On \( s \)-quasinormally embedded or weakly \( s \)-permutable subgroups of finite groups

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Abstract: Suppose that \( G \) is a finite group and \( H \) is a subgroup of \( G \). \( H \) is said to be \( s \)-quasinormally embedded in \( G \) if for each prime \( p \) dividing the order of \( H \), a Sylow \( p \)-subgroup of \( H \) is also a Sylow \( p \)-subgroup of some \( s \)-quasinormal subgroup of \( G \); \( H \) is said to be weakly \( s \)-permutable in \( G \) if there is a subnormal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_{sG} \), where \( H_{sG} \) is the subgroup of \( H \) generated by all those subgroups of \( H \) which are \( s \)-quasinormal in \( G \). We fix in every non-cyclic Sylow subgroup \( P \) of \( G \) some subgroup \( D \) satisfying \( 1 < |D| < |P| \) and study the structure of \( G \) under the assumption that every subgroup \( H \) of \( P \) with \( |H| = |D| \) is either \( s \)-quasinormally embedded or weakly \( s \)-permutable in \( G \). Some recent results are generalized and unified.

Key Words: \( s \)-quasinormally embedded subgroup; Weakly \( s \)-permutable subgroup; Solvable groups; Saturated formation; Finite groups.

1 Introduction

All groups considered in this paper are finite. We use conventional notions and notation. \( G \) always means a group, \( |G| \) denotes the order of \( G \) and \( \pi(G) \) denotes the set of all primes dividing \( |G| \). Let \( \mathcal{F} \) be a class of groups. We call \( \mathcal{F} \) a formation, provided that (1) if \( G \in \mathcal{F} \) and \( H \leq G \), then \( G/H \in \mathcal{F} \), and (2) if \( G/M \) and \( G/N \) are in \( \mathcal{F} \), then \( G/\langle M \cap N \rangle \) is in \( \mathcal{F} \) for any normal subgroups \( M, N \) of \( G \). A formation \( \mathcal{F} \) is said to be saturated if \( G/\Phi(G) \in \mathcal{F} \) implies that \( G \in \mathcal{F} \). In this paper, \( \mathcal{W} \) will denote the class of all supersolvable groups. Clearly, \( \mathcal{W} \) is a saturated formation.

A subgroup \( H \) of \( G \) is called \( s \)-quasinormal (or \( s \)-permutable, \( \pi \)-quasinormal) in \( G \) provided \( H \) permutes with all Sylow subgroups of \( G \), i.e., \( HP = PH \) for any Sylow subgroup \( P \) of \( G \). This concept was introduced by Kegel in [5] and has been studied extensively by Deskins [2] and Schmidt [12]. More recently, Ballester-Bolinches and Pedraza-Aguilera [1] generalized \( s \)-quasinormal subgroups to \( s \)-quasinormally embedded subgroups. A subgroup \( H \) is said to be \( s \)-quasinormally embedded in \( G \) if for each prime \( p \) dividing the order of \( H \), a Sylow \( p \)-subgroup of \( H \) is also a Sylow \( p \)-subgroup of some \( s \)-quasinormal subgroup of \( G \). Clearly, every \( s \)-quasinormal subgroup of \( G \) is an \( s \)-quasinormally embedded subgroup of \( G \), but the converse does not hold. Many authors consider minimal or maximal subgroups of a Sylow subgroup of a group when investigating the structure of \( G \), such as in [1-2], [5-10] and [12-16], etc. For example, Li, Wang and Wei in [10] provide the following result: Let \( G \) be a group and \( P \) a Sylow \( p \)-subgroup of \( G \), where \( p \) is a prime divisor of \( |G| \) with \( (|G|, p - 1) = 1 \). If every maximal subgroup of \( P \) is \( s \)-quasinormally embedded in \( G \), then \( G \) is \( p \)-nilpotent. Recently, Wei and Guo in [14] prove the following result: Let \( p \) be the smallest prime dividing the order of a group \( G \) and \( P \) a Sylow \( p \)-subgroup of \( G \). Then \( G \) is \( p \)-nilpotent if and only if there is a subgroup \( D \) such that \( 1 < |D| < |P| \) and every subgroup \( H \) of \( P \) with order \( |H| = |D| \) or with order \( 2|D| \) (if \( P \) is a nonabelian 2-group and \( |P : D| > 2 \)) is \( s \)-quasinormally embedded in \( G \).

