Optimal Investment and Consumption Decisions Under the Ho-Lee Interest Rate Model

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Abstract: In this paper, we consider an investment and consumption problem with stochastic interest rate, in which risk-free interest rate dynamics is driven by the Ho-Lee model,while risky asset price is supposed to follow a geometric Brownian motion and be correlated with interest rate dynamics. Our goal is to seek an optimal investment and consumption strategy to maximize the expected discounted utility of consumption and terminal wealth in the finite horizon. Firstly, we apply dynamic programming principle to derive Hamilton-Jacobi-Bellman(HJB) equation for the value function and take power utility and logarithm utility for our analysis. Secondly, by conjecturing the form of a solution and solving partial differential equations, we obtain the closed-form solutions to the optimal investment and consumption strategies. Finally, we provide a numerical example to demonstrate the impact of market parameters on the optimal investment and consumption strategy.

Key–Words: Investment and consumption problem; the Ho-Lee model; dynamic programming principle; HJB equation; closed-form solution;

1 Introduction

The first treatment of the investment and consumption problem in a continuous-time setting originated from the seminal papers of Merton(1969, 1971), in which an explicit optimal investment and consumption strategy was constructed for power and logarithm utility by applying dynamic programming approach and HJB equations technique. Since Merton, portfolio optimization problems with consumption have inspired hundreds of extensions and applications. For example, investment and consumption problems with transaction costs were investigated by Shreve and Soner(1994), Akian and Menaldi et al(1996), Janecek and Shreve(2004) respectively. Zariphopoulou et al(1991, 1997) and Yao and Zhang(2005) concerned an investment and consumption problems with borrowing constraints respectively. Further, the investment and consumption problems were extended to those in an incomplete market(referring to Chacko and Viceira(2005), Sasha and Zariphopoulou(2005)). But a drawback of these market models is the assumption of deterministic interest rate. In this paper,our main objective is to overcome this restriction. So we assume that interest rate dynamics follows an Ito process and particularly consider the case of the Ho-Lee model(Ho and Lee(1986)) for the short rate.

There were some results on portfolio selection problems with stochastic interest rates. For instance, Korn and Kraft(2001) considered an portfolio optimization problem with stochastic interest rates, in which interest rate dynamics was driven by the Ho-Lee model and the Vasicek model(Vasicek(1977)) repectively. Yet, consumption was ignored in their market model. Munk and Sørensen(2004, 2010) investigated an optimal consumption model, in which interest rate was supposed to follow the Vasicek model. Pang(2004, 2006) focused on an investment and consumption problem and interest rate is assumed to be driven by an ergodic Markov diffusion process, and an sub-supersolution approach was presented to prove the existence of solutions of HJB equations. However, they did not obtain the closed-form solutions to the optimal investment and consumption strategies. Liu(2007) generalized the investment and consumption problem with stochastic interest rate or stochastic volatility. Fleming(2003) extended the volatility of the stock to stochastic volatility and concerned an optimal consumption model, in which the infinite horizon expected utility of consumption was taken for the objective function. But the closed-form solution wasn't obtained as well. Noh and Kim(2011) considered an investment and consumption problem with stochastic volatility and stochastic interest rate, but they did not also derive the explicit solutions in the power and logarithm utility cases. In this paper, we assume that interest rate dynamics is governed by the Ho-Lee model and obtain the closed-form solutions to the optimal investment and consumption strategies in the power and logarithm utility cases by applying dynamic programming principle and HJB equation approach. Such problems have not been reported in the existing literatures.

In our real life, interest rate is not always fixed and stochastic interest rates are one of the most important factors an investor often faced with. So it is necessary for us to investigate an investment and consumption problem with stochastic interest rates. In this paper, we are concerned with an investment and consumption problem with stochastic interest rate, in which interest rate is supposed to follow the Ho-Lee model. The finite horizon expected discounted utility of consumption and terminal wealth is taken as the objective function. We apply dynamic programming to derive HJB equation and try to obtain an optimal investment and consumption strategy to maximize the objective function. Finally, the closed-form solutions in the power and logarithm cases are derived. Moreover, we provide a numerical example to illustrate the effect of market parameters on the optimal investment and consumption strategy. There are three main contributions: (i) We study an investment and consumption problem with the Ho-Lee interest rate model and obtain the closed-form solutions in the power and logarithm cases; (ii) We use the same approach as Liu(2007) to solve the equation (9); (iii) We provide a numerical example to illustrate our results and present some economic implications.

The rest of this paper is organized as follows. In section 2, we introduce the financial market and propose an investment and consumption problem with stochastic interest rate. In section 3, we derive the closed-form solutions to the optimal investment and consumption strategies in the power and logarithm utility cases by applying varable change technique. Section 4 provides a numerical example. Section 5 concludes the paper.

2 The financial market

In this section, we formulate an investment and consumption problem with stochastic interest rate and interest rate dynamics is governed by the Ho-Lee model.

We consider a financial market where two assets are traded continuously over $[0, T]$. One of the assets is a risk-free asset (i.e., bank acount) with price S_t at time t, whose price process S_t satisfies:

$$dS_t = r_t S_t dt, \quad S_0 = 1, \tag{1}$$

where $r_t > 0$ is risk-free interest rate at time t, and r_t is supposed to be stochastic process.

In this paper interest rate is assumed to be driven by the stochastic differential equation:

$$dr_t = a_t dt + b dB_t,$$

with initial condition $r_0 > 0$ and as explicit examples we will consider the Ho-Lee model given by $a_t = \tilde{\alpha}_t + b\tilde{\eta}_t$, where $b > 0$ is a constant and $\tilde{\alpha}_t$ and $\tilde{\eta}_t$ are assumed to be deterministic and continuous function of the time t.

