Fast Algorithms for Solving RFPrLR Circulant Linear Systems

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Abstract: In this paper, fast algorithms for solving RFPrLR circulant linear systems are presented by the fast algorithm for computing polynomials. The unique solution is obtained when the RFPrLR circulant matrix over the complex field $\mathbb{C}$ is nonsingular, and the special solution and general solution are obtained when the RFPrLR circulant matrix over the complex field $\mathbb{C}$ is singular. The extended algorithms is used to solve the RLPrFL circulant linear systems. Examples show the effectiveness of the algorithms.

Key–Words: RFPrLR circulant matrix, Linear system, Fast algorithm.

1 Introduction

Circulant matrix family have important applications in various disciplines including image processing [1], communications [2], signal processing [3], encoding [4], and preconditioner [5]. They have been put on firm basis with the work of P. Davis [6] and Z. L. Jiang [7].

The circulant matrices, long a fruitful subject of research [6, 7, 8, 9, 10, 11, 12], have in recent years been extended in many directions [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The $f(x)$-circulant matrices are another natural extension of this well-studied class, and can be found in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes [13]. The properties and structures of the $x^n - rx - 1$-circulant matrices, which are called RFPrLR circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good algorithms for solving the RFPrLR circulant linear system.

In this paper, the fast algorithms presented avoid the problems of error and efficiency produced by computing a great number of triangular functions by means of other general fast algorithms. There is only error of approximation when the fast algorithm is realized by computers, and only the elements in the first row of the RFPrLR circulant matrix and the constant term r are used by the fast algorithm, so the result of the computation is accurate in theory. Specially, the result computed by a computer is accurate over the rational field.

Definition 1. *A row first-plus-rlast right (*RFPrLR*) circulant matrix with the first row $(a_0, a_1, ..., a_{n-1})$ is meant a square matrix over the complex field $\mathbb{C}$ of the form*

$$A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \ldots, a_{n-1})$$

$$= \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 + ra_{n-1} & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 + ra_2 & \cdots & a_1 \\ a_1 & a_2 + ra_1 & \cdots & a_0 + ra_{n-1} \end{pmatrix}. \quad (1)$$

It can be seen that the matrix over the complex field $\mathbb{C}$ with an arbitrary first row and the following rule for obtaining any other row from the previous one: Get the i+1st row by adding r times the last element of the ith row to the first element of the ith row, and then shifting the elements of the ith row (cyclically) one position to the right.

Note that the RFPrLR circulant matrix is a $x^n - rx - 1$ circulant matrix [13], and when $r = 0$, A becomes a circulant matrix [6, 7], when $r = 1$, A becomes a RFPLR circulant matrix[17, 18, 19, 20], and when $r = -1$, A becomes a RFMLR circulant matrix [21, 22, 23]. Thus it is the extension of circulant matrix, RFPLR circulant matrix and RFMLR circulant matrix.

In this paper, let r be a complex number and satisfy $r^n \neq \frac{n^n}{(1-n)^{(n-1)}}$. It is easily verified that the polynomial $g(x) = x^n - rx - 1$ has no repeated roots in the complex field if $r^n \neq \frac{n^n}{(1-n)^{(n-1)}}$.

We define $\Theta_{(1,r)}$ as the basic RFPrLR circulant matrix, that is,

$$\Theta_{(1,r)} = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 1 & r & \dots & 0 & 0 \end{pmatrix}_{n\times n.} \quad (2)$$

It is easily verified that the polynomial $g(x) = x^n - rx - 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(1,r)}$, i.e., $\Theta_{(1,r)}$ is nonsingular nonderogatory. In addition,

$$\Theta^n_{(1,r)} = I_n + r\Theta_{(1,r)}.$$

In view of the structure of the powers of the basic RFPrLR circulant matrix $\Theta_{(1,r)}$, it is clear that

$$\begin{aligned} A &= \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \dots, a_{n-1}) \\ &= \sum_{i=0}^{n-1} a_i \Theta^i_{(1,r)}. \end{aligned} \quad (3)$$

Thus, A is a RFPrLR circulant matrix if and only if $A = f(\Theta_{(1,r)})$ for some polynomial $f(x)$. The polynomial $f(x) = \sum\limits_{i=0}^{n-1} a_i x^i$ will be called the representer of the RFPrLR circulant matrix A.

In addition, the product of two RFPrLR circulant matrices is a RFPrLR circulant matrix and the inverse of a nonsingular RFPrLR circulant matrix is also a RFPrLR circulant matrix. Furthermore, a RFPrLR circulant matrices commute under multiplication.

Definition 2. *A row last-plus-rfirst left (*RLPrFL*) circulant matrix with the first row* $(a_0, a_1, \dots, a_{n-1})$ *is meant a square matrix over the complex field* $\mathbb{C}$ *of the form*

$$A = \mathrm{RLP}r\mathrm{FLcircfr}(a_0, a_1, \dots, a_{n-1})$$

$$= \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & \dots & ra_0 + a_{n-1} & a_0 \\ \vdots & \iddots & \vdots & \vdots \\ a_{n-2} & \dots & ra_{n-3} + a_{n-4} & a_{n-3} \\ ra_0 + a_{n-1} & \dots & ra_{n-2} + a_{n-3} & a_{n-2} \end{pmatrix}. \quad (4)$$

It can be seen that the matrix over the complex field $\mathbb{C}$ with an arbitrary first row and the following rule for obtaining any other row from the previous one: Get the i+1st row by adding r times the first element of the ith row to the last element of the ith row, and then shifting the elements of the ith row (cyclically) one position to the left.

