Resolution of Stokes Equations with the $C_{a,b}$ Boundary Condition Using Mixed Finite Element Method

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Abstract: In this paper, we introduce the Stokes equations with a new boundary condition. In this context, we show the existence and uniqueness of the solution of the weak formulation associated with the proposed problem. To solve this problem, we use the discretization by mixed finite element method. In addition, two types of a posteriori error indicator are introduced and are shown to give global error estimates that are equivalent to the true error. In order to evaluate the performance of the method, the numerical results are compared with some previously published works and with others coming from commercial code like ADINA system.

Key–Words: Stokes equations, $C_{a,b}$ Boundary condition, Mixed finite element method, Residual error estimator, Adina system.

1 Introduction

This paper describes a numerical solution of Stocks equations with a new boundary generalizes the will known basis conditions, especially the Dirichlet and the Neumann conditions. So, we prove that the weak formulation of the proposed modeling has an unique solution. To calculate this latter, we use the discretization by mixed finite element method. Moreover, we propose two types of a posteriori error indicator which are shown to give global error estimates that are equivalent to the true error. To compare our solution with the some previously ones, as ADINA system, some numerical results are shown. In this modeling flow of porous media, it is essential to use a discretization method which satisfies the physics of the problem, i.e. conserve mass locally and preserve continuity of flux. The Raviart-Thomas Mixed Finite Element (MFE) method of lowest order satisfies these properties. Moreover, both the pressure and the velocity are approximated with the same order of convergence [4, 6]. The discretization of the velocity is based on the properties of Raviart-Thomas. Other works have been introduced by Brezzi, Fortin, Marini, Dougla and Robert [4, 5, 7]. This method was widely used for the prediction of the behavior of fluid in the hydrocarbons tank. A posteriori error analysis in problems related to fluid dynamics is a subject that has received a lot of attention during the last decades. In the conforming case there are several ways to define error estimators by using the residual equation. in particular, for the Stokes problem, M. Ainsworth, J. Oden [9], R.E. Bank, B.D. Welfert [10], C. Crestensen, S.A. Funken [11], D. Kay, D. Silvestre [12] and R. Verfurth [13], introduced several error estimators and provided that they are equivalent to the energy norm of the errors. Other works for the stationary Navier-Stokes problem have been introduced in [14, 17, 18, 20, 16]. This paper describes a numerical solution of Stokes equations with a boundary condition noted $C_{a,b}$. For the equations, we offer a choice of tow-dimensional domains on which the problem can be posed, along with boundary conditions and other aspects of the problem, and a choice of finite element discretization on a rectangular element mesh.

The plan of the paper is as follows. Section 2 presents the model problem used in this paper. The weak formulation is presented in section 3. In section 4, we show the existence and uniqueness of the solution. The discretization by mixed finite elements is described in section 5. Section 6 introduced two types of a posteriori error bounds of the computed solution. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in Section 7.

2 Governing Equations

We will consider the model of viscous incompressible flow in an idealized, bounded, connected domain in $\mathbb{R}^2$.

$$-\nabla^2 \overrightarrow{u} + \nabla p = \overrightarrow{f} \, in \, \Omega. \quad (1)$$

$$-\nabla . \overrightarrow{u} = 0 \, in \, \Omega. \quad (2)$$

by the $C_{a,b}$ boundary condition

$$a \overrightarrow{u} + b(\nabla \overrightarrow{u} - pI)\overrightarrow{n} = \overrightarrow{t} \, in \, \Gamma := \partial\Omega \quad (3)$$

We also assume that Ω has a polygonal boundary $\Gamma := \partial\Omega$, so $\overrightarrow{n}$ that is the usual outward-pointing normal. The vector field $\overrightarrow{u}$ is the velocity of the flow and the scalar variable p represents the pressure.

Our mathematical model is the Stoks system with a new boundary condition noted $C_{a,b}$, ∇ is the gradient, $\nabla\cdot$ is the divergence and ∇^2 is the Laplacien operator, $\overrightarrow{f}$, $\overrightarrow{t}$, a and b are the polynomials such that, $\overrightarrow{f}$ defined in Ω and a and b nonzero defined on $\partial\Omega$ verify:

There are two strictly positive constants α_1 and β_1 such that:

$$\alpha_1 \leq \frac{a(x)}{b(y)} \leq \beta_1 \, for \, all \, x \in \Gamma. \quad (4)$$

Remark . If $a \, and \, b$ are two strictly positive constants such that $a << b$ then $C_{a,b}$ is the Neumann boundary condition and if $b << a$ then $C_{a,b}$ is the Dirichlet boundary condition. for that a is called the Dirichlet coefficient and b is the Neumann coefficient.

3 The weak formulation

We define the spaces:

$$h^1 = \{\overrightarrow{u} : \Omega \to \mathbb{R} / \, \overrightarrow{u}; \, \frac{\partial \overrightarrow{u}}{\partial x}; \, \frac{\partial \overrightarrow{u}}{\partial y} \in L^2(\Omega)\} \quad (5)$$

$$H^1(\Omega) = [h^1]^2 \quad (6)$$

$$L_0^1(\Omega) = \{q \in L^2(\Omega) / \int_\Omega q = 0\} \quad (7)$$

The standard weak formulation of the Stokes flow problem (1) - (2)-(3) is the following:

Find $\overrightarrow{u} \in H^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that :

$$\begin{cases} \int_\Omega \nabla \overrightarrow{u} : \nabla \overrightarrow{v} + \int_\Gamma \frac{a}{b} \overrightarrow{u} . \overrightarrow{v} \\ - \int_\Omega p \nabla . \overrightarrow{v} = \int_\Omega \overrightarrow{f} . \overrightarrow{v} + \int_\Gamma \frac{1}{b} \overrightarrow{t} . \overrightarrow{v} \\ \int_\Omega q \nabla . \overrightarrow{u} = 0 \end{cases} \quad (8)$$

for all $\overrightarrow{v} \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$.

Let the bilinear forms

$$A(\overrightarrow{u}, \overrightarrow{v}) = \int_\Omega \nabla \overrightarrow{u} : \nabla \overrightarrow{v} + \int_\Gamma \frac{a}{b} \overrightarrow{u} . \overrightarrow{v} \quad (9)$$

$$B(\overrightarrow{u}, q) = - \int_\Omega q \nabla . \overrightarrow{u} \quad (10)$$

Given the functional $L : H^1(\Omega) \longrightarrow \mathbb{R}$

$$L(\overrightarrow{v}) = \int_\Omega \overrightarrow{f} . \overrightarrow{v} + \int_\Gamma \frac{1}{b} \overrightarrow{t} . \overrightarrow{v} \quad (11)$$

The underlying weak formulation (8) may be restated as:

Find $\overrightarrow{u} \in H^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that :

$$\begin{cases} A(\overrightarrow{u}, \overrightarrow{v}) + B(\overrightarrow{v}, p) = L(\overrightarrow{v}) \\ B(\overrightarrow{u}, q) = 0 \end{cases} \quad (12)$$

for all $\overrightarrow{v} \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$.