The other asset is a risky asset (i.e., stock) with price P_t at time t, whose price process P_t is given by the following stochastic differential equation:

$$dP_t = P_t[\mu_t dt + \sigma_t dW_t], \quad t \in [0, T], \tag{2}$$

where μ_t and σ_t are the appreciation rate and the volatility of the stock respectively. In addition, μ_t and σ_t are supposed to be deterministic and Borel-measurable bounded functions on $[0, T]$. B_t and W_t are all one-dimensional standard Brownian motions defined on complete probability space $(\Omega, P, \mathcal{F}_t, \{\mathcal{F}_t\}_{0 \le t \le T})$ and there are the correlation coefficient ρ, i.e., $dB_t dW_t = \rho dt$.

Assume that an investor invests the market value of his wealth π_t into the stock at time t, $t \in [0, T]$, Clearly, the amount invested in the bond is $X_t - \pi_t$, in which X_t represents the net wealth of the investor at time t. C_t is the consumption rate. Suppose that short-selling of stocks and borrowing at the interest rate of the bond is allowed and transaction cost is ignored. Wealth process X_t corresponding to trading strategy (π_t, C_t) is subjected to the following equation:

$$\begin{aligned} &dX_t = [r_t X_t + (\mu_t - r_t)\pi_t - C_t]dt + \pi_t \sigma_t dW_t, \\ &X_0 = x_0 > 0. \end{aligned} \tag{3}$$

Definition 1 *(Admissible strategy) An investment and consumption strategy (π_t, C_t) is admissible if the following conditions are satisfied:*

(i) (π_t, C_t) is progressively measurable and

$$\int_0^T (\pi_t)^2 dt < \infty, \int_0^T C_t dt < \infty;$$

(ii) $E[\int_0^T (\pi_t \sigma_t)^2 dt] < \infty$;

(iii) For arbitrary trading policy (π_t, C_t), the equation (3) has a pathwise unique strong solution.

Assume that the set of all admissible investment and consumption strategies (π_t, C_t) is denoted by $\Gamma =$

$\{(\pi_t, C_t) : 0 \le t \le T\}$. Mathematically, the investor wishes to maximize the following expected utility:

$$\max_{(\pi_t, C_t)\in\Gamma} \mathbb{E}[\alpha\int_0^T e^{-\beta t}U_1(C_t)dt+(1-\alpha)e^{-\beta T}U_2(X_T)], \tag{4}$$

where utility function $U(\cdot)$ is assumed to be strictly concave and continuously differentiable on $(-\infty, +\infty)$ and β is the subjective discount. The parameter α determines the relative importance of the intermediate consumption and the terminal wealth. Since $U(\cdot)$ is strictly concave, there exists a unique optimal trading strategy (π_t, C_t) which maximizes the problem (4).

3 Optimal investment and consumption decisions

In this section, we apply dynamic programming to obtain the HJB equation for the value function and investigate the optimal investment and consumption decisions in the power and logarithm utility cases.

The value function $H(t, r, x)$ is defined as

$$\begin{aligned} H(t,r,x) = \sup_{(\pi_t,C_t)\in\Gamma} & \mathbb{E}[\alpha \textstyle\int_0^T e^{-\beta t}U_1(C_t)dt \\ & +(1-\alpha)e^{-\beta T}U_2(X_T) \mid r_t = r, X_t = x], \end{aligned} \tag{5}$$

with boundary condition given by $H(T, r, x) = (1 - \alpha)e^{-\beta T}U_2(x)$.

According to dynamic programming principle(referring to Yong and Zhou(1999)), we derive the following theorem.

Theorem 2 *Assume that $H(t, r, x)$ is continuously differentiable with respect to $t \in [0, T]$, and twice continuously differentiable with respect to $(r, x) \in R \times R$, then $H(t, r, x)$ satisfies the following HJB equation:*

$$\begin{aligned} \sup_{(\pi_t,C_t)\in\Gamma} \Big\{ & \frac{\partial H}{\partial t} + rx\frac{\partial H}{\partial x} + a_t\frac{\partial H}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 H}{\partial r^2} + \\ & \alpha e^{-\beta t}U_1(C_t) - C_t\frac{\partial H}{\partial x} + (\mu_t - r)\pi_t\frac{\partial H}{\partial x} + \\ & \frac{1}{2}(\pi_t\sigma_t)^2\frac{\partial^2 H}{\partial x^2} + b\rho\pi_t\sigma_t\frac{\partial^2 H}{\partial r\partial x}\Big\} = 0, \end{aligned}$$

with boundary condition given by $H(T, r, x) = (1 - \alpha)e^{-\beta T}U_2(x)$, where $\frac{\partial H}{\partial t}, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial r}, \frac{\partial^2 H}{\partial x^2}, \frac{\partial^2 H}{\partial r^2}, \frac{\partial^2 H}{\partial x\partial r}$ represent partial derivatives of first-order and second-order with respect to the variables t, r, x respectively.

Proof. According to Theorem 3.1 of Lin and Li(2011), we assume that

$$H(t,r,x) = \sup_{(\pi_t,C_t)\in\Gamma} \mathbb{E}[H(\tilde{\theta}, r(\tilde{\theta}), X^{\pi,C}(\tilde{\theta}))].$$

For any stopping time $\tilde{\theta} \in \Re_{t,T}$, where $\Re_{t,T}$ represents the set of all stopping time valued in $[t, T]$.

Considering the time $\tilde{\theta} = t + h$, for arbitrary $(\pi_t, C_t) \in \Gamma$, we have

$$H(t, r, x) \geqslant \mathbb{E}[H(t + h, r(t + h), X^{\pi,C}(t + h))].$$

Applying $It\hat{o}'s$ formula between t and $t + h$, we have

$$\begin{aligned} H(t+h, r(t+h), X^{\pi_t,C_t}(t+h)) = H(t,r,x) + \\ \int_t^{t+h} (\frac{\partial H}{\partial t} + \ell^{\pi_t,C_t}H)(u, r(u), X^{\pi_t,C_t}(u))du \\ + \text{local martingale}, \end{aligned}$$

where $\ell^{\pi_t,C_t}H$ is defined by

$$\begin{aligned} \ell^{\pi_t,C_t}H = rX_t\frac{\partial H}{\partial x} + a_t\frac{\partial H}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 H}{\partial r^2} + \\ \alpha e^{-\beta t}U_1(C_t) - C_t\frac{\partial H}{\partial x} + (\mu_t - r)\pi_t\frac{\partial H}{\partial x} + \\ \frac{1}{2}(\pi_t\sigma_t)^2\frac{\partial^2 H}{\partial x^2} + b\rho\pi_t\sigma_t\frac{\partial^2 H}{\partial r\partial x}. \end{aligned}$$