Obviously, the RLPrFL circulant matrix over the complex field $\mathbb{C}$ is the extension of left circulant matrix[6, 7, 36].

For the convenience of application, we give the obvious results in following lemmas.

Lemma 3. *Let* $A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \dots, a_{n-1})$ *be a RFPrLR circulant matrix over* $\mathbb{C}$ *and let* $B = \mathrm{RLP}r\mathrm{FLcircfr}(a_{n-1}, a_{n-2}, \dots, a_1, a_0)$ *be a RLPrFL circulant matrix over* $\mathbb{C}$*. Then*

$$B\hat{I}_n = A$$

or

$$B = A\hat{I}_n,$$

where

$$\hat{I}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (5)$$

Lemma 4 ([16])**.** *Let* $\mathbb{C}[x]$ *be the polynomial ring over a field* $\mathbb{C}$*, and let* $f(x), g(x) \in \mathbb{C}[x]$*. Suppose that the polynomial matrix*

$$\begin{pmatrix} f(x) & 1 & 0 \\ g(x) & 0 & 1 \end{pmatrix}$$

is changed into the polynomial matrix

$$\begin{pmatrix} d(x) & u(x) & v(x) \\ 0 & s(x) & t(x) \end{pmatrix}$$

by a series of elementary row operations, then

$$(f(x), g(x)) = d(x)$$

and

$$f(x)u(x) + g(x)v(x) = d(x).$$

Lemma 5. $\mathbb{C}[x]/\langle x^n - rx - 1\rangle \cong \mathbb{C}[\Theta_{(1,r)}]$.

Proof. Consider the following $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \varphi:\ & \mathbb{C}[x] \to \mathbb{C}[\Theta_{(1,r)}] \\ & f(x) \to A = f(\Theta_{(1,r)}) \end{aligned}$$

for $f(x) \in \mathbb{C}[x]$. It is clear that φ is an $\mathbb{C}$-algebra epimorphism. So we have

$$\mathbb{C}[x]/\mathrm{ker}\varphi \cong \mathbb{C}[\Theta_{(1,r)}].$$

Since $\mathbb{C}[x]$ is a principal ideal integral domain, there is a monic polynomial $p(x) \in \mathbb{C}[x]$ such that ker$\varphi = \langle p(x)\rangle$. Since $x^n - rx - 1$ is the minimal polynomial of $\Theta_{(1,r)}$, then $p(x) = x^n - rx - 1$. $\square$

By Lemma 5, we have the following lemma.

Lemma 6. *Let $A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \ldots, a_{n-1})$ be a RFPrLR-circulant matrix. Then A is nonsingular if and only if*

$$(f(x), g(x)) = 1,$$

where

$$f(x) = \sum_{i=0}^{n-1} a_i x^i$$

is the representer of A and

$$g(x) = x^n - rx - 1.$$

Proof. A is nonsingular if and only if $f(x) + \langle x^n - rx - 1\rangle$ is an invertible element in $\mathbb{C}[x]/\langle x^n - rx - 1\rangle$. By Lemma 5, if and only if there exists

$$u(x) + \langle x^n - rx - 1\rangle \in \mathbb{C}[x]/\langle x^n - rx - 1\rangle$$

such that

$$f(x)u(x) + \langle x^n - rx - 1\rangle \equiv 1 + \langle x^n - rx - 1\rangle,$$

if and only if there exist $u(x), v(x) \in \mathbb{C}[x]$ such that

$$f(x)u(x) + (x^n - rx - 1)v(x) = 1$$

if and only if $(f(x), x^n - rx - 1) = 1$. □

In [37] for an $m \times n$ matrix A, any solution to the equation $AXA = A$ is called a *generalized inverse* of A. In addition, if X satisfies $X = XAX$, then A and X are said to be semi-inverses $A^{\{1,2\}}$.

In this paper, we only consider square matrices A. In [38] the smallest positive integer k for which rank(A^{k+1})= rank(A^k) holds is called the *index* of A. If A has index 1, the generalized inverse X of A is called the *group inverse* $A^{\#}$ of A. Clearly A and X are group inverses if and only if they are semi-inverses and $AX = XA$.

In [39] and [40] a semi-inverse X of A was considered in which the nonzero eigenvalues of X are the reciprocals of the nonzero eigenvalue of A. These matrices were called *spectral inverses*. It was shown in [40] that a nonzero matrix A has a unique spectral inverse, A^s, if and only if A has index 1: when A^s is the group inverse $A^{\#}$ of A.

2 Fast algorithms for solving the RFPrLR circulant linear system and the RLPrFL circulant linear system

Consider the RFPrLR circulant linear system

$$AX = b, \tag{6}$$

where A is given in Equation (1),

$$X = (x_1, x_2, \ldots, x_n)^T,$$

$$b = (b_{n-1}, \ldots, b_1, b_0)^T.$$

If A is nonsingular, then Equation (6) has a unique solution $X = A^{-1}b$.