4 The existence and uniqueness of the solution

In this section we will study the existence and uniqueness of the solution of problem (12), for that we need the following results:

Theorem 1 *There are two strictly positive constants c_1 and c_2 such that:*

$$c_1 \|\overrightarrow{v}\|_{1,\Omega} \leq \|\overrightarrow{v}\|_{J,\Omega} \leq c_2 \|\overrightarrow{v}\|_{1,\Omega} \quad (13)$$

$$where \quad \|\overrightarrow{v}\|_{J,\Omega} = [A(\overrightarrow{v}; \overrightarrow{v})]^{\frac{1}{2}} \quad (14)$$

$$\begin{aligned} \|\overrightarrow{v}\|_{1,\Omega} &= [\int_\Omega \nabla \overrightarrow{v} : \nabla \overrightarrow{v} + \int_\Omega \overrightarrow{v} . \overrightarrow{v}]^{\frac{1}{2}} \\ &= [\, |\overrightarrow{v}|_{1,\Omega}^2 + \|\overrightarrow{v}\|_{0,\Omega}^2]^{\frac{1}{2}}. \end{aligned} \quad (15)$$

Proof. The mapping $\gamma_0 : H^1(\Omega) \longrightarrow L^2(\Gamma)$, such that $\gamma_0(\overrightarrow{v}) = \overrightarrow{v}/\Gamma$, is continuous (see [2] theorem 1, 2), then there exists $c > 0$ such that:

$$\|\overrightarrow{v}\|_{0,\Gamma} \leq c \|\overrightarrow{v}\|_{1,\Omega} \, \forall \, \overrightarrow{v} \in H^1(\Omega).$$

Using this result and (4) give, $\|\overrightarrow{v}\|_{J,\Omega} \leq c_2 \|\overrightarrow{v}\|_{1,\Omega}$ for all $\overrightarrow{v} \in H^1(\Omega)$ with $c_2 = (\beta_1 c^2 + 1)^{\frac{1}{2}}$. On the other hand, according to 5.55 in [1], there exists a constant $\rho > 0$ such that:

$$\|\overrightarrow{v}\|_{0,\Omega}^2 \quad \leq \quad \rho(\|\nabla \overrightarrow{v}\|_{0,\Omega}^2 + \|\overrightarrow{v}\|_{0,\Gamma}^2 .)$$

Using (4) and $\|\nabla\overrightarrow{v}\|_{1,\Omega} \leq \|\overrightarrow{v}\|^2_{J,\Omega}$, give

$$c_1\|\overrightarrow{v}\|_{1,\Omega} \leq \|\overrightarrow{v}\|_{J,\Omega}\ \forall\overrightarrow{v} \in H^1(\Omega),$$

with $c_1 = (\frac{\rho C}{\alpha_1}+1)^{-\frac{1}{2}}$ and $C = \max\{\alpha_1; 1\}$. □

This result allows us to prove that $(H^1(\Omega), \|.\|_{J,\Omega}))$ is a Hilbert space witch is obliged condition for obtain the existence and uniqueness of the solution.

Theorem 2 *$(H^1(\Omega), \|.\|_{J,\Omega}))$ is a real Hilbert space.*

Proof: $(H^1(\Omega), \|.\|_{1,\Omega})$ is a real Hilbert space and $\|.\|_{1,\Omega}$ and $\|.\|_{J,\Omega}$ are equivalent norms, then $(H^1(\Omega), \|.\|_{J,\Omega}))$ is a real Hilbert space. □

Theorem 3 *B satisfies the inf-sup condition: there exists a constant $\beta > 0$ such that:*

$$\sup_{\overrightarrow{v}\in H^1(\Omega)} \frac{B(\overrightarrow{v}, q)}{\|\overrightarrow{v}\|_{J,\Omega}} \geq \beta\|q\|_{0,\Omega}. \tag{16}$$

for all $q \in L^2_0(\Omega)$.

Proof. The same proof of [2] it suffices to see that $H^1_0(\Omega) \subset H^1(\Omega)$ and $\|\overrightarrow{v}\|_{J,\Omega} = |\overrightarrow{v}|_{1,\Omega}\ in\ H^1_0(\Omega)$.

We define the "big" symmetric bilinear form

$$\begin{aligned} C[(\overrightarrow{u}, p); (\overrightarrow{v}, q)] &= A(\overrightarrow{u}, \overrightarrow{v}) + B(\overrightarrow{u}, q) \\ &\quad +B(\overrightarrow{v}, q) \end{aligned} \tag{17}$$

And the corresponding function $F(\overrightarrow{v}, q) = L(\overrightarrow{v})$ choosing the successive test vectors $(\overrightarrow{v}, 0)$ and $(\overrightarrow{0}, q)$ shows that the Stokes problem (12) can be rewritten in the form:

Find $(\overrightarrow{u}, p) \in H^1(\Omega) \times L^2_0(\Omega)$ such that

$$C[(\overrightarrow{u}, p); (\overrightarrow{v}, q)] = F(\overrightarrow{v}, q) \tag{18}$$

for all $(\overrightarrow{v}, q) \in H^1(\Omega) \times L^2_0(\Omega)$. □

The bilinear form A is positive continuous and $H^1(\Omega)-elliptic$ and the bilinear form b is continuous and satisfies the inf-sup condition. Then the problem (12) is well-posed, and the forms bilinear C and A satisfies the following propositions.

Proposition 4 *for all $(\overrightarrow{w}, s) \in H^1(\Omega) \times L^2_0(\Omega)$, we have*

$$\begin{aligned} &\sup_{(\overrightarrow{v},q)\in H^1\times L^2_0} \frac{C[(\overrightarrow{u}, p); (\overrightarrow{v}, q)]}{\|\overrightarrow{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \\ &\geq \delta(\|\overrightarrow{w}\|_{J,\Omega} + \|s\|_{0,\Omega}). \end{aligned} \tag{19}$$

Proof . See [1].

Proposition 5 *For all $(\overrightarrow{w}, s) \in H^1 \times L^2_0(\Omega)$, we have*

$$\begin{aligned} &\sup_{(\overrightarrow{v},q)\in H^1\times L^2_0} \frac{A(\overrightarrow{w}, \overrightarrow{v}) + d(s, q)}{\|\overrightarrow{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \\ &\geq \frac{1}{2}(\|\overrightarrow{w}\|_{J,\Omega} + \|s\|_{0,\Omega}), \end{aligned} \tag{20}$$

where $d(s, q) = \int_\Omega sq$.

Proof. Let $(\overrightarrow{w}, q) \in H^1(\Omega) \times L^2_0(\Omega)$. We will take $q = 0$ and $\overrightarrow{v} = \overrightarrow{w}$ in the first and $\overrightarrow{v} = \overrightarrow{0}$ and $q = s$ in the second, we obtain

$$\sup_{(\overrightarrow{v},q)} \frac{A(\overrightarrow{w}, \overrightarrow{v}) + d(s, q)}{\|\overrightarrow{v}\|_J + \|q\|_0} \geq \|\overrightarrow{w}\|_J, \tag{21}$$

$$\sup_{(\overrightarrow{v},q)} \frac{A(\overrightarrow{w}, \overrightarrow{v}) + d(s, q)}{\|\overrightarrow{v}\|_J + \|q\|_0} \geq \|s\|_{0,\Omega}, \tag{22}$$

we gather (21) and (22) to get (20). □

The bilinear form A is symmetric and continuous and semi positive definite on $H^1(\Omega)$, in this case we say the problem (12) is a type of saddle-point problem. The results (13)-(16) ensure the existence and uniqueness of the solution of the problem (12) (see Theorem 6. 2 in [1]). In the following section we will solve this problem by mixed finite element method.

5 Mixed finite element approximation

Let $T_h; h \geq 0$,be a family of rectangulations of Ω. For any $T \in T_h$, ω_T is of rectangles sharing at least one edge with element T, $\widetilde{\omega}_T$ is the set of rectangles sharing at least one vertex with T. Also, for an element edge E, ω_E denotes the union of rectangles sharing E,, while $\widetilde{\omega}_E$ is the set of rectangles sharing at least one vertex whit E. Next,∂T is the set of the four edges of T we denote by $\varepsilon(T)$ and N_T the set of its edges and vertices, respectively. We let $\varepsilon_h = \bigcup_{T\in T_k} \varepsilon(T)$ denotes the set of all edges split into interior and boundary edges. $\varepsilon_h = \varepsilon_{h,\Omega} \bigcup \varepsilon_{h,\Gamma}$ where

$$\varepsilon_{h,\Omega} = \{E \in \varepsilon_h : E \subset \Omega\}$$

$$\varepsilon_{h,\Gamma} = \{E \in \varepsilon_h : E \subset \partial\Omega\}$$

We denote by h_T the diameter of a simplex, by h_E the diameter of a face E of T, and we set $h = \max_{T\in T_k}\{h_T\}$. A discrete weak formulation is defined using finite dimensional spaces $X^1_h \subset H^1(\Omega)$ and $M^h \subset L^2_0(\Omega)$.