As another generalization of the normality, Skiba in [11] introduced the following concept: A subgroup \( H \) of \( G \) is said to be weakly \( s \)-permutable in \( G \) if there is a subnormal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_{sG} \), where \( H_{sG} \) is the subgroup of \( H \) generated by all those subgroups of \( H \) which are \( s \)-permutable in \( G \). Clearly, every \( s \)-permutable subgroup of \( G \) is an weakly \( s \)-permutable subgroup of \( G \), but the converse does not hold. He provides the following result: Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{W} \), the class of all supersolvable groups and \( G \) a group with \( E \) as a normal subgroup of \( G \) such that \( G/E \in \mathcal{F} \). Suppose that every non-cyclic Sylow subgroup \( P \) of \( F^n(E) \) has a subgroup \( D \) such that
1 \leq |D| < |P| and every subgroup \( H \) of \( P \) with order \( |H| = |D| \) or with order \( 2|D| \) (if \( P \) is a nonabelian 2-group and \( |P : D| > 2 \)) is weakly \( s \)-permutable in \( G \), where \( F^*(E) \) is the generalized Fitting subgroup of \( E \). Then \( G \in \mathcal{F} \).

The aim of this article is to unify and improve above Theorems using \( s \)-quasinormally embedded or weakly \( s \)-permutable subgroups. Our main theorem is the following result: Let \( \mathcal{F} \) be a saturated formation containing \( \mathcal{F} \), the class of all supersolvable groups and \( G \) a group with \( E \) as a normal subgroup of \( G \) such that \( G/E \in \mathcal{F} \). Suppose that every non-cyclic Sylow subgroup \( P \) of \( F^*(E) \) has a subgroup \( D \) such that \( 1 < |D| < |P| \) and every subgroup \( H \) of \( P \) with order \( |H| = |D| \) or with order \( 2|D| \) (if \( P \) is a nonabelian 2-group and \( |P : D| > 2 \)) is either \( s \)-quasinormally embedded or weakly \( s \)-permutable in \( G \), where \( F^*(E) \) is the generalized Fitting subgroup of \( E \). Then \( G \in \mathcal{F} \).

2 Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

**Lemma 1** ([11]) Suppose that \( U \) is \( s \)-quasinormally embedded in a group \( G \), and let \( H \leq G \) and \( K \leq G \). Then the following assertions hold.

(i) If \( U \leq H \), then \( U \) is \( s \)-quasinormally embedded in \( H \);

(ii) \( UK/K \) is \( s \)-quasinormally embedded in \( G \) and \( UK/K \) is \( s \)-quasinormally embedded in \( G/K \);

(iii) If \( K \leq H \) and \( H/K \) is \( s \)-quasinormally embedded in \( G/K \), then \( H \) is \( s \)-quasinormally embedded in \( G \).

**Lemma 2** ([11]) Let \( H \) be a weakly \( s \)-permutable subgroup of a group \( G \).

(i) If \( H \leq K \leq G \), then \( H \) is weakly \( s \)-permutable in \( K \);

(ii) If \( N \) is normal in \( G \) and \( N \leq H \leq G \), then \( H/N \) is weakly \( s \)-permutable in \( G/N \);

(iii) If \( H \) is a \( \pi \)-subgroup and \( N \) is a normal \( \pi' \)-subgroup of \( G \), then \( HN/N \) is weakly \( s \)-permutable in \( G/N \);

(iv) Suppose \( H \) is a \( p \)-group for some prime \( p \) and \( H \) is not \( s \)-permutable in \( G \). Then \( G \) has a normal subgroup \( M \) such that \( |G : M| = p \) and \( G = HM \).