If A is singular and Equation (6) has a solution, then the general solution of Equation (6) is

$$X = A^{\{1,2\}}b + (I_n - A^{\{1,2\}}A)Y,$$

where Y is an arbitrary n-dimension column vector.

The key of the problem is how to find $A^{-1}b$, $A^{\{1,2\}}b$ and $A^{\{1,2\}}A$. For this purpose, we at first prove the following results.

Theorem 7. *Let $A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0,a_1,\cdots, a_{n-1})$ be a nonsingular RFPrLR circulant matrix of order n over $\mathbb{C}$ and $b = (b_{n-1}, \ldots, b_1, b_0)^T$. Then there exists a unique RFPrLR circulant matrix*

$$C = \mathrm{RFP}r\mathrm{LRcircfr}(c_0, c_1, \ldots, c_{n-1})$$

of order n over $\mathbb{C}$ such that the unique solution of Equation (6) is the last column of C, i.e.

$$X = (c_{n-1}, \ldots, c_1, c_0 + rc_{n-1})^T.$$

Proof. Since matrix $A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \ldots, a_{n-1})$ is nonsingular, then by Lemma 6 we have $(f(x), g(x)) = 1$, where $f(x) = \sum_{i=0}^{n-1} a_i x^i$ is the representer of A and $g(x) = x^n - rx - 1$.

Let $B = \mathrm{RFP}r\mathrm{LRcircfr}(b_0 - rb_{n-1}, b_1, \ldots, b_{n-1})$ be the RFPrLR circulant matrix of order n constructed by $b = (b_{n-1}, \ldots, b_1, b_0)^T$. Then the representer of B is

$$b(x) = (b_0 - rb_{n-1}) + \sum_{i=1}^{n-1} b_i x^i.$$

Therefore, we can change the polynomial matrix

$$\begin{pmatrix} f(x) & \vdots & 1 & 0 & \vdots & b(x) \\ g(x) & \vdots & 0 & 1 & \vdots & 0 \end{pmatrix}$$

into the polynomial matrix

$$\begin{pmatrix} 1 & \vdots & u(x) & v(x) & \vdots & c(x) \\ 0 & \vdots & s(x) & t(x) & \vdots & c_1(x) \end{pmatrix}$$

by a series of elementary row operations. By Lemma 4, we have

$$\begin{pmatrix} u(x) & v(x) \\ s(x) & t(x) \end{pmatrix} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} u(x) & v(x) \\ s(x) & t(x) \end{pmatrix} \begin{pmatrix} b(x) \\ 0 \end{pmatrix} = \begin{pmatrix} c(x) \\ c_1(x) \end{pmatrix},$$

i.e.

$$f(x)u(x) + g(x)v(x) = 1, u(x)b(x) = c(x).$$

Substituting x by $\Theta_{(1,r)}$ in the above two equations respectively, we have

$$f(\Theta_{(1,r)})u(\Theta_{(1,r)}) + g(\Theta_{(1,r)})v(\Theta_{(1,r)}) = I_n,$$

$$u(\Theta_{(1,r)})b(\Theta_{(1,r)}) = c(\Theta_{(1,r)}).$$

Since

$$f(\Theta_{(1,r)}) = A, g(\Theta_{(1,r)}) = 0$$

and

$$b(\Theta_{(1,r)}) = B,$$

then

$$Au(\Theta_{(1,r)}) = I_n, \tag{7}$$

$$u(\Theta_{(1,r)})B = c(\Theta_{(1,r)}). \tag{8}$$

By Equation (7), we know that $u(\Theta_{(1,r)})$ is a unique inverse A^{-1} of A. According to Equation (8) and the characters of the RFPrLR circulant matrix, we know that the last column of C is

$$(c_{n-1}, \dots, c_1, c_0 + rc_{n-1})^T = A^{-1}b.$$

Since $AA^{-1}b = b$, then

$$A^{-1}b = (c_{n-1}, \dots, c_1, c_0 + rc_{n-1})^T$$

is the solution of Equation (6). Since both A^{-1} and B are unique, then $A^{-1}B$ is also unique. So

$$X = (c_{n-1}, \dots, c_1, c_0 + rc_{n-1})^T = A^{-1}b$$

is unique. □

Theorem 8. *Let $A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \dots, a_{n-1})$ be a singular RFPrLR circulant matrix of order n over $\mathbb{C}$ and $b = (b_{n-1}, \dots, b_1, b_0)^T$. If the solution of Equation (6) exists, then there exist a unique RFPrLR circulant matrix $C = \mathrm{RFP}r\mathrm{LRcircfr}(c_0, c_1, \dots, c_{n-1})$ and a unique RFPrLR circulant matrix*

$$E = \mathrm{RFP}r\mathrm{LRcircfr}(e_0, e_1, \dots, e_{n-1})$$

of order n over $\mathbb{C}$ such that

$$X_1 = (c_{n-1}, \dots, c_1, c_0 + rc_{n-1})^T$$

is a unique special solution of Equation (6) and

$$X_2 = X_1 + (I_n - E)Y$$

is a general solution of Equation (6), where Y is an arbitrary n-dimension column vector.