Then we obtain

$$\int_t^{t+h} (\frac{\partial H}{\partial t} + \ell^{\pi_t,C_t}H)(u, r(u), X^{\pi_t,C_t}(u))du \leqslant 0.$$

Dividing by h and letting $h \to 0$, this leads to

$$\frac{\partial H}{\partial t} + \ell^{\pi_t,C_t}H \leqslant 0.$$

On the other hand, suppose that $(\pi_t^*, C_t^*) \in \Gamma$ is an optimal investment and consumption policy, then we have

$$H(t, r, x) = \mathbb{E}[H(t + h, r(t + h), X^{\pi_t^*,C_t^*}(t + h))].$$

By similar arguments as above, we obtain

$$\frac{\partial H}{\partial t} + \ell^{\pi_t^*,C_t^*}H = 0.$$

Summarizing the above arguments, we obtain that $H(t, r, x)$ should satisfy

$$\frac{\partial H}{\partial t} + \sup_{(\pi_t,C_t)\in\Gamma} \ell^{\pi_t,C_t}H = 0.$$

This completes the proof of Theorem 1. □

The first-order maximizing conditions of the optimal value are

$$\begin{aligned}&\pi_t^* = -\frac{\mu_t - r_t}{\sigma_t^2}\cdot\frac{\partial H/\partial x}{\partial^2 H/\partial x^2} - \frac{b\rho}{\sigma_t}\cdot\frac{\partial^2 H/\partial x\partial r}{\partial^2 H/\partial x^2},\\ &U_1'(C_t^*) = \frac{\partial H/\partial x}{\alpha e^{-\beta t}}.\end{aligned} \quad (6)$$

Letting $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$, and putting (6) into the above HJB equation, we obtain a second-order nonlinear partial differential equation for the value function $H(t,r,x)$:

$$\begin{aligned}&\frac{\partial H}{\partial t} + rx\frac{\partial H}{\partial x} - \frac{1}{2}\theta_t^2\frac{(\partial H/\partial x)^2}{\partial^2 H/\partial x^2}\\ &-b\rho\theta_t\frac{(\partial H/\partial x)(\partial^2 H/\partial x\partial r)}{\partial^2 H/\partial x^2} - \frac{1}{2}b^2\rho^2\frac{(\partial^2 H/\partial x\partial r)^2}{\partial^2 H/\partial x^2}\\ &+a_t\frac{\partial H}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 H}{\partial r^2} - C_t^*\frac{\partial H}{\partial x} + \alpha e^{-\beta t}U_1(C_t^*) = 0.\end{aligned} \quad (7)$$

In this paper we choose power utility and logarithm utility function for our analysis. Note that the equation (7) is a non-linear second-order partial differential equation, which is hard to solve directly. In the subsection, we will apply variable change technique to solve it and obtain the optimal investment and consumption strategy in the power and logarithm utility cases.

3.1 Power utility

Power utility is defined by

$$U_1(x) = U_2(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1.$$

Assume that the value function $H(t,r,x)$ is conjectured to have the form

$$\begin{aligned}&H(t,r,x) = e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f(t,r)^\gamma,\\ &f(T,r) = (1-\alpha)^{\frac{1}{\gamma}}.\end{aligned} \quad (8)$$

Then we have

$$\begin{aligned}&\frac{\partial H}{\partial t} = -\beta e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f(t,r)^\gamma + \gamma e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f(t,r)^{\gamma-1}\frac{\partial f}{\partial t},\\ &\frac{\partial H}{\partial x} = e^{-\beta t}x^{-\gamma}f(t,r)^\gamma,\\ &\frac{\partial^2 H}{\partial x^2} = -\gamma e^{-\beta t}x^{-\gamma-1}f(t,r)^\gamma,\\ &\frac{\partial H}{\partial r} = \gamma e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f(t,r)^{\gamma-1}\frac{\partial f}{\partial r},\\ &\frac{\partial^2 H}{\partial x\partial r} = \gamma e^{-\beta t}x^{-\gamma}f(t,r)^{\gamma-1}\frac{\partial f}{\partial r},\\ &\frac{\partial^2 H}{\partial r^2} = \gamma(\gamma-1)e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f(t,r)^{\gamma-2}(\frac{\partial f}{\partial r})^2\\ &\quad +\gamma e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f(t,r)^{\gamma-1}\frac{\partial^2 f}{\partial r^2}.\end{aligned}$$

Further, the optimal consumption and the optimal portfolio are given by

$$\begin{aligned}&\pi_t^* = \left(\frac{1}{\gamma}\cdot\frac{\mu_t - r_t}{\sigma_t^2} + \frac{b\rho}{\sigma_t}\cdot\frac{\partial f/\partial r}{f}\right)x,\\ &C_t^* = \alpha^{\frac{1}{\gamma}}f^{-1}x.\end{aligned}$$

Introducing these derivatives into (7), we derive

$$\begin{aligned}&\gamma e^{-\beta t}\frac{x^{1-\gamma}}{1-\gamma}f^{\gamma-1}[\frac{\partial f}{\partial t} + \frac{1-\gamma}{\gamma}rf + (\frac{1-\gamma}{2\gamma^2}\theta_t^2 - \frac{\beta}{\gamma})f\\ &+(\frac{1-\gamma}{\gamma}b\rho\theta_t + a_t)\frac{\partial f}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial r^2}\\ &+\frac{1}{2}b^2(1-\gamma)(\rho^2-1)\frac{(\partial f/\partial r)^2}{f} + \alpha^{\frac{1}{\gamma}}] = 0.\end{aligned}$$