**Lemma 3** ([13]) Let \( G \) be a group, \( K \) an \( s \)-quasinormal subgroup of \( G \) and \( P \) a Sylow \( p \)-subgroup of \( K \), where \( p \) is a prime. If either \( P \leq O_p(G) \) or \( K_G = 1 \), then \( P \) is \( s \)-quasinormal in \( G \).

**Lemma 4** ([12]) If \( P \) is an \( s \)-quasinormal \( p \)-subgroup of a group \( G \) for some prime \( p \), then \( N_G(P) \geq O^p(G) \).

**Lemma 5** ([13]) Let \( G \) be a group and \( p \) a prime dividing \( |G| \) with \( (|G|, p - 1) = 1 \).

(i) If \( N \) is normal in \( G \) of order \( p \), then \( N \leq Z(G) \);

(ii) If \( G \) has cyclic Sylow \( p \)-subgroup, then \( G \) is \( p \)-nilpotent;

(iii) If \( M \leq G \) and \( |G : M| = p \), then \( M \leq G \).

**Lemma 6** ([10]) Let \( G \) be a group and \( P \) a Sylow \( p \)-subgroup of \( G \), where \( p \) is a prime divisor of \(|G|\) with \((|G|, p - 1) = 1 \). If every maximal subgroup of \( P \) is \( s \)-quasinormally embedded in \( G \), then \( G \) is \( p \)-nilpotent.

**Lemma 7** ([3, III, 5.2 and IV, 5.4]) Suppose that \( p \) is a prime and \( G \) is a minimal non-\( p \)-nilpotent group, i.e., \( G \) is not a \( p \)-nilpotent group but whose proper subgroups are all \( p \)-nilpotent.

(i) \( G \) has a normal Sylow \( p \)-subgroup \( P \) for some prime \( p \) and \( G = PQ \), where \( Q \) is a non-normal cyclic \( q \)-subgroup for some prime \( q \neq p \).

(ii) \( P/\Phi(P) \) is a minimal normal subgroup of \( G/\Phi(P) \).

(iii) The exponent of \( P \) is \( p \) or 4.

**Lemma 8** ([6]) Let \( H \) be a nilpotent subgroup of a group \( G \). Then the following statements are equivalent:

(i) \( H \) is \( s \)-quasinormal in \( G \);

(ii) \( H \leq F(G) \) and \( H \) is \( s \)-quasinormally embedded in \( G \).

**Lemma 9** ([14]) Let \( N \) be an elementary abelian normal \( p \)-subgroup of a group \( G \). If there exists a subgroup \( D \) in \( N \) such that \( 1 < |D| < |N| \) and every subgroup \( H \) of \( N \) with \( |H| = |D| \) is \( s \)-quasinormally embedded in \( G \), then there exists a maximal subgroup \( M \) of \( N \) such that \( M \) is normal in \( G \).

**Lemma 10** ([3, VI, 4.10]) Assume that \( A \) and \( B \) are two subgroups of a group \( G \) and \( G \neq AB \). If \( AB^g = B^gA \) holds for any \( g \in G \), then either \( A \) or \( B \) is contained in a nontrivial normal subgroup of \( G \).

The generalized Fitting subgroup \( F^*(G) \) of \( G \) is the unique maximal normal quasinilpotent subgroup of \( G \). Its definition and important properties can be found in [4, X, 13]. We would like to give the following basic facts we will use in our proof.
Lemma 11 ([4, X.13]) Let $G$ be a group and $M$ a subgroup of $G$.

(i) If $M$ is normal in $G$, then $F^*(M) \leq F^*(G)$;
(ii) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \text{Soc}(F(G)C_G(F(G))/F(G))$;
(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(F^*(G)) = F(G)$.

Lemma 12 ([111]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup $P$ of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P:D| > 2$) is weakly $s$-permutable in $G$, where $F^*(E)$ is the generalized Fitting subgroup of $E$. Then $G \in \mathcal{F}$.