Proof. Since $A = \mathrm{RFP}r\mathrm{LRcircfr}(a_0, a_1, \dots, a_{n-1})$, then the representer of A is $f(x) = \sum_{i=0}^{n-1} a_i x^i$ and the characteristic polynomial of $\Theta_{(1,r)}$ is

$$g(x) = x^n - rx - 1.$$

We can change the polynomial matrix $\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$ into the polynomial matrix $\begin{pmatrix} d(x) \\ 0 \end{pmatrix}$ by a series of elementary row operations. Since A is singular, by Lemma 4 and Lemma 6, we know that $d(x)$ is the largest common factor, which is not equal to 1, of $f(x)$ and $g(x)$. Let

$$f(x) = d(x)f_1(x)$$

and

$$g(x) = d(x)g_1(x),$$

then

$$(f_1(x), g_1(x)) = 1.$$

Since $(d(x), g_1(x)) = 1$, we have

$$(f(x), g_1(x)) = (d(x)f_1(x), g_1(x)) = 1.$$

Since $(d(x), g_1(x)) = 1$ and $(f(x), g_1(x)) = 1$, we have

$$(f(x)d(x), g_1(x)) = 1.$$

Let $B = \mathrm{RFP}r\mathrm{LRcircfr}(b_0 - rb_{n-1}, b_1, \dots, b_{n-1})$ be the RFPrLR circulant matrix of order n constructed by $b = (b_{n-1}, \dots, b_1, b_0)^T$. Then the representer of B is

$$b(x) = (b_0 - rb_{n-1}) + \sum_{i=1}^{n-1} b_i x^i.$$

Therefore, we can change the polynomial matrix

$$\begin{pmatrix} f(x)d(x) & \vdots & 1 & 0 & \vdots & d(x)b(x) & \vdots & d(x)f(x) \\ g_1(x) & \vdots & 0 & 1 & \vdots & 0 & \vdots & 0 \end{pmatrix}$$

into the polynomial matrix

$$\begin{pmatrix} 1 & \vdots & u(x) & v(x) & \vdots & c(x) & \vdots & e(x) \\ 0 & \vdots & s(x) & t(x) & \vdots & c_1(x) & \vdots & e_1(x) \end{pmatrix}$$

by a series of elementary row operations. Then, by Lemma 4, we have

$$f(x)d(x)u(x) + g_1(x)v(x) = 1, \tag{9}$$

$$d(x)u(x)b(x) = c(x), \tag{10}$$

$$d(x)u(x)f(x) = e(x). \quad (11)$$

Therefore

$$f(x)d(x)u(x)f(x) + g(x)f_1(x)v(x) = f(x). \quad (12)$$

Substituting x by $\Theta_{(1,r)}$ in Equations (10), (11) and (12), respectively, we have

$$d(\Theta_{(1,r)})u(\Theta_{(1,r)})b(\Theta_{(1,r)}) = c(\Theta_{(1,r)}),$$

$$d(\Theta_{(1,r)})u(\Theta_{(1,r)})f(\Theta_{(1,r)}) = e(\Theta_{(1,r)}),$$

$$f(\Theta_{(1,r)})d(\Theta_{(1,r)})u(\Theta_{(1,r)})f(\Theta_{(1,r)})+$$
$$g(\Theta_{(1,r)})f_1(\Theta_{(1,r)})v(\Theta_{(1,r)}) = f(\Theta_{(1,r)}).$$

Since $f(\Theta_{(1,r)}) = A$, $g(\Theta_{(1,r)}) = 0$ and $b(\Theta_{(1,r)}) = B$, then

$$d(\Theta_{(1,r)})u(\Theta_{(1,r)})B = c(\Theta_{(1,r)}), \quad (13)$$

$$d(\Theta_{(1,r)})u(\Theta_{(1,r)})A = e(\Theta_{(1,r)}), \quad (14)$$

$$Ad(\Theta_{(1,r)})u(\Theta_{(1,r)})A = A. \quad (15)$$

In the same way, we have

$$\begin{aligned} &d(\Theta_{(1,r)})u(\Theta_{(1,r)})Ad(\Theta_{(1,r)})u(\Theta_{(1,r)}) \\ &= d(\Theta_{(1,r)})u(\Theta_{(1,r)}). \end{aligned} \quad (16)$$

By Equation (15) and (16), we know $T = d(\Theta_{(1,r)})u(\Theta_{(1,r)})$ is a semi-inverses $A^{\{1,2\}}$ of A.

Let

$$\begin{aligned} C &= TB = c(\Theta_{(1,r)}) \\ &= \text{RFP}r\text{LRcircfr}(c_0, c_1, \dots, c_{n-1}). \end{aligned}$$

and let

$$\begin{aligned} E &= TA = e(\Theta_{(1,r)}) \\ &= \text{RFP}r\text{LRcircfr}(e_0, e_1, \dots, e_{n-1}). \end{aligned}$$

According to Equation (13) and the characters of the RFPrLR circulant matrix, we know that the last column of C is

$$(c_{n-1}, \dots, c_1, c_0 + rc_{n-1})^T = Tb.$$

Since $AX = b$ has a solution, then $ATb = b$, i.e., Tb is a solution of Equation (6). We have

$$\begin{aligned} A[Tb + (I_n - E)Y] &= ATb + A(I_n - E)Y \\ &= b + AY - AEY \\ &= b + AY - ATAY \\ &= b + AY - AY = b, \end{aligned}$$

so $X_2 = X_1 + (I_n - E)Y$ is the general solution of Equation (6), where Y is an arbitrary n-dimension column vector and $X_1 = Tb$.