Eliminating the dependence on x, we obtain

$$\begin{aligned}&\frac{\partial f}{\partial t} + \frac{1-\gamma}{\gamma}rf + (\frac{1-\gamma}{2\gamma^2}\theta_t^2 - \frac{\beta}{\gamma})f + \frac{1}{2}b^2\frac{\partial^2 f}{\partial r^2}\\ &+(\frac{1-\gamma}{\gamma}b\rho\theta_t + a_t)\frac{\partial f}{\partial r} + \frac{1}{2}b^2(1-\gamma)(\rho^2-1)\frac{(\partial f/\partial r)^2}{f}\\ &+\alpha^{\frac{1}{\gamma}} = 0,\\ &f(T,r) = (1-\alpha)^{\frac{1}{\gamma}}.\end{aligned} \quad (9)$$

This is a nonlinear second-order partial differential equation, which is difficult to solve. In this paper, inspired by the paper of Liu(2007), we conjecture a solution to (9) with the following structure

$$f(t,r) = \alpha^{\frac{1}{\gamma}}\int_t^T \hat{f}(u,r)du + (1-\alpha)^{\frac{1}{\gamma}}\hat{f}(t,r). \quad (10)$$

After some calculation, we find that $\hat{f}(t,r)$ satisfies

$$\begin{aligned}&\frac{\partial \hat{f}}{\partial t} + \frac{1-\gamma}{\gamma}r\hat{f} + (\frac{1-\gamma}{2\gamma^2}\theta_t^2 - \frac{\beta}{\gamma})\hat{f}\\ &+(\frac{1-\gamma}{\gamma}b\rho\theta_t + a_t)\frac{\partial \hat{f}}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 \hat{f}}{\partial r^2}\\ &+\frac{1}{2}b^2(1-\gamma)(\rho^2-1)\frac{(\partial \hat{f}/\partial r)^2}{\hat{f}} = 0,\\ &\hat{f}(T,r) = 1.\end{aligned} \quad (11)$$

In fact, when we define a differential operator ∇ on any function $f(t,r)$ by

$$\begin{aligned}\nabla f &= \frac{1-\gamma}{\gamma}rf + (\frac{1-\gamma}{2\gamma^2}\theta_t^2 - \frac{\beta}{\gamma})f\\ &\quad +(\frac{1-\gamma}{\gamma}b\rho\theta_t + a_t)\frac{\partial f}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial r^2}\\ &\quad +\frac{1}{2}b^2(1-\gamma)(\rho^2-1)\frac{(\partial f/\partial r)^2}{f}.\end{aligned}$$

We obtain

$$\frac{\partial \hat{f}}{\partial t} + \nabla\hat{f} = 0, \quad \hat{f}(T,r) = 1.$$

On the other hand, we find that

$$\begin{aligned}&\frac{\partial f}{\partial t} + \nabla f = -\alpha^{\frac{1}{\gamma}}\hat{f}(t,r) + \alpha^{\frac{1}{\gamma}}\int_t^T \nabla\hat{f}(u,r)du\\ &= -\alpha^{\frac{1}{\gamma}}\hat{f}(t,r) - \alpha^{\frac{1}{\gamma}}\int_t^T \frac{\partial \hat{f}(u,r)}{\partial u}du\\ &= -\alpha^{\frac{1}{\gamma}}\hat{f}(t,r) - \alpha^{\frac{1}{\gamma}}[\hat{f}(T,r) - \hat{f}(t,r)]du = -\alpha^{\frac{1}{\gamma}}.\end{aligned}$$

Namely, $f(t,r)$ satisfies

$$\frac{\partial f}{\partial t}+\nabla f+\alpha^{\frac{1}{\gamma}}=0, \quad f(T,r)=(1-\alpha)^{\frac{1}{\gamma}}.$$

Therefore, the equation (11) holds.

For the equation (11), we try to fit a solution of the form

$$\begin{aligned}\hat{f}(t,r)&=\exp\{D(t)+E(t)r\},\\ D(T)&=0, \quad E(T)=0.\end{aligned} \tag{12}$$

Putting (12) into (11), we derive

$$\hat{f}[X]=0$$

where

$$\begin{aligned}X \;=\;& D'(t)+(\tfrac{1-\gamma}{\gamma}b\rho\theta_t+a_t)E(t)\\ &+\tfrac{1}{2}b^2(\rho^2-\gamma\rho^2+\gamma)E^2(t)+\tfrac{1-\gamma}{2\gamma^2}\theta_t^2\\ &-\tfrac{\beta}{\gamma}+r\left(E'(t)+\tfrac{1-\gamma}{\gamma}\right).\end{aligned}$$

Furthermore, we obtain the following two ordinary differential equations:

$$E'(t)+\frac{1-\gamma}{\gamma}=0, \quad E(T)=0;$$

$$\begin{aligned}&D'(t)+(\tfrac{1-\gamma}{\gamma}b\rho\theta_t+a_t)E(t)\\ &+\tfrac{1}{2}b^2(\rho^2-\gamma\rho^2+\gamma)E^2(t)+\tfrac{1-\gamma}{2\gamma^2}\theta_t^2-\tfrac{\beta}{\gamma}=0,\\ &D(T)=0.\end{aligned}$$

Solving the above two equations, we obtain

$$E(t)=\frac{1-\gamma}{\gamma}(T-t), \tag{13}$$

$$D(t)=\int_t^T[(\frac{1-\gamma}{\gamma}b\rho\theta_s+a_s)E(s)+$$

$$\frac{1}{2}b^2(\rho^2-\gamma\rho^2+\gamma)E^2(s)+\frac{1-\gamma}{2\gamma^2}\theta_s^2-\frac{\beta}{\gamma}]ds. \tag{14}$$

Finally, we can summarize the optimal portfolio of the problem (4) under power-type utility function in the following theorem.