3 Main results

Theorem 13 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

Proof. Assume that the theorem is not true and let $G$ be a counterexample of minimal order. We derive a contradiction in several steps.

By Lemmas 1 and 2, the following two steps are obvious.

Step 1. $O_p'(G) = 1$.

Step 2. $G$ has a unique minimal normal subgroup $N$ and $G/N$ is $p$-nilpotent. Moreover, $\Phi(G) = 1$.

Step 3. $O_p(G) = 1$.

If $O_p(G) \neq 1$, then step 2 yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, $G$ has a maximal subgroup $M$ such that $G = MN$ and $G/N \cong M$ is $p$-nilpotent. Since $O_p(G) \cap N$ is normalized by $N$ and $M$, we conclude that $O_p(G) \cap M$ is normal in $G$. The uniqueness of $N$ yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore, $P \cap M < P$, and, thus there exists a maximal subgroup $P_1$ of $P$ such that $P \cap M < P_1$. Hence, $P = NP_1$. By hypothesis, $P_1$ is $s$-quasinormally embedded or weakly $s$-permutable in $G$. Suppose first $P_1$ is $s$-quasinormally embedded in $G$. Then there is an $s$-quasinormal subgroup $K$ of $G$ such that $P_1 \in \text{Syl}_p(K)$. If $K_G \neq 1$, then $N \leq K$. Since $N$ is a normal $p$-subgroup of $K$ and $P_1 \in \text{Syl}_p(K)$, we have that $N \leq P_1$, a contradiction. Hence $K_G = 1$, and so by Lemma 3 $P_1$ is $s$-quasinormal in $G$. By Lemma 4, $O^p(G) \leq N_G(P_1)$, $P_1 \leq G$. Then $N \cap P_1 = 1$ and $|N| = p$. By Lemma 5, $N \leq Z(G)$ and hence $G$ is $p$-nilpotent, a contradiction. Therefore, we may assume that $P_1$ is weakly $s$-permutable in $G$. Then there is a subnormal subgroup $T$ of $G$ such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)s_G \leq O_p(G) = N \leq O^p(G)$$

because $N$ is the unique minimal normal subgroup of $G$. Since $|G:T|$ is a power of $p$, $O^p(G) \leq T$. Hence,

$$P_1 \cap T \leq (P_1)s_G \leq O^p(G) \cap P_1 \leq T \cap P_1,$$

and so

$$P_1 \cap T = (P_1)s_G = O^p(G) \cap P_1.$$

Consequently, $G = O^p(G)$ implies that $(P_1)s_G$ is normal in $G$ by Lemma 4. By the minimality of $N$, we have $(P_1)s_G = N$ or $(P_1)s_G = 1$. If $(P_1)s_G = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)s_G = 1$, and so $|T_p| = p$. Then $T$ is $p$-nilpotent. Let $T_{p'}$ be the normal $p'$-complement of $T$. Then $T_{p'}$ is normal in $G$ and $T_{p'}$ is a $p'$-Hall subgroup of $G$. It follows that $T_{p'}$ is the normal $p'$-complement of $G$, a contradiction.

Step 4. The final contradiction.

If $P$ has a maximal subgroup $P_1$ which is weakly $s$-permutable in $G$, then there is a subnormal subgroup $T$ of $G$ such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)s_G \leq O_p(G) = 1.$$

Then $P_1 \cap T = 1$. Hence $|T_p| = p$. Therefore, $T$ is $p$-nilpotent. Thus $G$ is $p$-nilpotent, a contradiction. Now we may assume that all maximal subgroups of $P$ are $s$-quasinormally embedded in $G$. Then $G$ is $p$-nilpotent by Lemma 6, a contradiction. □

The following corollaries is immediate from Theorem 13.

Corollary 14 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is $s$-quasinormally embedded in $G$. Then $G$ is $p$-nilpotent.

Corollary 15 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is weakly $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 16 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is $s$-permutable in $G$. Then $G$ is $p$-nilpotent.
Corollary 17 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is permutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 18 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is normal in $G$. Then $G$ is $p$-nilpotent.