Since both A and T are RFPrLR circulant matrices, then $AT = TA$. If there exists another RFPrLR circulant matrix T_1 such that

$$AT_1A = A, T_1AT_1 = T_1, T_1A = AT_1.$$

Let $AT = TA = H$ and $AT_1 = T_1A = F$. Clearly $H^2 = H$ and $F^2 = F$. Thus we have

$$H = AT = AT_1AT = FH,$$

$$F = T_1A = T_1ATA = FH.$$

So $H = F$. Hence

$$T = TAT = HT = FT = T_1AT$$

$$= T_1H = T_1F = T_1AT_1 = T_1.$$

So T is unique. Hence $TB = C, TA = E$ and Tb are also unique. □

By Theorem 7 and Theorem 8, we have the following fast algorithm for solving the RFPrLR circulant linear system (6):

Step 1. From the RFPrLR circulant linear system (6), we get the polynomial

$$f(x) = \sum_{i=0}^{n-1} a_i x^i, g(x) = x^n - rx - 1$$

and

$$b(x) = (b_0 - rb_{n-1}) + \sum_{i=1}^{n-1} b_i x^i;$$

Step 2. Change the polynomial matrix $\begin{pmatrix} f(x) & b(x) \\ g(x) & 0 \end{pmatrix}$ into the polynomial matrix $\begin{pmatrix} d(x) & c(x) \\ 0 & c_1(x) \end{pmatrix}$ by a series of elementary row operations;

Step 3. If $d(x) = 1$, then the RFPrLR circulant linear system (6) has a unique solution. Substituting x by $\Theta_{(1,r)}$ in polynomial $c(x)$, we obtain a RFPrLR circulant matrix

$$C = c(\Theta_{(1,r)}) = \text{RFP}r\text{LRcircfr}(c_0, c_1, \dots, c_{n-1}).$$

So the unique solution of $AX = b$ is

$$(c_{n-1}, \dots, c_1, c_0 + rc_{n-1})^T;$$

Step 4. If $d(x) \neq 1$, $d(x)$ dividing $g(x)$, we get the quotient $g_1(x)$ and change the polynomial matrix

$$\begin{pmatrix} f(x)d(x) & d(x)b(x) & d(x)f(x) & d(x)f(x)b(x) \\ g_1(x) & 0 & 0 & 0 \end{pmatrix}$$

into the polynomial matrix

$$\begin{pmatrix} 1 & c(x) & e(x) & r(x) \\ 0 & c_1(x) & e_1(x) & r_1(x) \end{pmatrix}$$

by a series of elementary row operations;

Step 5. Substituting x by $\Theta_{(1,r)}$ in polynomial $r(x)$, we get a RFPrLR circulant matrix

$$R = r(\Theta_{(1,r)}) = AA^{\{1,2\}}B.$$

If the last column of R isn't b, then $AX = b$ has no solution. Otherwise, the RFPrLR circulant linear system $AX = b$ has a solution. Substituting x by $\Theta_{(1,r)}$ in polynomial $c(x)$ and $e(x)$, we have two RFPrLR circulant matrices

$$C = c(\Theta_{(1,r)}) = A^{\{1,2\}}B$$

and

$$E = e(\Theta_{(1,r)}) = A^{\{1,2\}}A.$$

So the unique special solution of $AX = b$ is

$$X_1 = (c_{n-1}, \ldots, c_1, c_0 + rc_{n-1})^T$$

and the general solution of $AX = b$ is

$$X_2 = X_1 + (I_n - E)Y,$$

where Y is an arbitrary n-dimension column vector.

The advantage of the above algorithm is that it can solve $AX = b$ whether the coefficient matrix of $AX = b$ is singular or nonsingular.

By Lemma 3 and Theorem 7, we have the following theorem.

Theorem 9. *Let $A = \mathrm{RLP}r\mathrm{FLcircfr}(a_{n-1}, \ldots, a_1, a_0)$ be a nonsingular RLPrFL circulant matrix of order n over $\mathbb{C}$ and $b = (b_{n-1}, \ldots, b_1, b_0)^T$. Then there exists a unique RFPrLR circulant matrix*

$$C = \mathrm{RFP}r\mathrm{LRcircfr}(c_0, c_1, \ldots, c_{n-1})$$

of order n over $\mathbb{C}$ such that the unique solution of $AX = b$ is

$$X = (c_0 + rc_{n-1}, c_1, \ldots, c_{n-2}, c_{n-1})^T.$$

By Lemma 3 and Theorem 8, we have the following theorem.

