then $\sigma(t)=t$ and $e_{-\delta}(t,T)=1$. Calculating the upper right derivatives of $V(t)$ along the solution of system (1), it follows from (11)-(14), (H_4) and (H_5) that for $t\in[T,+\infty)_{\mathbb{T}}$,

$$\begin{aligned}
& D^+V(t) \\
= \ & \operatorname{sgn}(x_1(t)-x_2(t))\bigg[\frac{x_1^\Delta(t)}{x_1(t)}-\frac{x_2^\Delta(t)}{x_2(t)}\bigg] \\
& +\operatorname{sgn}(y_1(t)-y_2(t))\bigg[\frac{y_1^\Delta(t)}{y_1(t)}-\frac{y_2^\Delta(t)}{y_2(t)}\bigg] \\
= \ & \operatorname{sgn}(x_1(t)-x_2(t))\bigg[-p_1(t)(x_1(t)-x_2(t)) \\
& -d_1(t)(x_1(t)-x_2(t)) \\
& -k_1(t)\bigg(\frac{y_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{y_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\bigg)\bigg] \\
& +\operatorname{sgn}(y_1(t)-y_2(t))\bigg[-p_2(t)(y_1(t)-y_2(t)) \\
& -d_2(t)(y_1(t)-y_2(t)) \\
& +k_2(t)\bigg(\frac{x_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{x_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\bigg)\bigg] \\
\le \ & \operatorname{sgn}(x_1(t)-x_2(t))\bigg[-p_1(t)(x_1(t)-x_2(t)) \\
& -d_1(t)(x_1(t)-x_2(t)) \\
& -k_1(t)\bigg(\frac{y_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{y_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& +\frac{y_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{y_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\bigg)\bigg] \\
& +\operatorname{sgn}(y_1(t)-y_2(t))\bigg[-p_2(t)(y_1(t)-y_2(t)) \\
& -d_2(t)(y_1(t)-y_2(t)) \\
& +k_2(t)\bigg(\frac{x_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{x_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& +\frac{x_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{x_2(t)}{a(t)+b(t)x_2(t)+c(t)y_2(t)}\bigg) \\
\le \ & -\bigg(p_1(t)-\frac{k_1(t)b(t)y_2(t)}{a^2(t)} \\
& -\frac{k_2(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& +\frac{k_2(t)b(t)x_2(t)}{\Psi(t)}\bigg)|x_1(t)-x_2(t)| \\
& -\bigg(p_2(t)+\frac{k_1(t)}{a(t)+b(t)x_1(t)+c(t)y_1(t)} \\
& -\frac{k_1(t)c(t)y_2(t)}{a^2(t)} \\
& +\frac{k_2(t)c(t)x_2(t)}{\Psi(t)}\bigg)|y_1(t)-y_2(t)| \\
\le \ & -\Gamma_1(m_1-\varepsilon_0)|L_{\mathbb{T}}(x_1(t)(1+f(t))) \\
& -L_{\mathbb{T}}(x_2(t)(1+f(t)))| \\
& -\Gamma_2(m_2-\varepsilon_0)|L_{\mathbb{T}}(y_1(t)(1+g(t))) \\
& -L_{\mathbb{T}}(y_2(t)(1+g(t)))| \\
= \ & -\gamma V(t). \qquad (17)
\end{aligned}$$

By the comparison theorem and (17), we have

$$\begin{aligned}
V(t) \ & \le \ e_{-\gamma}(t,T)V(T) \\
& < \ 2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+\frac{M_2+\varepsilon_0}{m_2-\varepsilon_0}\bigg)e_{-\gamma}(t,T),
\end{aligned}$$

that is,

$$\begin{aligned}
& |L_{\mathbb{T}}(x_1(t)(1+f(t)))-L_{\mathbb{T}}(x_2(t)(1+f(t)))| \\
& +|L_{\mathbb{T}}(y_1(t)(1+g(t)))-L_{\mathbb{T}}(y_2(t)(1+g(t)))| \\
& < 2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+\frac{M_2+\varepsilon_0}{m_2-\varepsilon_0}\bigg)e_{-\gamma}(t,T),
\end{aligned}$$

then

$$\frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+\frac{1}{M_2+\varepsilon_0}|y_1(t)-y_2(t)| < 2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+\frac{M_2+\varepsilon_0}{m_2-\varepsilon_0}\bigg)e_{-\gamma}(t,T). \quad (18)$$

It follows from (18) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

This completes the proof. □

Theorem 9. *Assume that the conditions $(H_1)-(H_5)$ hold, then system (1) has a unique globally attractive positive almost periodic solution.*