Theorem 3 *If utility function is given by* $U_1(x)=U_2(x)=\frac{x^{1-\gamma}}{1-\gamma}$, $\gamma>0$ *and* $\gamma\neq 1$, *then the optimal investment and consumption decision for the problem (4) is*

$$\begin{aligned}\pi_t^*&=\left(\tfrac{1}{\gamma}\cdot\tfrac{\mu_t-r_t}{\sigma_t^2}+\tfrac{b\rho}{\sigma_t}\cdot\tfrac{\partial f/\partial r}{f}\right)X_t,\\ C_t^*&=\alpha^{\frac{1}{\gamma}}f^{-1}X_t,\end{aligned} \tag{15}$$

where
$f=\alpha^{\frac{1}{\gamma}}\int_t^T e^{D(s)+E(s)r}ds+(1-\alpha)^{\frac{1}{\gamma}}e^{D(t)+E(t)r}$, $D(t)$ *and* $E(t)$ *are determined by (14) and (13) respectively.*

Remark 4 *Noting that these assumptions* $\gamma>0$, $0<\alpha<1$ *and obvious conclusion* $f(t,r)>0$, *we get*
$\frac{\partial f(t,r)}{\partial r}=\alpha^{\frac{1}{\gamma}}\int_t^T E(s)e^{D(s)+E(s)r}ds+(1-\alpha)^{\frac{1}{\gamma}}E(t)e^{D(t)+E(t)r}$, *where* $E(t)=\frac{1-\gamma}{\gamma}(T-t)$.

This implies that as the risk aversion coefficient
$0<\gamma<1\Rightarrow E(t)>0$, *and* $\frac{\partial f(t,r)}{\partial r}>0$.
$1<\gamma\Rightarrow E(t)<0$, *and* $\frac{\partial f(t,r)}{\partial r}<0$.

From the explicit solution (15) we can see that the optimal investment strategy π_t^* consists of two terms. The first term coincides with the classical optimal one in Merton(1969, 1971) . The second term can be interpreted as a correction term which is determined by ρ and b. The correction term exists when there is additional interest rate risk in the financial market. The correction term vanishes when risk-free interest rate is not stochastic process.

In order to compare the results obtained by Theorem 1 with those in the existing literatures, we analyze several special cases.

Special case 1. When $\alpha=0$ and $\beta=0$, the problem (4) is reduced to a dynamic asset allocation problem with stochastic interest rates. Then the optimal consumption strategy is $C_t^*=0$, and $f(t,r)=e^{D(t)+E(t)r}$, we obtain $\frac{\partial f(t,r)}{\partial r}=E(t)e^{D(t)+E(t)r}$ and $\frac{\partial f/\partial r}{f}=E(t)$. So the optimal investment strategy is given by

$$\pi_t^*=\left(\frac{1}{\gamma}\frac{\mu_t-r_t}{\sigma_t^2}+\frac{b\rho}{\sigma_t}\frac{\partial f/\partial r}{f}\right)X_t.$$

This implies that an investor is not consume any wealth and invests all his wealth in stock and bond. In addition, if risk-free interest rate is a constant or deterministic function of the time t, i.e., $b=0$ or $\rho=0$, then we have

$$\pi_t^*=\frac{1}{\gamma}\cdot\frac{\mu_t-r_t}{\sigma_t^2}X_t.$$

This is the well-known result in the power utility theory.

Special case 2. When $b=0$ and $a_t=0$, risk-free interest rate is a constant and the problem (4) is degenerated to an investment and consumption problem with constant interest rate. Then we have

$$D(t)=\int_t^T\left[\frac{1-\gamma}{2\gamma^2}\theta_s^2-\frac{\beta}{\gamma}\right]ds,$$

$$E(t)=\frac{1-\gamma}{\gamma}(T-t).$$

Denote $\eta(t)$ by

$$\eta(t) = \frac{1-\gamma}{2\gamma^2}\theta_t^2 - \frac{\beta}{\gamma} + \frac{1-\gamma}{\gamma}r,$$

and we obtain

$$f(t,r) = \alpha^{\frac{1}{\gamma}} \int_t^T e^{\int_s^T \eta(z)dz} ds + (1-\alpha)^{\frac{1}{\gamma}} e^{\int_s^T \eta(z)dz}.$$

Therefore, the optimal investment and consumption strategy is given by

$$\pi_t^* = \frac{1}{\gamma} \cdot \frac{\mu_t - r_t}{\sigma_t^2} X_t,$$

$$C_t^* = \left[\int_t^T e^{\int_s^T \eta(z)dz} ds + (\frac{1-\alpha}{\alpha})^{\frac{1}{\gamma}} e^{\int_s^T \eta(z)dz}\right]^{-1} X_t.$$

Special case 3. If $\gamma \to 1$, we have

$$E(t) = 0, \quad D(t) = \beta(t-T),$$

$$f(t,r) = \frac{\alpha}{\beta}(1 - e^{\beta(t-T)}) + (1-\alpha)e^{\beta(t-T)}.$$

Therefore, (15) can be rewritten as

$$\begin{aligned} \pi_t^* &= \tfrac{\mu_t - r_t}{\sigma_t^2} X_t, \\ C_t^* &= \tfrac{\alpha}{\frac{\alpha}{\beta}(1-e^{\beta(t-T)}) + (1-\alpha)e^{\beta(t-T)}} X_t. \end{aligned} \tag{16}$$

It is well known that when $\gamma \to 1$ power utility is degenerated to logarithm utility function. So we find that the optimal strategy (16) is consistent with those under logarithm utility function in the following subsection.

3.2 Logarithm utility

If logarithm utility is defined as

$$U_1(x) = U_2(x) = \ln x,$$

we try to fit a solution to (7) with the following form

$$H(t,r,x) = g(t)e^{-\beta t} \ln x + h(t,r),$$

$$g(T) = 1-\alpha, \ \ h(T,r) = 0.$$

Then we have

$$\frac{\partial H}{\partial t} = g'(t)e^{-\beta t} \ln x - \beta g(t)e^{-\beta t} \ln x + \frac{\partial h}{\partial t},$$

$$\frac{\partial H}{\partial x} = g(t)e^{-\beta t}\frac{1}{x}, \quad \frac{\partial^2 H}{\partial x^2} = -g(t)e^{-\beta t}\frac{1}{x^2},$$

$$\frac{\partial H}{\partial r} = \frac{\partial h}{\partial r}, \quad \frac{\partial^2 H}{\partial r^2} = \frac{\partial^2 h}{\partial r^2}, \quad \frac{\partial^2 H}{\partial x \partial r} = 0.$$