Theorem 10. *Let $A = \mathrm{RLP}r\mathrm{FLcircfr}(a_{n-1}, \ldots, a_1, a_0)$ be a singular RLPrFL circulant matrix of order n over $\mathbb{C}$ and $b = (b_{n-1}, \ldots, b_1, b_0)^T$. If the solution of $AX = b$ exists, then there exists a unique RFPrLR circulant matrix*

$$C = \mathrm{RFP}r\mathrm{LRcircfr}(c_0, c_1, \ldots, c_{n-1})$$

and a unique RFPrLR circulant matrix

$$E = \mathrm{RFP}r\mathrm{LRcircfr}(e_0, e_1, \ldots, e_{n-1})$$

of order n over $\mathbb{C}$ such that

$$X_1 = (c_0 + rc_{n-1}, c_1, \ldots, c_{n-2}, c_{n-1})^T$$

is the unique special solution of $AX = b$ and

$$X_2 = X_1 + \hat{I}_n(I_n - E)Y$$

is the general solution of $AX = b$, where Y is an arbitrary n-dimension column vector and $\hat{I}_n$ is given in Equation (5).

By Theorem 9 and Theorem 10, we can get the fast algorithm for solving the RLPrFL circulant linear system $AX = b$, where

$$A = \mathrm{RLP}r\mathrm{FLcircfr}(a_{n-1}, \ldots, a_1, a_0),$$
$$X = (x_1, x_2, \ldots, x_n)^T,$$
$$b = (b_{n-1}, \ldots, b_1, b_0)^T.$$

Step 1. From the RLPrFL circulant linear system $AX = b$, we get the polynomial

$$f(x) = \sum_{i=0}^{n-1} a_i x^i, g(x) = x^n - rx - 1$$

and

$$b(x) = (b_0 - rb_{n-1}) + \sum_{i=1}^{n-1} b_i x^i;$$

Step 2. Change the polynomial matrix $\begin{pmatrix} f(x) & b(x) \\ g(x) & 0 \end{pmatrix}$ into the polynomial matrix $\begin{pmatrix} d(x) & c(x) \\ 0 & c_1(x) \end{pmatrix}$ by a series of elementary row operations;

Step 3. If $d(x) = 1$, then the RLPrFL circulant linear system $AX = b$ has a unique solution. Substituting x by $\Theta_{(1,r)}$ in polynomial $c(x)$, we have a RFPrLR circulant matrix

$$C = c(\Theta_{(1,r)}) = \mathrm{RFP}r\mathrm{LRcircfr}(c_0, c_1, \ldots, c_{n-1}).$$

So the unique solution of $AX = b$ is

$$(c_0 + rc_{n-1}, c_1, \ldots, c_{n-2}, c_{n-1})^T;$$

Step 4. If $d(x) \neq 1$, dividing $g(x)$ by $d(x)$, we get the quotient $g_1(x)$ and change the polynomial matrix

$$\begin{pmatrix} f(x)d(x) & d(x)b(x) & d(x)f(x) & d(x)f(x)b(x) \\ g_1(x) & 0 & 0 & 0 \end{pmatrix}$$

into the polynomial matrix

$$\begin{pmatrix} 1 & c(x) & e(x) & r(x) \\ 0 & c_1(x) & e_1(x) & r_1(x) \end{pmatrix}$$

by a series of elementary row operations;

Step 5. Substituting x by $\Theta_{(1,r)}$ in polynomial $r(x)$, we get a RFPrLR circulant matrix

$$R = r(\Theta_{(1,r)}) = AA^{\{1,2\}}B.$$

If the last column of R isn't b, then $AX = b$ has no solution. Otherwise, the RLPrFL circulant linear system $AX = b$ has a solution. Substituting x by $\Theta_{(1,r)}$ in polynomial $c(x)$ and $e(x)$, we have two RFPrLR circulant matrices

$$C = c(\Theta_{(1,r)}) = A^{\{1,2\}}B$$

and

$$E = e(\Theta_{(1,r)}) = A^{\{1,2\}}A.$$

So the unique special solution of $AX = b$ is

$$X_1 = (c_0 + rc_{n-1}, c_1, \ldots, c_{n-2}, c_{n-1})^T$$

and

$$X_2 = X_1 + \hat{I}_n(I_n - E)Y$$

is the general solution of $AX = b$, where Y is an arbitrary n-dimension column vector.

3 Examples

Example 1. Solve the RFP3LR circulant linear system $AX = b$, where

$$A = \mathrm{RFP3LRcircfr}(2, 1, 0, 1)$$

and

$$b = (0, 1, 2, 1)^T.$$

From $A = \mathrm{RFP3LRcircfr}(2, 1, 0, 1)$ and $b = (0, 1, 2, 1)^T$, we get the polynomial

$$f(x) = 2 + x + x^3, \quad g(x) = -1 - 3x + x^4$$

and

$$b(x) = 1 + 2x + x^2.$$

Then

$$A(x) = \begin{pmatrix} f(x) & b(x) \\ g(x) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + x + x^3 & 1 + 2x + x^2 \\ -1 - 3x + x^4 & 0 \end{pmatrix}.$$