Theorem 19 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

Proof. Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

Step 1. $O_p'(G) = 1$.

If $O_p'(G) \neq 1$, Lemma 1 (ii) and Lemma 2 (iii) guarantee that $G/O_p'(G)$ satisfies the hypotheses of the theorem. Thus $G/O_p'(G)$ is $p$-nilpotent by the choice of $G$. Then $G$ is $p$-nilpotent, a contradiction.

Step 2. $|D| > p$.

Suppose that $|D| = p$. Since $G$ is not $p$-nilpotent, $G$ has a minimal non-$p$-nilpotent subgroup $G_1$. By Lemma 7 (i), $G_1 = [P_1]Q$, where $P_1 \in \text{Syl}_p(G_1)$ and $Q \in \text{Syl}_q(G_1)$, $p \neq q$. Let $x \in P_1$ and $L = \langle x \rangle$. Then $L$ is of order $p$ or $4$ by Lemma 7 (iii). By the hypotheses, $L$ is either $s$-quasinormally embedded or weakly $s$-permutable in $G$, thus in $G_1$ by Lemma 1 (i) and 2 (i). First, suppose that $L$ is weakly $s$-permutable in $G_1$. Then there is a subnormal subgroup $T$ of $G_1$ such that $G_1 = LT$ and $L \cap T \leq L_1$. Hence $P_1 = P_1 \cap G_1 = P_1 \cap LT = L(P_1 \cap T)$. Since $P_1/\Phi(P_1)$ is abelian, we have $(P_1 \cap T)\Phi(P_1)/\Phi(P_1)$ is normal in $G_1/\Phi(P_1)$. Since $P_1/\Phi(P_1)$ is the minimal normal subgroup of $G_1/\Phi(P_1)$, we have that $P_1 \cap T \leq \Phi(P_1)$ or $P_1 = (P_1 \cap T)\Phi(P_1) = P_1 \cap T$. If $P_1 \cap T \leq \Phi(P_1)$, then $L = P_1$ is normal in $G_1$. It follows that $G_1$ is $p$-nilpotent, a contradiction. If $P_1 \cap T = \Phi(P_1)$ and $T = G_1$ and so $L = L_1G_1$ is $s$-permutable in $G_1$. For any element $x$ in $P_1$, now we have $(x)Q$ is a proper subgroup of $G_1$, then $(x)Q = (x) \times Q$. This implies that $G_1 = P_1 \times Q$, a contradiction. Therefore, $L = \langle x \rangle$ is $s$-quasinormally embedded in $G_1$ for every element $x$ in $P_1$, then by Lemma 8 $(x)Q$ is $s$-quasinormally embedded in $G_1$. Thus $LQ \leq G_1$. Therefore, $LQ = L \times Q$. Then $G_1 = P_1 \times Q$, a contradiction.


By Theorem 13.

Step 4. $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Assume that $H \leq P$ such that $|H| = |D|$ and $H$ is weakly $s$-permutable in $G$. Then there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap K \leq H_G$. By Lemma 2 (iv), we may assume $G$ has a normal subgroup $M$ such that $[G : M] = p$ and $G = HM$. Since $|P : D| > p$ by Step 3, $M$ satisfies the hypotheses of the theorem. The choice of $G$ yields that $M$ is $p$-nilpotent. It is easy to see that $G$ is $p$-nilpotent, contrary to the choice of $G$.

Step 5. If $N \leq P$ and $N$ is minimal normal in $G$, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, $N$ is elementary abelian. By Lemma 9, $N$ has a maximal subgroup which is normal in $G$, contrary to the minimality of $N$.

Step 6. Suppose that $N \leq P$ and $N$ is minimal normal in $G$. Then $G/N$ is $p$-nilpotent.