The discrete version of (12) is:

$$\begin{cases} find \ \overrightarrow{u}_h \in X_h^1 \ and \ p_h \in M^h \ such \ that: \\ A(\overrightarrow{u}_h, \overrightarrow{v}_h) + B(\overrightarrow{v}_h, p_h) = L(\overrightarrow{v}_h), \\ B(\overrightarrow{v}_h, p_h), = 0 \end{cases} \quad (23)$$

for all $\overrightarrow{v}_h \in X_h^1$ and $q_h \in M_h$.

We use a set of vector-valued basis functions $\{\overrightarrow{\varphi_i}\}_{i,j=1,\dots,n_u}$ so that

$$\overrightarrow{u}_h = \sum_{j=1}^{n_u} u_j \overrightarrow{\varphi_j} \quad (24)$$

We introduce a set of pressure basis functions $\{\psi_k\}_{k=1,\dots,n_p}$ and set

$$p_h = \sum_{k=1}^{n_p} p_k \psi_k \quad (25)$$

where n_u and n_p are the numbers of velocity and pressure basis functions, respectively.

We find that the discrete formulation (23) can be expressed as a system of linear equations

$$\begin{pmatrix} A_0 & B_0^T \\ B_0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (26)$$

The system is referred to as the discrete Newton problem. The matrix A_0 is the vector Laplacian matrix and B_0 is the divergence matrix

$$A_0 = [a_{i,j}]; \quad (27)$$

$$a_{i,j} = \int_\Omega \nabla \overrightarrow{\varphi}_i : \nabla \overrightarrow{\varphi}_j + \int_{\partial\Omega} \frac{a}{b} \overrightarrow{\varphi}_i . \overrightarrow{\varphi}_j .$$

$$B_0 = [b_{k,j}]; b_{k,j} = -\int_\Omega \psi_k \nabla . \overrightarrow{\varphi}_j \quad (28)$$

$$f = [f_i]; \ f_i = \nu \int_\Omega \overrightarrow{f} . \overrightarrow{\varphi}_i + \int_{\partial\Omega} \frac{1}{b} \overrightarrow{t} \overrightarrow{\varphi}_i \quad (29)$$

for $i, j = 1, ..., n_u$, $k = 1, ..., n_p$, and the function pair $(\overrightarrow{u_h}, p_h)$ obtained by substituting the solution vectors $U \in \mathbb{R}^{n_u}$ and $P \in \mathbb{R}^{n_p}$ into (24) and (25) is the mixed finite element solution. The system (26)-(29) is henceforth referred to as the discrete stokes problem. We use the iterative methods Minimum Residual Method (MINRES) for solving the symmetric system.

6 A Posteriori error estimator

In this section, we propose two types of a posteriori error indicator; The first one is the residual error estimator and the second one is the local Poisson problem estimator. Which are shown to give global error estimates that are equivalent to the true error.

6.1 A Residual error estimator

The bubble functions on the reference element $\widetilde{T} = (0,1) \times (0,1)$ are defined as follows

$$\begin{aligned} b_{\widetilde{T}} &= 2^4 x(1-x)y(1-y), \\ b_{\widetilde{E}_1,\widetilde{T}} &= 2^2 x(1-x)(1-y), \\ b_{\widetilde{E}_2,\widetilde{T}} &= 2^2 y(1-y)x, \\ b_{\widetilde{E}_3\widetilde{T}} &= 2^2 y(1-x)x, \\ b_{\widetilde{E}_4,\widetilde{T}} &= 2^2 y(1-y)(1-x). \end{aligned}$$

Here; $b_{\widetilde{T}}$ is the reference element bubble function, and $b_{\widetilde{E}_i,\widetilde{T}}$, i=1:4 are reference edge bubble functions. For any $T \in T_h$, the element bubble functions is $b_T = b_{\widetilde{T}} \circ F_T$ and the element edge bubble function is $b_{E_i,T} = b_{\widetilde{E}_i,\widetilde{T}} \circ F_T$ where F_T the affine map form $\widetilde{T}$ to T.

For an interior edge $E \in \varepsilon_{h,\Omega}$, b_E is defined piecewise, so that $b_{E/T_i} = b_{E,T_i}$, i=1:2 , where $E = \overline{T}_1 \cap \overline{T}_2$. For a boundary edge $E \in \varepsilon_{h,\Gamma}$, $b_E = b_{E,T}$, where T is the rectangle such that $E \in \partial T$.

With these bubble functions, ceruse et al ([19], lemma 4.1) established the following lemma.

Lemma 6 *Let T be an arbitrary rectangle in τ_h and $E \in \partial T$. For any $\overrightarrow{v}_T \in P_{k_0}(T)$ and $\overrightarrow{v}_E \in P_{k_1}(E)$, the following inequalities hold.*

$$c_k \|\overrightarrow{v}_T\|_{0,T} \le \|\overrightarrow{v}_T b_T^{\frac{1}{2}}\|_{0,T} \le C_k \|\overrightarrow{v}_T\|_{0,T} \quad (30)$$

$$|\overrightarrow{v}_T b_T|_{1,T} \le C_k h_T^{-1} \|\overrightarrow{v}_T\|_{0,T} \quad (31)$$

$$c_k \|\overrightarrow{v}_E\|_{0,E} \le \|\overrightarrow{v}_E b_E^{\frac{1}{2}}\|_{0,E} \le C_k \|\overrightarrow{v}_E\|_{0,E} \quad (32)$$

$$\|\overrightarrow{v}_E b_E\|_{0,T} \le C_k h_E^{\frac{1}{2}} \|\overrightarrow{v}_E\|_{0,E} \quad (33)$$

$$|\overrightarrow{v}_E b_E|_{1,T} \le C_k h_E^{-\frac{1}{2}} \|\overrightarrow{v}_E\|_{0,E}, \quad (34)$$

where c_k and C_k are tow constants which only depend on the element aspect ratio and the polynomial degrees k_0 and k_1.

Here, k_0 and k_1 are fixed and c_k and C_k can be associated with generic constants c and C. In addition, $\overrightarrow{v}_E$ which is only defined on the edge E also denotes its natural extension to the element T. From the inequalities (33) and (34) we will establish the following lemma:

Lemma 7 *Let T be an rectangle and $E \in \partial T \bigcap \varepsilon_{h,\Gamma}$. For any $\overrightarrow{v}_E \in P_{k_1}(E)$, the following inequalities hold.*

$$\|\overrightarrow{v}_E b_E\|_{J,T} \leq Ch_E^{-\frac{1}{2}}\|\overrightarrow{v}_E\|_{0,E}. \tag{35}$$

Proof. Since $\overrightarrow{v}_E b_E = \overrightarrow{0}$ in the other three edges of rectangle T, it can be extended to the whole of Ω by setting $\overrightarrow{v}_E b_E = \overrightarrow{0}$ in $\Omega \backslash \overline{T}$, then

$$\|\overrightarrow{v}_E b_E\|_{1,T} = \|\overrightarrow{v}_E b_E\|_{1,\Omega}$$
$$\text{and } \|\overrightarrow{v}_E b_E\|_{J,T} = \|\overrightarrow{v}_E b_E\|_{J,\Omega}.$$