We transform the polynomial matrix $A(x)$ by a series of elementary row operations as follows:

$$A(x) = \begin{pmatrix} 2 + x + x^3 & 1 + 2x + x^2 \\ -1 - 3x + x^4 & 0 \end{pmatrix}$$

$$\Big\Downarrow (2) - x(1)$$

$$\begin{pmatrix} 2 + x + x^3 & 1 + 2x + x^2 \\ -1 - 5x - x^2 & p_3(x) \end{pmatrix}$$

$$\Big\Downarrow (1) + x(2)$$

$$\begin{pmatrix} 2 - 5x^2 & 1 + 2x - 2x^3 - x^4 \\ -1 - 5x - x^2 & -x - 2x^2 - x^3 \end{pmatrix}$$

$$\Big\Downarrow (1) - 5(2)$$

$$\begin{pmatrix} 7 + 25x & p_1(x) \\ -1 - 5x - x^2 & -x - 2x^2 - x^3 \end{pmatrix}$$

$$\Big\Downarrow (2) + \tfrac{1}{25}x(1)$$

$$\begin{pmatrix} 7 + 25x & p_1(x) \\ -1 - \frac{118}{25}x & p_2(x) \end{pmatrix}$$

$$\Big\Downarrow 118(1) + 625(2)$$

$$\begin{pmatrix} 201 & p_3(x) \\ -1 - \frac{118}{25}x & p_2(x) \end{pmatrix}$$

$$\Big\Downarrow \tfrac{1}{201}(1)$$

$$\begin{pmatrix} 1 & p_4(x) \\ -1 - \frac{118}{25}x & p_2(x) \end{pmatrix}.$$

where

$$\begin{aligned} p_1(x) &= 1 + 7x + 10x^2 + 3x^3 - x^4, \\ p_2(x) &= -\tfrac{24}{25}x - \tfrac{43}{25}x^2 - \tfrac{15}{25}x^3 + \tfrac{3}{25}x^4 - \tfrac{1}{25}x^5, \\ p_3(x) &= 118 + 226x + 105x^2 - 21x^3 - 43x^4 - 25x^5, \\ p_4(x) &= \tfrac{118}{201} + \tfrac{226}{201}x + \tfrac{35}{67}x^2 - \tfrac{7}{67}x^3 - \tfrac{43}{201}x^4 - \tfrac{25}{201}x^5. \end{aligned}$$

Since $d(x) = 1$, then the RFP3LR circulant linear system $AX = b$ has a unique solution. On the other hand,

$$c(x) = \frac{118}{201} + \frac{226}{201}x + \frac{35}{67}x^2 - \frac{7}{67}x^3 - \frac{43}{201}x^4 - \frac{25}{201}x^5.$$

Substituting x by $\Theta_{(1,3)}$ in polynomial $c(x)$, we have the RFP3LR circulant matrix

$$C = c(\Theta_{(1,3)}) = \mathrm{RFP3LRcircfr}(\frac{25}{67}, \frac{24}{67}, \frac{10}{67}, -\frac{7}{67}).$$

So the unique solution of $AX = b$ is the last column of C, i.e.,

$$X = (-\frac{7}{67}, \frac{10}{67}, \frac{24}{67}, \frac{4}{67})^T.$$

Example 2. Solve the RFP6LR circulant linear system

$$AX = b,$$

where

$$A = \mathrm{RFP6LRcircfr}(1, 3+\sqrt{10})$$

and

$$b = (1, 3+\sqrt{10})^T.$$

From $A = \mathrm{RFP6LRcircfr}(1, 3+\sqrt{10})$ and $b = (1, 3+\sqrt{10})^T$, we get the polynomial

$$f(x) = 1 + (3+\sqrt{10})x, \quad g(x) = -1 - 6x + x^2$$

and

$$b(x) = \sqrt{10} - 3 + x.$$

Then

$$A(x) = \begin{pmatrix} f(x) & b(x) \\ g(x) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+(3+\sqrt{10}) & \sqrt{10}-3+x \\ -1-6x+x^2 & 0 \end{pmatrix}.$$

We transform the polynomial matrix $A(x)$ by a series of elementary row operations as follows:

$$A(x) = \begin{pmatrix} 1+(3+\sqrt{10})x & \sqrt{10}-3+x \\ -1-6x+x^2 & 0 \end{pmatrix}$$

$$\Big\Downarrow \; (2) - ((\sqrt{10}-3)x-1)(1)$$

$$\begin{pmatrix} 1+(3+\sqrt{10})x & \sqrt{10}-3+x \\ 0 & q_1(x) \end{pmatrix}$$

$$\Big\Downarrow \; (\sqrt{10}-3)(1)$$

$$\begin{pmatrix} \sqrt{10}-3+x & q_1(x) \\ 0 & q_2(x) \end{pmatrix},$$

where

$$q_1(x) = 19 - 6\sqrt{10} + (\sqrt{10}-3)x,$$

$$q_2(x) = -3 + \sqrt{10} + (6\sqrt{10}-18)x - (\sqrt{10}-3)x^2.$$

It is obvious that $d(x) = \sqrt{10} - 3 + x \neq 1$, it denoted that A is singular.