So the optimal investment and consumption strategy is

$$\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} x, \quad C_t^* = \frac{\alpha}{g(t)} x.$$

Introducing these derivatives into (7), we derive

$$\begin{aligned} &e^{-\beta t} \ln x\, (g'(t) - \beta g(t) + \alpha) + \tfrac{\partial h}{\partial t} + a_t \tfrac{\partial h}{\partial r} \\ &+ \tfrac{1}{2}b^2 \tfrac{\partial^2 h}{\partial r^2} + (r + \tfrac{1}{2}\theta_t^2)e^{-\beta t} g(t) \\ &+ \alpha e^{-\beta t}(\ln\alpha - \ln g(t) - 1) = 0. \end{aligned}$$

So we can obtain the following two equations

$$g'(t) - \beta g(t) + \alpha = 0, \quad g(T) = 1-\alpha; \tag{17}$$

$$\begin{aligned} &\tfrac{\partial h}{\partial t} + a_t \tfrac{\partial h}{\partial r} + \tfrac{1}{2}b^2 \tfrac{\partial^2 h}{\partial r^2} + (r + \tfrac{1}{2}\theta_t^2)e^{-\beta t} g(t) \\ &+ \alpha e^{-\beta t}(\ln\alpha - \ln g(t) - 1) = 0, \\ &h(T,r) = 0. \end{aligned} \tag{18}$$

The solution to (17) is given by

$$g(t) = (1-\alpha)e^{-\beta(T-t)} - \frac{\alpha}{\beta}\left(e^{-\beta(T-t)} - 1\right).$$

Suppose that the solution to (18) is of the structure

$$h(t,r) = A(t) + B(t)r, \quad A(T) = 0, \quad B(T) = 0.$$

Putting this in (18), we derive

$$\begin{aligned} &r(B'(t) + g(t)e^{-\beta t}) + A'(t) + \tfrac{1}{2}\theta_t^2 g(t)e^{-\beta t} \\ &+ a_t B(t) + \alpha e^{-\beta t}(\ln\alpha - \ln g(t) - 1) = 0. \end{aligned} \tag{19}$$

We can split (19) into two equations in order to eliminate the dependence on r.

$$B'(t) + g(t)e^{-\beta t} = 0, \quad B(T) = 0;$$

$$A'(t) + \frac{1}{2}\theta_t^2 g(t)e^{-\beta t} + a_t B(t) +$$

$$\alpha e^{-\beta t}(\ln\alpha - \ln g(t) - 1) = 0, \quad A(T) = 0.$$

Solving the equations, we obtain

$$B(t) = \int_t^T g(t)e^{-\beta t} dt,$$

$$A(t) = \int_t^T (\frac{1}{2}\theta_t^2 g(t)e^{-\beta t} + a_t B(t) +$$

$$\alpha e^{-\beta t}(\ln\alpha - \ln g(t) - 1))dt.$$

Finally, we conclude the results of the problem (4) in the logarithm utility case.

Theorem 5 *If utility function is given by $U_1(x) = U_2(x) = \ln x$, then the optimal investment and consumption strategy of the problem (4) is*

$$\begin{aligned} \pi_t^* &= \frac{\mu_t - r_t}{\sigma_t^2} X_t, \\ C_t^* &= \frac{\alpha\beta}{\beta(1-\alpha)e^{-\beta(T-t)} - \alpha\left(e^{-\beta(T-t)} - 1\right)} X_t. \end{aligned} \tag{20}$$

Remark 6 *It can be seen from (20) that the optimal investment strategy under logarithm utility function is coincides with the classical optimal one in Merton (1969, 1971). It implies that additional risk of interest rate cannot produce an effect on an investor. Therefore, when the risk preference of an investor is determined by logarithm utility function, an investor need not take additional risk of interest rate into consideration.*

4 Numerical analysis

This section provides a numerical example to demonstrate the effect of market parameters on the optimal investment and consumption strategy. Take power utility for example, through the numerical analysis, unless otherwise stated, the basic parameters are given by: $a_t = 0.06$, $b = 0.12$, $r_0 = 0.05$, $\mu_t = 0.12$, $\sigma_t = 0.20$, $\rho = 0.6$, $t = 0$, $T = 1$, $\alpha = 0.6$, $\beta = 0.5$, $\gamma = 2$, $x_0 = 100$.

4.1 The impact on the optimal investment strategy

Fig.1-Fig.6 provides the effect of main market parameters on the optimal investment strategy. We can obtain some sensitivity analysis and economic implications from the graphs of the Fig.1-Fig.6.

(a1) The optimal investment strategy π_t^* decreases with respect to the parameter b in the positive correlation setting, while π_t^* increases with respect to the value of b in the negative correlation setting. This is to say, when $\rho > 0$, the larger the value of b, the less the amount invested in the stock; when $\rho < 0$, the amount invested in the stock is increasing as the value of b becomes bigger. From the Ho-Lee model, we can see that the parameter b represents the risk of interest rate. When $\rho > 0$, the bigger the value of b, the bigger the risk of interest rate. As the same time, the volatility of the stock will become bigger. These factors lead to that the investor will invest less money in the stock. In addition, we can conclude that the amount invested in the stock in the negative correlation setting is more than that in the positive correlation setting.

(a2) π_t^* decreases in the value of σ_t, and the amount invested in the stock in the negative correlation setting is larger than that in the positive correlation setting. When the value of σ_t becomes more and more bigger, the amount investing the stock becomes more and more less. In fact, the parameter σ_t represents the volatility of the stock. The bigger the value of σ_t, the more investment risk the investor is faced with. Therefore, the investor wishes to invest less money in the stock.

(a3) π_t^* increases with respect to μ_t. It means that when the value of μ_t becomes more and more bigger, the amount invested in the stock becomes bigger as well.

(a4) π_t^* decreases with respect to the time t when $\rho < 0$, while π_t^* increases with respect to the time t, when $\rho > 0$.