Using the inequalities (13), (33) and (34), gives

$$\begin{aligned}
\|\overrightarrow{v}_E b_E\|_{J,T} &= \|\overrightarrow{v}_E b_E\|_{J,\Omega} \\
&\leq c_2\|\overrightarrow{v}_E b_E\|_{1,\Omega} \\
&= c_2\|\overrightarrow{v}_E b_E\|_{1,T} \\
&= c_2(\|\overrightarrow{v}_E b_E\|_{0,T}^2 + |\overrightarrow{v}_E b_E|_{1,T}^2)^{\frac{1}{2}} \\
&\leq c_2 C_k(h_E + h_E^{-1})^{\frac{1}{2}}\|\overrightarrow{v}_E\|_{0,E} \\
&\leq c_2 C_k(D^2+1)^{\frac{1}{2}} h_E^{-\frac{1}{2}}\|\overrightarrow{v}_E\|_{0,E} \\
&\leq Ch_E^{-\frac{1}{2}}\|\overrightarrow{v}_E\|_{0,E}
\end{aligned}$$

with D is the diameter of Ω and $C = c_2 C_k(D^2+1)^{\frac{1}{2}}$. $\square$

Lemma 8 *Clement interpolation estimate: Given $\overrightarrow{v} \in H^1(\Omega)$, let $\overrightarrow{v}_h \in X_h^1$ be the quasi-interpolant of $\overrightarrow{v}$ defined by averaging as in [20]. For any $T \in T_h$ and for all $E \in \partial T$,*

$$\|\overrightarrow{v} - \overrightarrow{v}_h\|_{0,T} \leq Ch_T|\overrightarrow{v}|_{1,\widetilde{\omega}_T} \tag{36}$$

$$\|\overrightarrow{v} - \overrightarrow{v}_h\|_{0,E} \leq Ch_E^{\frac{1}{2}}|\overrightarrow{v}|_{1,\widetilde{\omega}_E}. \tag{37}$$

We let $(\overrightarrow{u}, p)$ denote the solution of (12) and let $(\overrightarrow{u}_h, p_h)$ denote the solution of (23) with an approximation on a rectangular subdivision T_h.

Our aim is to estimate the velocity and the pressure errors $\overrightarrow{u} - \overrightarrow{u}_h \in H^1(\Omega)$ and $p - p_h \in L_0^2(\Omega)$. The element contribution $\eta_{R,T}$ of the residual error estimator η_R is given by

$$\begin{aligned}
\eta_{R,T}^2 &= h_T^2\|\overrightarrow{R}_T\|_{0,T}^2 + \|R_T\|_{0,T}^2 \\
&\quad + \sum_{E\in\partial T} h_E\|\overrightarrow{R}_E\|_{0,E}^2
\end{aligned} \tag{38}$$

and the components in (38) are given by

$$\overrightarrow{R}_T = \{\overrightarrow{f} + \nabla^2\overrightarrow{u}_h - \nabla p_h\}|_T,$$

$$R_T = \{\nabla.\overrightarrow{u}_h\}|_T,$$

$$\vec{R}_E = \begin{cases} \frac{1}{2}[|\nabla\vec{u}_h - p_h I|] \text{ if } E \in \varepsilon_{h,\Omega} \\ \frac{1}{b}\overrightarrow{t} - [\frac{a}{b}\overrightarrow{u}_h + (\nabla\overrightarrow{u}_h - p_h I)\overrightarrow{n}] \text{ if } E \in \varepsilon_{h,\Gamma} \end{cases}$$

With the key contribution coming from the stress jump associated with an edge E adjoining elements T and S:

$$\begin{aligned}
&[[\nabla\overrightarrow{u}_h - p_h I]] \\
&= ((\nabla\overrightarrow{u}_h - p_h I)|_T - (\nabla\overrightarrow{u}_h - p_h I)|_S)\overrightarrow{n}_{E,T}.
\end{aligned}$$

The global residual error estimator is given by:

$$\eta_R = (\textstyle\sum_{T\in\tau_h}\eta_{R,T}^2)^{\frac{1}{2}}.$$

Our aim is to bound $\|\overrightarrow{u} - \overrightarrow{u}_h\|_X$ and $\|p - p_h\|_M$ with respect to the norm $\|.\|_J$ for velocity $\|\overrightarrow{v}\|_X = \|\overrightarrow{v}\|_{J,\Omega}$ and the quotient norm for the pressure $\|p\|_X = \|p\|_{0,\Omega}$.
For any $T \in T_h$, and $E \in \partial T$, we define the following two functions:

$$\overrightarrow{w}_T = \overrightarrow{R}_T b_T;\ \overrightarrow{w}_E = \overrightarrow{R}_E b_E$$

- $\overrightarrow{w}_T = \overrightarrow{0}$ on ∂T.
- If $E \in \partial T \cap \varepsilon_{h,\Omega}$

then $\overrightarrow{w}_E = \overrightarrow{0}$ on $\partial\omega_E$,
- If $E \in \partial T \cap \varepsilon_{h,\Gamma}$ *then* $\overrightarrow{w}_E = \overrightarrow{0}$ in the other three edges of rectangle T.
- $\overrightarrow{w}_T$ and $\overrightarrow{w}_E$ can be extended to whole of Ω by setting:

$\overrightarrow{w}_T = \overrightarrow{0}$ in $\Omega - \overline{T}$
$\overrightarrow{w}_E = \overrightarrow{0}$ in $\Omega - \overline{\omega_E}$ if $E \in \partial T \cap \varepsilon_{h,\Omega}$.
$\overrightarrow{w}_T = \overrightarrow{0}$ in $\Omega - \overline{T}$ if $E \in \partial T \cap \varepsilon_{h,\Gamma}$.

With these two functions we have the following lemmas:

Lemma 9 *For any $T \in T_h$ we have:*

$$\int_T \overrightarrow{f}.\overrightarrow{w}_T = \int_T(\nabla\overrightarrow{u} - pI) : \nabla\overrightarrow{w}_T \tag{39}$$

Proof. By applying the Green formula and $\overrightarrow{w}_T = \overrightarrow{0}$ on ∂T, and using (1) we obtain

$$\begin{aligned}
\int_T(\nabla\overrightarrow{u} - pI) : \nabla\overrightarrow{w}_T &= \int_T(-\nabla^2\overrightarrow{u} + \nabla p).\overrightarrow{w}_T \\
&\quad + \int_{\partial T}(\nabla\overrightarrow{u} - pI)\overrightarrow{n}\,\overrightarrow{w}_T \\
&= \int_T \overrightarrow{f}.\overrightarrow{w}_T.
\end{aligned}$$

Lemma 10 *i) if $E \in \partial T \cap \varepsilon_{h,\Omega}$,we have:*

$$\int_{\omega_E}\overrightarrow{f}.\overrightarrow{w}_E = \int_{\omega_E}(\nabla\overrightarrow{u} - pI) : \nabla\overrightarrow{w}_E. \tag{40}$$

ii) if $E \in \partial T \cap \varepsilon_{h,\Gamma}$, we have:

$$\int_T \overrightarrow{f}.\overrightarrow{w}_E = \int_T (\nabla \overrightarrow{u} - pI) : \nabla \overrightarrow{w}_E + \int_{\partial T} (\frac{a}{b}\overrightarrow{u} - \frac{1}{b}\overrightarrow{t}).\overrightarrow{w}_E. \quad (41)$$

Proof. i) The same proof of (39).

ii) if $E \in \partial T \cap \varepsilon_{h,\Gamma}$, we have: Using (1) gives

$$\int_T \overrightarrow{f}.\overrightarrow{w}_E = \int_T (-\nabla^2 \overrightarrow{u} + \nabla p).\overrightarrow{w}_E.$$