If $|N| < |D|$, $G/N$ satisfies the hypotheses of the theorem by Lemma 1 (ii). Thus $G/N$ is $p$-nilpotent by the minimal choice of $G$. So we may suppose that $|N| = |D|$ by Step 5. We will show that every cyclic subgroup of $P/N$ of order $p$ or order 4 (when $P/N$ is a non-abelian 2-group) is $s$-quasinormally embedded in $G/N$. Let $K \leq P$ and $|K/N| = p$. By Step 2, $N$ is non-cyclic, so are all subgroups containing $N$. Hence there is a maximal subgroup $L \neq N$ of $K$ such that $K = NL$. Of course, $|N| = |D| = |L|$. Since $L$ is $s$-quasinormally embedded in $G$ by the hypotheses, $K/N = LN/N$ is $s$-quasinormally embedded in $G/N$ by Lemma 1 (ii). If $p = 2$ and $P/N$ is non-abelian, take a cyclic subgroup $X/N$ of $P/N$ of order 4. Let $K/N$ be maximal in $X/N$. Then $K$ is maximal in $X$ and $|K/N|=2$. Since $X$ is non-cyclic and $X/N$ is cyclic, there is a maximal subgroup $L$ of $X$ such that $N$ is not contained in $L$. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, $L$ is $s$-quasinormally embedded in $G$. By Lemma 1 (ii), $X/N = LN/N$ is $s$-quasinormally embedded in $G/N$. Hence $G/N$ satisfies the hypotheses. By the minimal choice of $G$, $G/N$ is $p$-nilpotent.

Step 7. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup $N$ of $G$ contained in $O_p(G)$. By Step 6, $G/N$ is $p$-nilpotent. It is easy to see that $N$ is the unique minimal normal subgroup of $G$ contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. 

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Hence $O_p(G)$ is an elementary abelian $p$-group. On the other hand, $G$ has a maximal abelian subgroup $M$ such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is $p$-nilpotent. Then $G$ can be written as $G = N(M \cap P)M_p'$, where $M_p'$ is the normal $p$-complement of $M$. Pick a maximal subgroup $S$ of $M_p = P \cap M$. Then $NSM_p$ is a subgroup of $G$ with index $p$. Since $p$ is the minimal prime in $\pi(G)$, we know that $NSM_p$ is normal in $G$. Now by Step 3 and the induction, we have $NSM_p$ is $p$-nilpotent. Therefore, $G$ is $p$-nilpotent, a contradiction.

Step 8. The minimal normal subgroup $L$ of $G$ is not $p$-nilpotent.

If $L$ is $p$-nilpotent, then it follows from the fact that $L_p'$ char $L \leq G$ that $L_p' \leq O_p(G) = 1$. Thus $L$ is a $p$-group. Whence $L \leq O_p(G) = 1$ by Step 7, a contradiction.

Step 9. $G$ is a non-abelian simple group.

Suppose that $G$ is not a simple group. Take a minimal normal subgroup $L$ of $G$. Then $L < G$. If $|L|_p > |D|$, then $L$ is $p$-nilpotent by the minimal choice of $L$, contrary to Step 8. If $|L|_p \leq |D|$, take $P_s \geq L \cap P$ such that $|P_s| = p|D|$. Hence $P_s$ is a Sylow $p$-subgroup of $P_sL$. Since every maximal subgroup of $P_s$ is of order $|D|$, every maximal subgroup of $P_s$ is $s$-quasinormally embedded in $G$ by hypotheses, thus in $P_sL$ by Lemma 1 (i). Now applying Theorem 13, we get $P_sL$ is $p$-nilpotent. Therefore, $L$ is $p$-nilpotent, contrary to Step 8.

Step 10. The final contradiction.

Suppose that $H$ is a subgroup of $P$ with $|H| = |D|$ and $Q$ is a Sylow $q$-subgroup with $q \neq p$. Then $HQ^g = Q^gH$ for any $g \in G$ by the hypotheses that $H$ is $s$-quasinormally embedded in $G$ and Lemma 8. Since $G$ is simple by Step 9, $G = HQ$ from Lemma 10, the final contradiction.

The following corollaries are immediate from Theorem 19.