(a5) π_t^* is decreasing as the value of γ is increasing. Meantime, the amount invested in the stock in the positive correlation setting is bigger than that in the negative correlation setting when $0 < \gamma < 1$; the amount invested in the stock is contrary to it when $\gamma > 1$. On the other hand, it is well known that power utility is degenerated to logarithm utility when $\gamma \to 1$. In a result, it can be seen that when $0 < \gamma < 1$, the optimal investment strategy in the power utility case is larger than that in the logarithm utility case. From economic implication of the parameter γ, when the value of γ become bigger, the risk aversion of the investor is more and more bigger, which leads to that the investor will invest less amount of the wealth in the stock.

(a6) π_t^* increases with respect to the parameter ρ when $0 < \gamma < 1$; π_t^* decreases with respect to the parameter ρ when $\gamma > 1$.

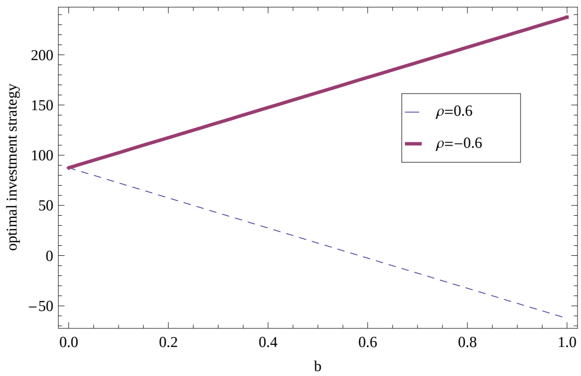

Figure 1: The impact of b on π_t^*

4.2 The impact on the optimal consumption strategy

The graphs in the Fig.7-Fig.12 illustrate the effect of main market parameters on the optimal consumption strategy. We can obtain the following conclusions from the Fig.7-Fig.12.