Proof. By Lemma 7, there exists a bounded positive solution $u(t)=(u_1(t),u_2(t))\in S(\mathbb{T})$, then there exists a sequence $\{t'_k\}$, $\{t'_k\}\to+\infty$ as $k\to+\infty$, such that $(u_1(t+t'_k),u_2(t+t'_k))$ is a solution of the following system:

$$\begin{aligned} x^{\Delta}(t) &= x(t)[r_1(t+t'_k)-p_1(t+t'_k)x(t)\\ &\quad -d_1(t+t'_k)x(\sigma(t+t'_k))]\\ &\quad -\frac{k_1(t+t'_k)x(t)y(t)}{a(t+t'_k)+b(t+t'_k)x(t)+c(t+t'_k)y(t)},\\ y^{\Delta}(t) &= y(t)[-r_2(t+t'_k)-p_2(t+t'_k)y(t)\\ &\quad -d_2(t+t'_k)y(\sigma(t+t'_k))]\\ &\quad +\frac{k_2(t+t'_k)x(t)y(t)}{a(t+t'_k)+b(t+t'_k)x(t)+c(t+t'_k)y(t)}. \end{aligned}$$

From the above discussion and Lemma 1, we have that not only $\{u_i(t+t'_k)\}, i=1,2$ but also $\{u_i^{\Delta}(t+t'_k)\}, i=1,2$ are uniformly bounded, thus $\{u_i(t+t'_k)\}, i=1,2$ are uniformly bounded and equi-continuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{u_i(t+t_k)\}\subseteq\{u_i(t+t'_k)\}$ such that for any $\varepsilon>0$, there exists a $N(\varepsilon)>0$ with the property that if $m,k>N(\varepsilon)$ then

$$|u_i(t+t_m)-u_i(t+t_k)|<\varepsilon,\ i=1,2.$$

It shows that $u_i(t), i=1,2$ are asymptotically almost periodic functions, then, $\{u_i(t+t_k)\}, i=1,2$ are the sum of an almost periodic function $q_i(t+t_k), i=1,2$ and a continuous function $p_i(t+t_k), i=1,2$ defined on $\mathbb{T}$, that is

$$u_i(t+t_k)=p_i(t+t_k)+q_i(t+t_k),\ \forall t\in\mathbb{T},$$

where

$$\lim_{k\to+\infty}p_i(t+t_k)=0,\ \lim_{k\to+\infty}q_i(t+t_k)=q_i(t),$$

$q_i(t)$ is an almost periodic function. It means that $\lim_{k\to+\infty}u_i(t+t_k)=q_i(t), i=1,2.$

On the other hand

$$\begin{aligned} &\lim_{k\to+\infty}u_i^{\Delta}(t+t_k)\\ &= \lim_{k\to+\infty}\lim_{h\to 0}\frac{u_i(t+t_k+h)-u_i(t+t_k)}{h}\\ &= \lim_{h\to 0}\lim_{k\to+\infty}\frac{u_i(t+t_k+h)-u_i(t+t_k)}{h}\\ &= \lim_{h\to 0}\frac{q_i(t+h)-q_i(t)}{h}. \end{aligned}$$

So, the limit $q_i(t), i=1,2$ exist.

Now we shall prove that $(q_1(t),q_2(t))$ is an almost solution of system (1).

From the properties of almost periodic function, there exists a sequence $\{t_n\}$, $t_n\to+\infty$ as $n\to+\infty$, such that $a(t+t_n)\to a(t), b(t+t_n)\to b(t), c(t+t_n)\to c(t), r_i(t+t_n)\to r(t), p_i(t+t_n)\to p_i(t), d_i(t+t_n)\to d_i(t), k_i(t+t_n)\to k_i(t), \sigma(t+t_n)\to\sigma(t), i=1,2$ as $n\to+\infty$ uniformly on $\mathbb{T}$.

It is easy to know that $u_i(t+t_n)=q_i(t), i=1,2$ as $n\to+\infty$, then we have

$$\begin{aligned} &q_1^{\Delta}(t)\\ &= \lim_{n\to+\infty}u_1^{\Delta}(t+t_n)\\ &= \lim_{n\to+\infty}u_1(t+t_n)[r_1(t+t_n)\\ &\quad -p_1(t+t_n)u_1(t+t_n)\\ &\quad -d_1(t+t_n)u_1(\sigma(t+t_n))]\\ &\quad -\frac{k_1(t+t_n)u_1(t+t_n)u_2(t+t_n)}{\Theta(t+t_n)}\\ &= q_1(t)[r_1(t)-p_1(t)q_1(t)-d_1(t)q_1(\sigma(t))]\\ &\quad -\frac{k_1(t)q_1(t)q_2(t)}{a(t)+b(t)q_1(t)+c(t)q_2(t)},\\ &q_2^{\Delta}(t)\\ &= \lim_{n\to+\infty}u_2^{\Delta}(t+t_n)\\ &= \lim_{n\to+\infty}u_2(t+t_n)[-r_2(t+t_n)\\ &\quad -p_2(t+t_n)u_2(t+t_n)\\ &\quad -d_2(t+t_n)y(\sigma(t+t_n))]\\ &\quad +\frac{k_2(t+t_n)u_1(t+t_n)u_2(t+t_n)}{\Theta(t+t_n)}\\ &= q_2(t)[-r_2(t)-p_2(t)q_2(t)-d_2(t)q_2(\sigma(t))]\\ &\quad +\frac{k_2(t)q_1(t)q_2(t)}{a(t)+b(t)q_1(t)+c(t)q_2(t)}, \end{aligned}$$

where

$$\begin{aligned} \Theta(t+t_n) &= a(t+t_n)+b(t+t_n)u_1(t+t_n)\\ &\quad +c(t+t_n)u_2(t+t_n). \end{aligned}$$