**Corollary 20** Suppose that $G$ is a group. If every non-cyclic Sylow subgroup of $G$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$, then $G$ has a Sylow tower of supersolvable type.

**Corollary 21** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-quasinormally embedded in $G$. Then $G$ is $p$-nilpotent.

**Corollary 22** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

**Corollary 23** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

**Corollary 24** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

**Corollary 25** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. If $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is normal in $G$. Then $G$ is $p$-nilpotent.

**Corollary 26** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Suppose that every cyclic subgroup of $P$ of prime order or order 4 is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G$ is $p$-nilpotent.

**Corollary 27** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Suppose that every cyclic subgroup of $P$ of prime order or order 4 is $s$-quasinormally embedded in $G$. Then $G$ is $p$-nilpotent.

**Corollary 28** Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Suppose that every cyclic subgroup of $P$ of prime order or order 4 is weakly $s$-permutable in $G$. Then $G$ is $p$-nilpotent.
Corollary 30 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Suppose that every cyclic subgroup of $P$ of prime order or order 4 is permutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 31 Let $p$ be the smallest prime dividing the order of a group $G$ and $P$ be a Sylow $p$-subgroup of $G$. Suppose that every cyclic subgroup of $P$ of prime order or order 4 is normal in $G$. Then $G$ is $p$-nilpotent.

Theorem 32 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow $p$-subgroup of $E$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

Proof. Suppose that $P$ is a non-cyclic Sylow $p$-subgroup of $E$, $\forall p \in \pi(E)$. Since $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$ by hypotheses, thus in $E$ by Lemma 1 (i). Applying Corollary 20, we conclude that $E$ has a Sylow tower of supersolvable type. Let $q$ be the maximal prime divisor of $|E|$ and $Q \in \text{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup $H$ of $Q$ with $|H| = |D|$, since $Q \trianglelefteq O_q(G)$, $H$ is either $s$-quasinormal or weakly $s$-permutable in $G$ by Lemma 8. Since $s$-quasinormal implies weakly $s$-permutable and $F^*(Q) = Q$ by Lemma 11, we get $G \in \mathcal{F}$ by applying Lemma 12.

The following corollaries are immediate from Theorem 32.

Corollary 33 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow $p$-subgroup of $E$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-quasinormally embedded in $G$. Then $G \in \mathcal{F}$.

Corollary 34 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow $p$-subgroup of $E$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

Corollary 35 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow $p$-subgroup of $E$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

Corollary 36 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow $p$-subgroup of $E$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is permutable in $G$. Then $G \in \mathcal{F}$.

Corollary 37 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow $p$-subgroup of $E$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is normal in $G$. Then $G \in \mathcal{F}$.

Corollary 38 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of $E$ is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

Corollary 39 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of $E$ of prime order or order 4 is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

Corollary 40 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of $E$ is $s$-quasinormally embedded in $G$. Then $G \in \mathcal{F}$.

Corollary 41 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal
of prime order or order 4 is weakly $E$-quasinormal in $G$. Then $G \in \mathcal{F}$.

**Corollary 42** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of $E$ is weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

**Corollary 43** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of $E$ is weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

**Corollary 44** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of $E$ of prime order or order 4 is cyclic subgroup of any non-cyclic Sylow subgroup of $E$. Then $G \in \mathcal{F}$.

**Corollary 45** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every cyclic subgroup of any non-cyclic Sylow subgroup of $E$ of prime order or order 4 is $s$-quasinormal in $G$. Then $G \in \mathcal{F}$.

**Corollary 46** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of $E$ is quasinormal in $G$. Then $G \in \mathcal{F}$.

**Theorem 50** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

**Proof.** We distinguish two cases:

**Case 1.** $\mathcal{F} = \mathcal{U}$.

Let $G$ be a minimal counter-example.