By applying the Green formula, we obtain

$$\int_T \overrightarrow{f}.\overrightarrow{w}_E = \int_T (\nabla \overrightarrow{u} - pI) : \nabla \overrightarrow{w}_E - \int_{\partial T} (\frac{\partial \overrightarrow{u}}{\partial n} - \overrightarrow{n}p).\overrightarrow{w}_E.$$

Since $\overrightarrow{w}_E = \overrightarrow{0}$ in the other three edges of rectangle T and we have by (3), $a\overrightarrow{u} + b(\frac{\partial \overrightarrow{u}}{\partial n} - \overrightarrow{n}p) = \overrightarrow{t}$ in E $\subset \partial T \cap \partial\Omega$, then

$$\int_T \overrightarrow{f}.\overrightarrow{w}_E = \int_T (\nabla \overrightarrow{u} - pI) : \nabla \overrightarrow{w}_E + \int_{\partial T} (\frac{a}{b}\overrightarrow{u} - \frac{1}{b}\overrightarrow{t}).\overrightarrow{w}_E$$

Theorem 11 *For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids T_h, the residual estimator η_R satisfies:*

$$\|\overrightarrow{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C_\Omega \eta_R$$

$$\eta_{R,T} \leq C(\sum_{T' \in \omega_T} \{\|\overrightarrow{e}\|^2_{J,T'} + \|\varepsilon\|^2_{0,T'}\})^{\frac{1}{2}}.$$

Note that the constant C in the local lower bound is independent of the domain, and
$\|\overrightarrow{e}\|^2_{J,T'} = \int_{T'} \nabla \overrightarrow{e} : \nabla \overrightarrow{e} + \int_{\partial T'} \frac{a}{b} \overrightarrow{e}.\overrightarrow{e}$ *and* $\|\varepsilon\|^2_{0,T'} = \int_{T'} \varepsilon^2$.

Proof: To establish the upper bound we let $(\overrightarrow{v}, q) \in H^1(\Omega) \times L^2_0(\Omega)$ and $\overrightarrow{v} \in X^1_n$ be the clement interpolant of $\overrightarrow{v}$, then
$C[(\overrightarrow{e}, \varepsilon); (\overrightarrow{v}, q)]$

$$\begin{aligned}
&= C[(\overrightarrow{e}, \varepsilon); (\overrightarrow{v} - \overrightarrow{v}_h, q)] \\
&= C[(\overrightarrow{u}, p); (\overrightarrow{v} - \overrightarrow{v}_h, q)] \\
&\quad - C[(\overrightarrow{u}_h, p_h); (\overrightarrow{v} - \overrightarrow{v}_h, q)] \\
&= L(\overrightarrow{v} - \overrightarrow{v}_h) - A(\overrightarrow{u}_h, \overrightarrow{v} - \overrightarrow{v}_h) \\
&\quad - B(\overrightarrow{v} - \overrightarrow{v}_h, p_h) - B(\overrightarrow{u}_h, q).
\end{aligned}$$

$$\begin{aligned}
&= \sum_{T \in T_h} \{(\overrightarrow{R}_T, \overrightarrow{v} - \overrightarrow{v}_h)_T \\
&\quad - \sum_{E \in \varepsilon(T)} (\overrightarrow{R}_E, \overrightarrow{v} - \overrightarrow{v}_h)_E + (R_T, q)_T\}. \\
&\leq \sum_{T \in T_h} \{\|\overrightarrow{R}_T\|_{0,T} \|\overrightarrow{v} - \overrightarrow{v}_h\|_{0,T} + \|R_T\|_{0,T} \|q\|_{0,T} \\
&\quad + \sum_{E \in \varepsilon(T)} \|\overrightarrow{R}_E\|_{0,E} \|\overrightarrow{v} - \overrightarrow{v}_h\|_{0,E}\} \\
&\leq (\sum_{T \in T_h} h_T^2 \|\overrightarrow{R}\|^2_{0,T})^{\frac{1}{2}} (\sum_{T \in T_h} \frac{1}{h_T^2} \|\overrightarrow{v} - \overrightarrow{v}_h\|^2_{0,T})^{\frac{1}{2}} \\
&\quad + (\sum_{T \in T_h} \sum_{E \in \partial T} h_E \|\overrightarrow{R_E}\|^2_{0,T})^{\frac{1}{2}} \\
&\quad \times (\sum_{T \in T_h} \sum_{E \in \partial T} \frac{1}{h_E} \|\overrightarrow{v} - \overrightarrow{v}_h\|^2_{0,E})^{\frac{1}{2}} \\
&\quad + (\sum_{T \in T_h} \|q\|^2_{0,T})^{\frac{1}{2}} (\sum_{T \in T_h} \|\nabla.\overrightarrow{u}_h\|^2_{0,T})^{\frac{1}{2}}.
\end{aligned}$$

Using (36) and (37), then gives

$$C[(\overrightarrow{e}, \varepsilon); (\overrightarrow{v}, q)] \leq C'(\sum_{T \in \tau_h} \|\overrightarrow{v}\|^2_{J,T} + \|q\|^2_{0,T})^{\frac{1}{2}}$$

$$\times (\sum_{T \in T_h} \{h_T^2 \|\overrightarrow{R}_T\|^2_{0,T} + \sum_{E \in \partial T} h_E \|\overrightarrow{R}_E\|^2_{0,E} + \|R_T\|^2_{0,T})^{\frac{1}{2}}$$

Finally, using (19), gives:
$\|\overrightarrow{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq$

$$C'(\sum_{T \in \tau_h} \{h_T^2 \|\overrightarrow{R}_T\|^2_{0,T} + \sum_{E \in \partial T} h_E \|\overrightarrow{R}_E\|^2_{0,E} + \|R_T\|^2_{0,T}\})^{\frac{1}{2}}$$

This establishes the upper bound. Turning to the local lower bound. First, for the element residual part, we have

$$\begin{aligned}
&\int_T \overrightarrow{R}_T.\overrightarrow{w}_T \\
&= \int_T (\overrightarrow{f} + \nabla^2 \overrightarrow{u}_h - \nabla p_h).\overrightarrow{w}_T \\
&= \int_T \overrightarrow{f}.\overrightarrow{w}_T - \int_T (\nabla \overrightarrow{u}_h - p_h I) : \nabla \overrightarrow{w}_T \\
&\quad + \int_{\partial T} (\nabla \overrightarrow{u}_h - p_h I)\overrightarrow{n}.\overrightarrow{w}_T
\end{aligned}$$

Using (39), (31) and $\overrightarrow{w}_T = \overrightarrow{0}$ *in* ∂T, give:

$$\begin{aligned}
\int_T \overrightarrow{R}_T.\overrightarrow{w}_T &= \int_T (\nabla \overrightarrow{e} - \varepsilon I) : \nabla \overrightarrow{w}_T \\
&\leq (|\overrightarrow{e}|_{1,T} + \|\varepsilon\|_{0,T}) |\overrightarrow{w_T}|_{1,T} \\
&\leq C(\|\overrightarrow{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T}) |\overrightarrow{w_T}|_{1,T} \\
&\leq C(\|\overrightarrow{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T})^{\frac{1}{2}} h_T^{-1} \|\overrightarrow{R}_T\|_{0,T}
\end{aligned}$$

In addition, from the inverse inequality (30)

$$\int_T \overrightarrow{R}_T.\overrightarrow{w}_T = \|\overrightarrow{R}_T.b_T^{\frac{1}{2}}\|^2_{0,T} \geq c\|\overrightarrow{R}_T\|^2_{0,T}.$$

Thus,

$$h_T^2\|\overrightarrow{R}_T\|^2_{0,T} \leq C(\|\overrightarrow{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T}). \tag{42}$$