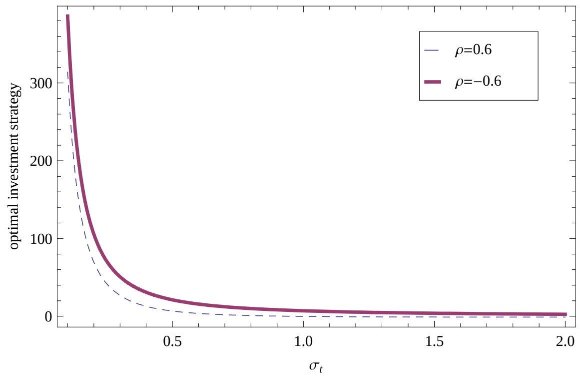

Figure 2: The impact of σ_t on π_t^*

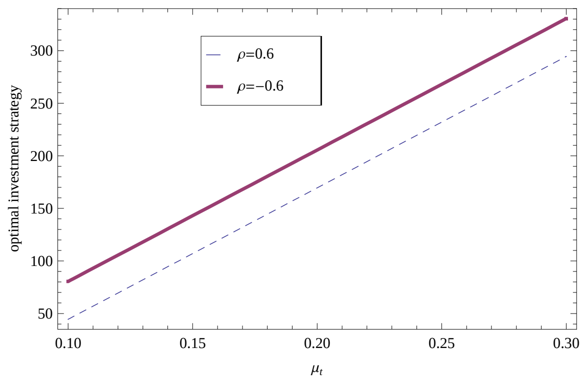

Figure 3: The impact of μ_t on π_t^*

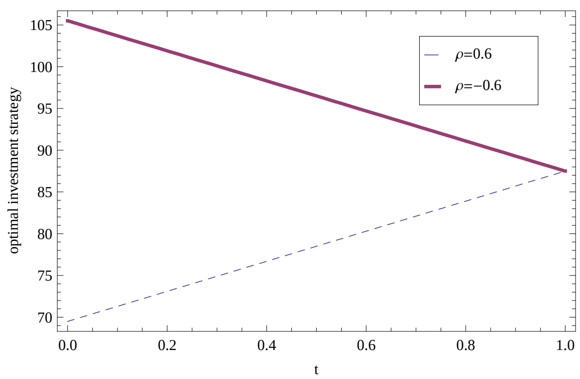

Figure 4: The impact of the time t on π_t^*

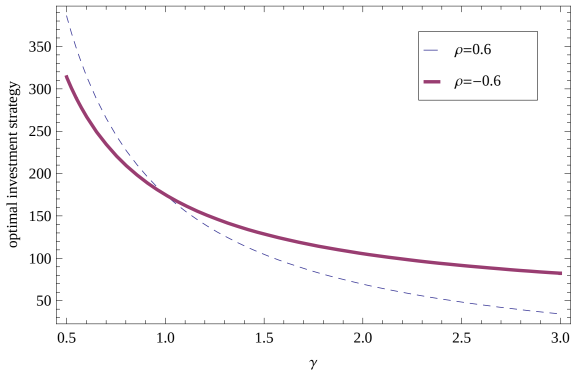

Figure 5: The impact of γ on π_t^*

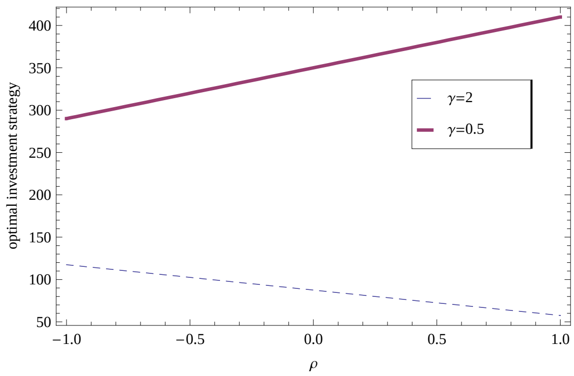

Figure 6: The impact of ρ on π_t^*

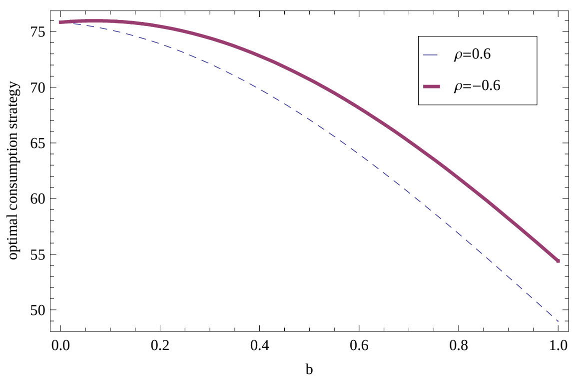

Figure 7: The impact of b on C_t^*

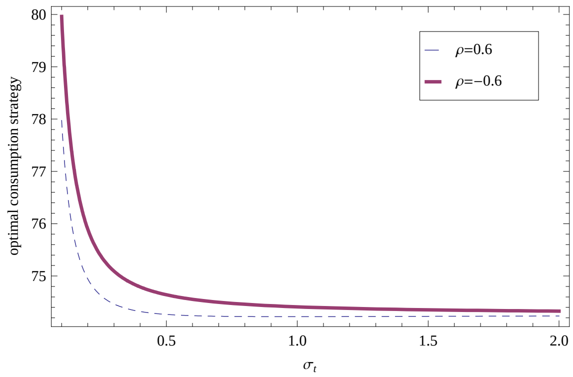

Figure 8: The impact of σ_t on C_t^*

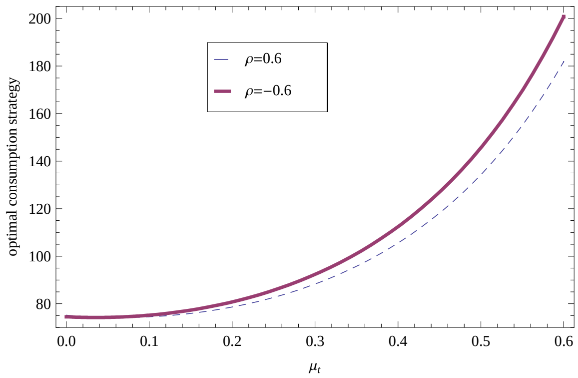

Figure 9: The impact of μ_t on C_t^*

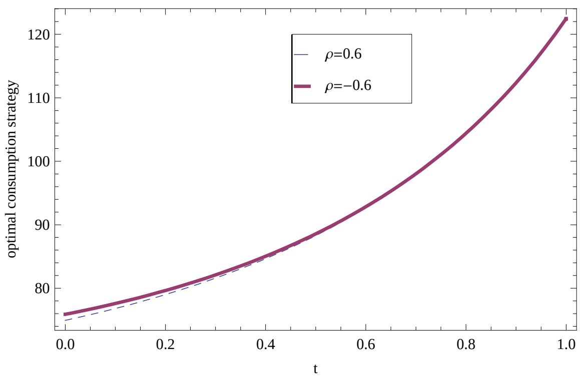

Figure 10: The impact of the time t on C_t^*

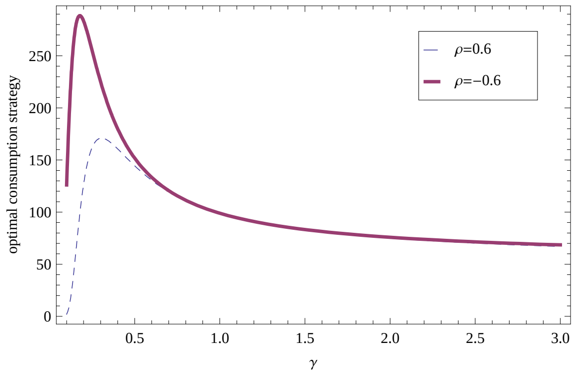

Figure 11: The impact of γ on C_t^*

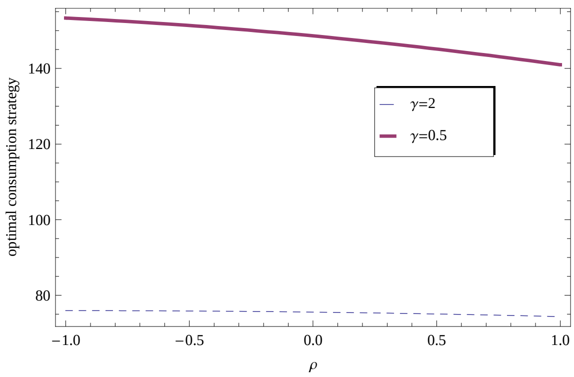

Figure 12: The impact of ρ on C_t^*

(b1) The optimal consumption strategy C_t^* decreases with respect to the parameter b, and the amount the investor consumes in the negative correlation setting is larger than that in the positive correlation setting. It tells us that the more the risk of interest rate, the less the investor wishes to consume.

(b2) C_t^* is decreasing as the value of σ_t is increasing. In addition, the amount invested in the stock in the negative correlation setting is more than that in the positive correlation setting. It displays that the bigger the volatility of the stock, the less the amount consumed by the investor.

(b3) C_t^* increases with respect to the parameter μ_t. It implies that when the appreciation rate of the stock becomes more and more bigger, the amount invested in the stock is increasing as well.

(b4) C_t^* is increasing as the value of the time t becomes bigger.

(b5) C_t^* first increases then slowly decreases with respect to the parameter γ. It means that the investor consumes initially the more amount of money and later slowly decreases the amount to consume as the value of γ becomes more and more bigger.

(b6) C_t^* is slowly decreasing in the parameter ρ. It demonstrates that the correlation of interest rate and stock price becomes bigger, the investor wishes to consume less money.

5 Conclusion

This paper investigates an investment and consumption problem with stochastic interest rate, in which interest rate is assumed to follow the Ho-Lee model and be correlated with stock price. The optimal investment and consumption strategies for power and logarithm utility function are derived by applying dynamic programming and variable change technique. A numerical example is given to illustrate the impact of market parameters on the investment and consumption strategy.

In future research, we try to treat the solutions of the problem with other stochastic interest rate models, such as the investment and consumption problem with Cox-Ingersoll-Ross model(CIR(1985)) and stochastic volatility, which will lead to a more sophisticated HJB equation and cannot tackle it. We leave this point for future research.

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