**Step 1.** Every proper normal subgroup $N$ of $G$ containing $F^*(E)$ (if it exists) is supersolvable.

If $N$ is a proper normal subgroup of $G$ containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 11 (iii), $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup $P$ of $F^*(E \cap N) = F^*(E)$, $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$ by hypotheses, thus in $N$ by Lemma 1 (i) and Lemma 2 (i). So $N$ and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of $G$ implies that $N$ is supersolvable.

**Step 2.** $E = G$.

If $E < G$, then $E \in \mathcal{U}$ by Step 1. Hence $F^*(E) = F(E)$ by Lemma 11. It follows that every Sylow subgroup of $F^*(E)$ is normal in $G$. By Lemma 8, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormal or weakly $s$-permutable in $G$. Applying Lemma 12 for the special case $\mathcal{F} = \mathcal{U}$, $G \in \mathcal{F}$, a contradiction.

**Step 3.** $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathcal{F}$ by Theorem 32, contrary to the choice of $G$. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathcal{U}$ and $F^*(G) = F(G)$ by Lemma 11.

**Step 4.** The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormal or weakly $s$-permutable in $G$ by Lemma 8. Applying Lemma 12, $G \in \mathcal{U}$, a contradiction.
Case 2. $\mathcal{F} \neq \mathcal{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormally embedded or weakly $s$-permutable in $G$, thus in $E$ Lemma 1 (i) and Lemma 2 (i). Applying Case 1, $E \in \mathcal{U}$. Then $F^*(E) = F(E)$ by Lemma 11. It follows that each Sylow subgroup of $F^*(E)$ is normal in $G$. By Lemma 8, each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is either $s$-quasinormal or weakly $s$-permutable in $G$.

Applying Lemma 12, $G \in \mathcal{F}$. These complete the proof of the theorem.

The following corollaries are immediate from Theorem 50.

**Corollary 51** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-quasinormally embedded in $G$. Then $G \in \mathcal{F}$.

**Corollary 52** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is weakly $s$-permutable in $G$. Then $G \in \mathcal{F}$.

**Corollary 53** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-permutable in $G$. Then $G \in \mathcal{F}$.

**Corollary 54** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-quasinormally embedded or weakly $s$-permutable in $G$.

**Corollary 55** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and $G$ a group with $E$ as a normal subgroup of $G$ such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup $D$ such that $1 < |D| < |P|$ and every subgroup $H$ of $P$ with order $|H| = |D|$ or with order $2|D|$ (if $P$ is a nonabelian 2-group and $|P : D| > 2$) is $s$-quasinormally embedded or weakly $s$-permutable in $G$.

**Corollary 56** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is either $s$-quasinormally embedded or weakly $s$-permutable in $G$.

**Corollary 57** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is $s$-quasinormally embedded or weakly $s$-permutable in $G$.

**Corollary 58** ([9, Theorem 1.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$.

**Corollary 59** ([9, Theorem 1.2]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$.

**Corollary 60** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$.

**Corollary 61** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$.
Corollary 62 ([7, Theorem 3.4]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^\ast(E)$ is $s$-quasinormal in $G$.

Corollary 63 ([8, Theorem 3.3]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^\ast(E)$ of prime order or order 4 is $s$-quasinormal in $G$.

Corollary 64 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^\ast(E)$ is quasinormal in $G$.

Corollary 65 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^\ast(E)$ of prime order or order 4 is quasinormal in $G$.

Corollary 66 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^\ast(E)$ is normal in $G$.

Corollary 67 Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^\ast(E)$ of prime order or order 4 is normal in $G$.

4 Conclusion

The results explained in the previous sections show that the method that we replace conditions for all maximal subgroups or all minimal subgroups of Sylow subgroups of $G$ by conditions referring to only some subgroups of Sylow subgroups of $G$ in order to investigate the structure of a finite group is very useful. Results of this type are interesting. In addition, there are many other generalizations of the normality, for example, $SS$-quasinormal subgroups in [6]; $c^\ast$-normality in [13]; $X$-semipermutable subgroups in [17]; $c$-supplemented subgroups in [18]. As an application, we may consider using the above special subgroups to characterize the structure of finite groups.

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