Next comes the divergence part,

$$\begin{aligned} &\|R_T\|_{0,T} = \|\nabla.\overrightarrow{u}_h\|_{0,T} \\ =\ & \|\nabla.(\overrightarrow{u} - \overrightarrow{u}_h)\|_{0,T} \\ \leq\ & \sqrt{2}|\overrightarrow{u} - \overrightarrow{u}_h|_{1,T} \\ \leq\ & \sqrt{\frac{2}{\nu}}\|\overrightarrow{e}\|_{J,T}. \end{aligned} \tag{43}$$

Finally, we need to estimate the jump term. For an edge $E \in \partial T \cap \varepsilon_{h,\Omega}$. We have

$$\begin{aligned} 2\int_E \overrightarrow{R}_E.\overrightarrow{w}_E &= \sum_{i=1:2}\int_{\partial T_i}(\nabla\overrightarrow{u}_h - p_h I)\overrightarrow{n}.\overrightarrow{w}_E \\ &= \int_{\omega_E}(\nabla\overrightarrow{u}_h - p_h I) : \nabla\overrightarrow{w}_E \\ &\quad + \sum_{i=1:2}\int_{T_i}(\nabla^2\overrightarrow{u}_h - \nabla p_h)\overrightarrow{w}_E \end{aligned}$$

Using (40) and $\overrightarrow{w}_E = \overrightarrow{0}$ in $\partial\omega_E$, gives:

$$\begin{aligned} 2\int_E \overrightarrow{R}_E.\overrightarrow{w}_E &= -\int_{\omega_E}(\nabla\overrightarrow{e} - \varepsilon I) : \nabla\overrightarrow{w}_E \\ &\quad + \sum_{i=1:2}\int_{T_i}\vec{R}_{T_i}\overrightarrow{w}_E \\ &\leq (|\overrightarrow{e}|_{1,\omega_E} + \|\varepsilon\|_{0,\omega_E})|\overrightarrow{w}_E|_{1,w_E} \\ &\quad + \sum_{i=1:2}\|\overrightarrow{R}_{T_i}\|_{0,T_i}\|\overrightarrow{w}_E\|_{0,T_i} \end{aligned}$$

Using (33) and (34), give

$$\begin{aligned} \int_E \overrightarrow{R}_E.\overrightarrow{w}_E &\leq C(|\overrightarrow{e}|^2_{1,\omega_E} + \|\varepsilon\|^2_{0,\omega_E})^{\frac{1}{2}} h_E^{\frac{-1}{2}}\|\overrightarrow{R}_E\|_{0,E} \\ &\quad + \sum_{i=1:2}\|\overrightarrow{R}_{T_i}\|_{0,T_i} h_E^{\frac{1}{2}}\|\overrightarrow{R}_E\|_{0,E}. \end{aligned}$$

Using (42), gives

$$\begin{aligned} &\int_E \overrightarrow{R}_E.\overrightarrow{w}_E \\ \leq\ & C(\|\overrightarrow{e}\|^2_{J,\omega_E} + \|\varepsilon\|^2_{0,\omega_E})^{\frac{1}{2}} h_E^{\frac{-1}{2}}\|\overrightarrow{R}_E\|_{0,E} \end{aligned} \tag{44}$$

Using (32) gives

$$\int_E \overrightarrow{R}_E.\overrightarrow{w}_E = \|\overrightarrow{R}_E b_E^{\frac{1}{2}}\|^2_{0,E} \geq c\|\overrightarrow{R}_E\|^2_{0,E},$$

and thus using (44) gives

$$h_E\|\overrightarrow{R}_E\|^2_{0,E} \leq C(\|\overrightarrow{e}\|^2_{J,\omega_E} + \|\varepsilon\|^2_{0,\omega_E}). \tag{45}$$

We also need to show that (45) holds for boundary edges. For an $E \in \partial T \cap \varepsilon_{h,\Gamma}$, we have

$$\begin{aligned} &\int_E \overrightarrow{R}_E.\overrightarrow{w}_E \\ &= \int_{\partial T}[\frac{a}{b}\overrightarrow{u}_h + (\nabla\overrightarrow{u}_h - p_h I)\overrightarrow{n} - \frac{1}{b}\overrightarrow{t}].\overrightarrow{w}_E \\ =\ & \int_{\partial T}[\frac{a}{b}\overrightarrow{u}_h - \frac{1}{b}\overrightarrow{t}].\overrightarrow{w}_E \\ &+ \int_T(\nabla\overrightarrow{u}_h - p_h I) : \nabla\overrightarrow{w}_E \\ &+ \int_T(\nabla^2\overrightarrow{u}_h - \nabla p_h).\overrightarrow{w}_E \end{aligned}$$

Using (41) and (4), gives

$$\begin{aligned} &\int_E \overrightarrow{R}_E.\overrightarrow{w}_E = -\int_T(\nabla\overrightarrow{e} - \varepsilon I) : \nabla\overrightarrow{w}_E \\ &\quad - \int_{\partial T}\frac{a}{b}\overrightarrow{e}\,\overrightarrow{w}_E + \int_T \overrightarrow{R}_T.\overrightarrow{w}_E \\ \leq\ & (|\overrightarrow{e}|_{1,T} + \|\varepsilon\|_{0,T})|\overrightarrow{w}_E|_{1,T} \\ &\quad + \beta_1\|\overrightarrow{e}\|_{0,\partial T}\|\overrightarrow{wE}\|_{0,\partial T} \\ &\quad + \|\overrightarrow{R}_T\|_{0,T}\|\overrightarrow{w}_E\|_{0,T} \\ \leq\ & C(\|\overrightarrow{e}\|_{J,T} + \|\varepsilon\|_{0,T})\|\overrightarrow{w}_E\|_{J,T} \\ &\quad + \|\overrightarrow{R}_T\|_{0,T}\|\overrightarrow{w}_E\|_{0,T} \end{aligned}$$

Using (33) and (35) gives

$$\begin{aligned} \int_E \overrightarrow{R}_E.\overrightarrow{w}_E &\leq C''(\|\overrightarrow{e}\|_{J,T} + \|\varepsilon\|_{0,T})h_E^{-\frac{1}{2}}\|\overrightarrow{R}_E\|_{0,E} \\ &\quad + \|\overrightarrow{R}_T\|_{0,T} h_E^{\frac{1}{2}}\|\overrightarrow{R}_E\|_{0,E}. \end{aligned}$$

Using (42) gives

$$\int_E \overrightarrow{R}_E.\overrightarrow{w}_E \leq C(\|\overrightarrow{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T})^{\frac{1}{2}} h_E^{\frac{-1}{2}}\|\overrightarrow{R}_E\|_{0,E} \tag{46}$$

Using (32), gives

$$\int_E \overrightarrow{R}_E.\overrightarrow{w}_E = \|\overrightarrow{R}_E b_E^{\frac{1}{2}}\|^2_{0,E} \geq c\|\overrightarrow{R}_E\|^2_{0,E},$$

and thus using (46) gives

$$h_E\|\overrightarrow{R}_E\|^2_{0,E} \leq C(\|\overrightarrow{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T}). \tag{47}$$

Finally, combining (42), (43), (45) and (47) establishes the local lower bound.