This proves that $(q_1(t),q_2(t))$ is a positive almost periodic solution of system (1). By Lemma 8, it follows that system (1) has a unique globally attractive positive almost periodic solution. This completes the proof. □

4 An Example

Consider the following almost periodic predator-prey system with Beddington-DeAngelis functional response on time scales

$$
\begin{aligned}
x^{\Delta}(t) &= x(t)[10+\cos(\sqrt{2}t) - x(t) - x(\sigma(t))] \\
&\quad - \frac{(1+0.5\cos t)x(t)y(t)}{1+x(t)+(0.03+0.01\cos t)y(t)}, \\
y^{\Delta}(t) &= y(t)[-2-\cos(\sqrt{2}t) - 2y(t) - y(\sigma(t))] \\
&\quad + \frac{(3+\cos t)x(t)y(t)}{1+x(t)+(0.03+0.01\cos t)y(t)}. \quad (19)
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
& r_1^u = 11, r_1^l = 9, r_2^u = 3, r_2^l = 1, p_1^u = p_1^l = 1, \\
& p_2^u = p_2^l = 2, d_1^u = d_1^l = 1, d_2^u = d_2^l = 1, \\
& k_1^u = 1.5, k_1^l = 0.5, k_2^u = 4, k_2^l = 2, \\
& a^u = a^l = 1, b^u = b^l = 1, c^u = 0.04, c^l = 0.02.
\end{aligned}
$$

By a direct calculation, we can get

(H_1) $k_2^u M_1 - a^l r_2^l = 0.1634 > 0$;

(H_2) $a^l(r_1^l - p_1^u M_1) - k_1^u M_2 = 0.0012 > 0$;

(H_3) $k_2^l m_1 - a^u(r_2^u + p_2^u M_2) = 0.1232 > 0$;

(H_4) $p_1^l + \frac{k_2^l b^l m_1}{(a^u+b^u M_1+c^u M_2)^2} - \frac{k_1^u b^u M_2}{(a^l)^2} - \frac{k_2^u}{a^l}$ $= 0.0019 > 0$;

(H_5) $p_2^l + \frac{k_1^l}{a^u+b^u M_1+c^u M_2} + \frac{k_2^l c^l m_1}{(a^u+b^u M_1+c^u M_2)^2}$ $- \frac{k_1^u c^u M_2}{(a^l)^2} = 0.046 > 0$.

That is, the conditions $(H_1) - (H_5)$ hold. According to Theorems 9, system (19) has a unique globally attractive positive almost periodic solution. For dynamic simulations of system (19) with $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, see Figures 1 and 2, respectively.

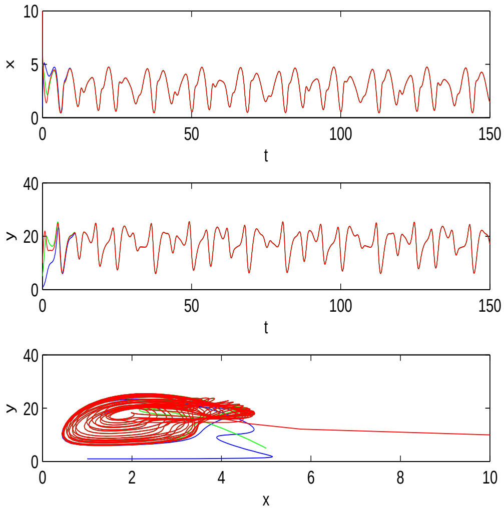

Figure 1: $\mathbb{T} = \mathbb{R}$. Dynamics behavior of system (19) with initial condition $(x(0), y(0)) = \{(1,1); (5,5); (10,10)\}$.

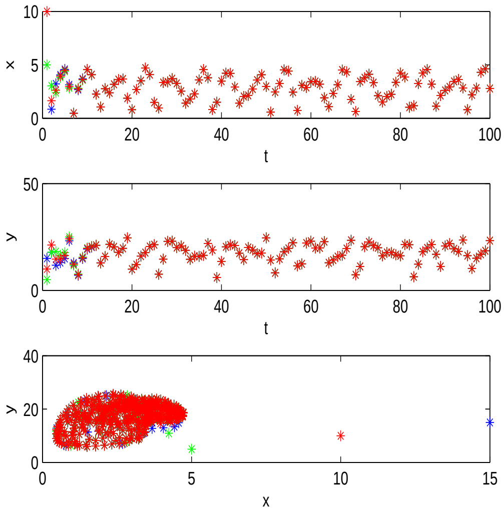

Figure 2: $\mathbb{T} = \mathbb{Z}$. Dynamics behavior of system (19) with initial condition $(x(1), y(1)) = \{(5,5); (10,10); (15,15)\}$.

5 Conclusion

This paper is focused on the existence of a unique globally attractive positive almost periodic solution of a predator-prey system with Beddington-DeAngelis functional response on time scales. The methods used in this paper are completely new, and the methods that can be applied to many other ecosystems.

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