Then

$$g_1(x) = g(x)/d(x) = -\sqrt{10} - 3 + x,$$

$$d(x)f(x) = \sqrt{10} - 3 + 2x + (\sqrt{10}+3)x^2,$$

$$d(x)b(x) = 19 - 6\sqrt{10} + (2\sqrt{10}-6)x + x^2,$$

$$\begin{aligned} d(x)f(x)b(x) &= 19 - 6\sqrt{10} + 3(\sqrt{10}-3)x \\ &+ 3x^2 + (3+\sqrt{10})x^3, \end{aligned}$$

Thus we structure matrix $B(x)$ and transform the polynomial matrix $B(x)$ by a series of elementary row operations as follows:

$$B(x) = \begin{pmatrix} f(x)d(x) & d(x)b(x) & d(x)f(x) & d(x)f(x)b(x) \\ g_1(x) & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} r_1(x) & s_1(x) & s_2(x) & s_3(x) \\ -\sqrt{10}-3+x & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\Downarrow \; (1) - (3+\sqrt{10})x(2)$$

$$\begin{pmatrix} r_2(x) & s_1(x) & s_2(x) & s_3(x) \\ -\sqrt{10}-3+x & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\Downarrow \; (1) - (21+6\sqrt{10})(2)$$

$$\begin{pmatrix} 40\sqrt{10}+120 & s_1(x) & s_2(x) & s_3(x) \\ -3-\sqrt{10}+x & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\Downarrow \; \tfrac{1}{40}(\sqrt{10}-3)(1)$$

$$\begin{pmatrix} 1 & s_4(x) & s_5(x) & s_6(x) \\ -3-\sqrt{10}+x & 0 & 0 & 0 \end{pmatrix}.$$

where

$$\begin{aligned} r_1(x) &= \sqrt{10} - 3 + 2x + (\sqrt{10}+3)x^2, \\ r_2(x) &= \sqrt{10} - 3 + (21+6\sqrt{10})x, \\ s_1(x) &= 19 - 6\sqrt{10} + (2\sqrt{10}-6)x + x^2, \\ s_2(x) &= \sqrt{10} - 3 + 2x + (\sqrt{10}+3)x^2, \\ s_3(x) &= 19 - 6\sqrt{10} + 3(\sqrt{10}-3)x + 3x^2 + (3+\sqrt{10})x^3, \\ s_4(x) &= \tfrac{37\sqrt{10}-117}{40} + \tfrac{19-6\sqrt{10}}{20}x + \tfrac{\sqrt{10}-3}{40}x^2, \\ s_5(x) &= \tfrac{19-6\sqrt{10}}{40} + \tfrac{\sqrt{10}-3}{20}x + \tfrac{1}{40}x^2, \\ s_6(x) &= \tfrac{37\sqrt{10}-117}{40} + \tfrac{57-18\sqrt{10}}{40}x \\ &\quad + \tfrac{3\sqrt{10}-9}{40}x^2 + \tfrac{1}{40}x^3. \end{aligned}$$

Substituting x by $\Theta_{(1,6)}$ in polynomial

$$\begin{aligned} r(x) &= \frac{37\sqrt{10}-117}{40} + \frac{57-18\sqrt{10}}{40}x \\ &+ \frac{3\sqrt{10}-9}{40}x^2 + \frac{1}{40}x^3, \end{aligned}$$

we get the RFP6LR circulant matrix

$$\begin{aligned} R &= r(\Theta_{(1,6)}) = AA^{\{1,2\}}B \\ &= \mathrm{RFP6LRcircfr}(\sqrt{10}-3, 1). \end{aligned}$$

Since the last column of R is b, the RFP6LR circulant linear system $AX = b$ has a solution.

Substituting x by $\Theta_{(1,6)}$ in polynomial

$$c(x) = \frac{37\sqrt{10} - 117}{40} + \frac{19 - 6\sqrt{10}}{20}x + \frac{\sqrt{10} - 3}{40}x^2$$

and

$$e(x) = \frac{19 - 6\sqrt{10}}{40} + \frac{\sqrt{10} - 3}{20}x + \frac{1}{40}x^2,$$

we have the RFP6LR circulant matrices

$$\begin{aligned} C &= c(\Theta_{(1,6)}) = A^{\{1,2\}}B \\ &= \text{RFP6LRcircfr}(\frac{19\sqrt{10} - 60}{20}, \frac{10 - 3\sqrt{10}}{20}) \end{aligned}$$

and

$$\begin{aligned} E &= e(\Theta_{(1,6)}) = A^{\{1,2\}}A \\ &= \text{RFP6LRcircfr}(\frac{10 - 3\sqrt{10}}{20}, \frac{\sqrt{10}}{20}). \end{aligned}$$

So the unique special solution of $AX = b$ is the last column of C, i.e.,

$$X_1 = (\frac{10 - 3\sqrt{10}}{20}, \frac{\sqrt{10}}{20})^T$$

and a general solution of $AX = b$ is

$$\begin{aligned} X_2 &= X_1 + (I_n - E)Y \\ &= \begin{pmatrix} \frac{10-3\sqrt{10}}{20} + \frac{10+3\sqrt{10}}{20}k_1 - \frac{\sqrt{10}}{20}k_2 \\ \frac{\sqrt{10}}{20} - \frac{\sqrt{10}}{20}k_1 + \frac{(10-3\sqrt{10})}{20}k_2 \end{pmatrix}, \end{aligned}$$

where $Y = (k_1, k_2)^T \in \mathbb{C}$.

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