Remark. Theorem 11 also holds for stable (and unstable) mixed approximations defined on a triangular subdivision if we take the obvious interpretation of The Proof is identical except for the need to define appropriate element and edge bubble functions

6.2 A Local Poisson Problem Estimator

The local Poisson problem estimator:

$$\eta_P = \sqrt{\sum_{T \in T_h} \eta_{P,T}^2}$$

as follows

$$\eta_{P,T}^2 = \|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2 \tag{48}$$

Let

- $V_T = H^1(T)$
- $A_T(\overrightarrow{e}_{P,T}, \overrightarrow{v}) = \int_T \nabla \overrightarrow{e}_{P,T} : \nabla \overrightarrow{v} + \int_{\partial T} \frac{a}{b} \overrightarrow{e}_{P,T} \overrightarrow{v}$

$\vec{e}_{P,T} \in V_T$ satisfies the uncoupled Poisson problems

$$A_T(\overrightarrow{e}_{P,T}, \overrightarrow{v}) = (\vec{R}_T, \vec{v})_T - \sum_{E \in \varepsilon(T)} (\vec{R}_E, \vec{v})_E \tag{49}$$

$$\varepsilon_{P,T} = \nabla.\vec{u}_h / T \tag{50}$$

for any $\overrightarrow{v} \in V_T$.

Theorem 12 *The estimator $\eta_{P,T}$ is equivalent to the $\eta_{R,T}$ estimator:*

$$c\eta_{P,T} \leq \eta_{R,T} \leq C\eta_{P,T}$$

Proof. For the upper bound, we first let $\vec{w}_T = \vec{R}_T b_T$ (b_T is an element interior bubble function). From (49),

$$(\vec{R}_T, \vec{w}_T)_T = (\vec{e}_{P,T}, \vec{w}_T)_T \leq |\vec{e}_{P,T}|_{1,T}.|\vec{w}_T|_{1,T}$$

Using (31) we get

$$\begin{aligned} &(\vec{R}_T, \vec{w}_T)_T \leq \\ &C\frac{1}{h_T}\|\vec{R}\|_{0,T}(\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2)^{\frac{1}{2}} \end{aligned} \tag{51}$$

In addition, from the inverse inequalities (30), $\|\vec{R}\|_{0,T}^2 \leq C.(\vec{R}_T, \vec{w}_T)_T$ and using (51), to get

$$h_T^2\|\vec{R}_T\|_{0,E}^2 \leq C(\|\vec{e}_{P,T}\|_{J,\Omega}^2 + \|\varepsilon_{P,T}\|_{0,T}^2) \tag{52}$$

Next, we let $\vec{w}_E = \vec{R}_E\, b_E$ is an edge bubble function.

If $E \in \partial T \bigcap \varepsilon_{h,\Gamma}$ using (49), (33) and (35) give

$$\begin{aligned} &(\vec{R}_E, \vec{w}_E)_E \\ = \;& -A_T(\overrightarrow{e}_{P,T}, \overrightarrow{w}_E) + (\vec{R}_T, \vec{w}_E)_T \\ \leq \;& \|\vec{e}_{P,T}\|_{J,T}\|\vec{w}_E\|_{J,T} + \|\vec{R}_E\|_{0,T}\|\vec{w}_E\|_{0,T} \\ \leq \;& Ch_E^{-\frac{1}{2}}\|\vec{R}_E\|_{0,E}(\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2)^{\frac{1}{2}} \end{aligned}$$

If $E \in \partial T \bigcap \varepsilon_{h,\Omega}$, see that a and b defined just in Γ, then we can posed $a = 0$ in $\Omega - \Gamma$.

Using (49), (33), (34) and (52) give

$$\begin{aligned} &(\vec{R}_E, \vec{w}_E)_E \\ = \;& -(\nabla\vec{e}_{P,T}, \nabla\vec{w}_E)_T + (\vec{R}_T, \vec{w}_E)_T \\ \leq \;& |\vec{e}_{P,T}|_{1,T}|\vec{w}_E|_{1,T} + \|\vec{R}_E\|_{0,T}\|\vec{w}_E\|_{0,T} \\ \leq \;& Ch_E^{-\frac{1}{2}}\|\vec{R}_E\|_{0,T}(\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2)^{\frac{1}{2}} \end{aligned}$$

Finally, for any $T \in T_h$ and any $E \in \partial T$, we have

$$\begin{aligned} &(\vec{R}_E, \vec{w}_E)_E \\ \leq \;& Ch_E^{-\frac{1}{2}}\|\vec{R}_E\|_{0,T}(\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2)^{\frac{1}{2}} \end{aligned} \tag{53}$$

From the inverse inequalities (32), $\|\vec{R}_E\|_{0,E}^2 \leq C.(\vec{R}_E, \vec{w}_E)_E$ and using (53) give

$$h_E\|\vec{R}_E\|_{0,E}^2 \leq C(\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2) \tag{54}$$

We have also

$$\begin{aligned} \|R_T\|_{0,T} &= \|\nabla.\vec{u}_h\|_{0,T} = \|\varepsilon_{P,T}\|_{0,T} \\ &\leq (\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2)^{\frac{1}{2}} \end{aligned} \tag{55}$$

Combining (52), (54) and (55), establishes the upper bound in the equivalence relation. For the lower , we need to use (20):

$$\begin{aligned} \eta_{P,T} &= (\|\vec{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2)^{\frac{1}{2}} \\ &\leq \|\vec{e}_{P,T}\|_{J,T} + \|\varepsilon_{P,T}\|_{0,T} \\ &\leq 2 \sup_{(\vec{v},q) \in V_T \times L_0^2(\Omega)} \frac{A(\vec{e}_{P,T}, \vec{v}) + d(\varepsilon_{P,T}, q)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \end{aligned}$$

Using (49) and (50) give

$$\begin{aligned} \eta_{P,T} &\leq 2 \sup_{(\vec{v},q) \in V_T \times L_0^2(\Omega)} \frac{\lambda(\overrightarrow{v}, T)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \\ &\leq 2 \sup_{(\vec{v},q) \in V_T \times L_0^2(\Omega)} \frac{\theta(\overrightarrow{v}, T)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \end{aligned} \tag{56}$$

where

$$\lambda(\overrightarrow{v}, T) = (\overrightarrow{R}_T, \vec{v})_T - \sum_{E \in \varepsilon(T)} (\overrightarrow{R}_E, \vec{v}) + (R_T, q)$$

and

$$\begin{aligned} \theta(\overrightarrow{v}, T) = \;& \|\vec{R}\|_{0,T}\|\vec{v}\|_{0,T} + \sum_{E \in \varepsilon(T)} \|\vec{R}_E\|_{0,E}\|\vec{v}\|_{0,E} \\ & + \|\nabla.\vec{u}_h\|_{0,T}\|q\|_{0,T} \end{aligned}$$

Now, since $\vec{v}$ is zero at the four vertices of T, a scaling argument and the usual trace theorem, see e.g. [14, Lemma 1.5], shows that $\vec{v}$ satisfies

$$\|\vec{v}\|_{0,E} \leq C h_E^{\frac{1}{2}} |\vec{v}|_{1,T} \tag{57}$$
$$\|\vec{v}\|_{0,E} \leq C h_T |\vec{v}|_{1,T} \tag{58}$$

Combining these two inequalities with (56) immediately gives the lower bound in the equivalence relation

Theorem 13 *For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids T_h the estimator η_P satisfies:*

$$\|\vec{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C\eta_P$$

and

$$\eta_{P,T} \leq C\Big(\sum_{T\in T_h} \{\|\vec{e}\|^2_{J,T'} + \|\varepsilon\|^2_{0,T'}\Big)^{\frac{1}{2}}$$

Note that the constant C in the local lower bound is independent of the domain.

7 Numerical simulation

In this section some numerical results of calculations with mixed finite element Method will presented in two examples [14, 16].

Example 14 *Square domain, enclosed flow boundary condition.*

This is a classic test problem used in fluid dynamics, known as driven-cavity flow. It is a model of the flow in a square cavity with the lid moving from left to right. Let the computational model:

$\{y = 1; -1 \leq x \leq 1/u_x = 1-x^2\}$, a regularized cavity.

The $C_{a,b}$ condition is satisfied, just take a and b two real number strictly positive such that $a \succ\succ b$, $\overrightarrow{t} = (a(1-x^2); 0)$ on $\Gamma_1 = (y = 1; -1 \leq x \leq 1)$ and $\overrightarrow{t} = (0; 0)$ on the other three boundary of the square domain.

The streamlines are computed from the velocity solution by solving the Poisson equation numerically subject to a zero Dirichlet boundary condition.

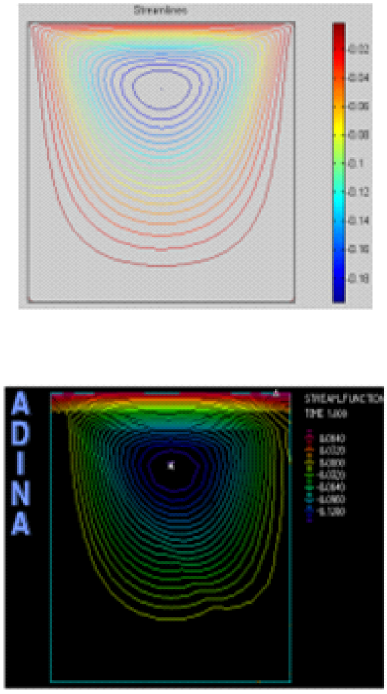

Fig. 1. Uniform streamline plot by MFE (left) associated with a 64×64 square grid, $P_1 - P_0$ approximation, and uniform streamline plot (right) computed with ADINA system.

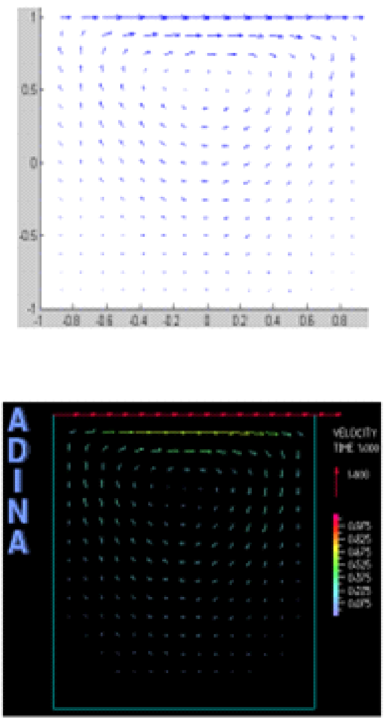

Fig. 2. Velocity vectors solution by MFE (left) associated with a 64×64 square grid, $P_1 - P_0$

approximation and velocity vectors solution (right) computed with ADINA system.

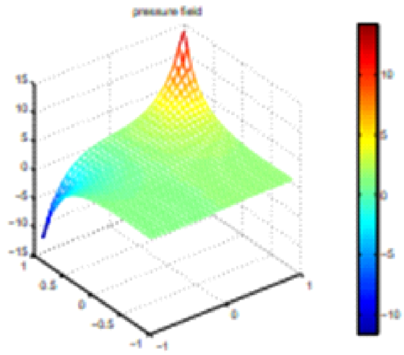

Fig. 3. Pressure plot for the flow with a 64×64 square grid.

Table 1. η_R is the residual error estimator and η_P is the local Poisson problem error estimator for a colliding flow.

grid	η_R	η_P
8×8	2.616759 e-001	8.722532e-001
16×16	1.695069e-001	6.053819e-001
32×32	1.020879e-001	3.781035e-001
64×64	6.063286e-002	2.220984e-001

Example 15 *Square domain Ω, analytic solution. This analytic test problem is associated with the following solution of the Stokes equation system:*

$$u_x = 20xy^3;\ u_y = 20x^4 - 5y^4; \tag{59}$$

$$p = 60x^2y - 20y^3 + constant.$$

It is a simple model of colliding flow, and a typical solution of streamline is illustrated in Figure 4. To solve this problem numerically, the finite element interpolant of the velocity in (59) is specified everywhere on $\partial\Omega$. The Dirichlet boundary condition for the stream function calculation is the interpolant of the exact stream function: $\psi(x, y) = 5xy^4 - x^5$. See that the $C_{a,b}$ condition is satisfied with $a \succ\succ b$ a real number very large and $\overrightarrow{t} = (20axy^3; (20x^4 - 5y^4)a)$ on Γ

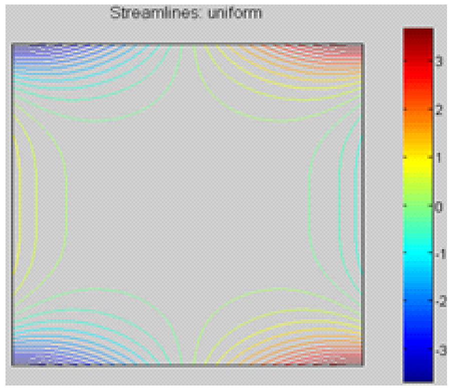

Fig. 4. Uniform streamline plot by MFE associated with a 64×64 square grid.

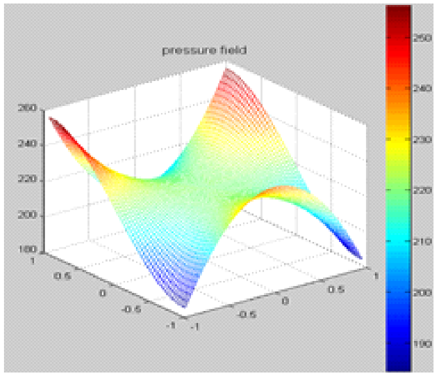

Fig. 5. Pressure plot for the flow with a 64×64 square grid.

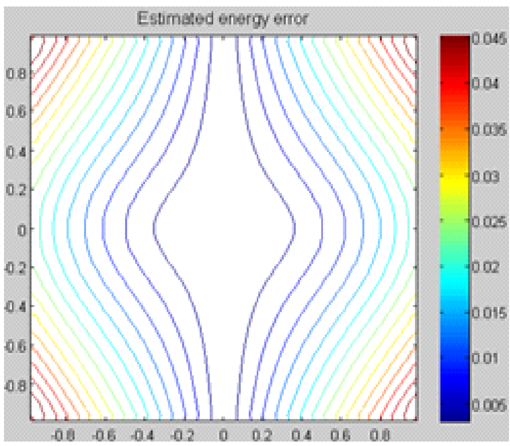

Fig. 6. Estimated error $\eta_{P,T}$ associated with 64×64 square grid.

Figure 5 shows the pressure plot and figure 6 shows the estimated error associated with 64×64 square grid.

Table 2. η_R is the residual error estimator and η_P is the local Poisson problem error estimator for a colliding flow.

grid	η_R	η_P
8×8	2.371793e+000	8.147694e+000
16×16	1.303621e+000	4.599936e+000
32×32	7.205132e-001	2.401784e+000
64×64	3.407043e-001	1.218978e+000

8 Conclusion

In this work, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We introduced the Stokes equations with a new boundary condition noted $C_{a,b}$.

The weak formulation obtained is a problem of saddle point type. We have shown the existence and uniqueness of the solution of this problem. We used the discretization by mixed finite element method with two type of the posteriori error estimation of the computed solutions. Both types proved that the larger the grid is, the better the approximation is.

Numerical experiments were carried out and compared with satisfaction with other numerical results, either resulting from the literature, or resulting from calculation with commercial software like Adina system.

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References:

[1] Alexandre Ern, Aide-mémoire Eléments Finis, Dunod, Paris, 2005.

[2] P. A. Raviart, J. Thomas, *Introduction l'analyse numérique des équations aux dérivées partielles*, Masson, Paris, 1983.

[3] R. Dautray et J. L. Lions, *Mathematical Analysis and Numerical Methods for science and Technology*, Vol.4. Integral equations and numerical methods, Springer-Verlag, Berlin, Allemagne, 1990.

[4] F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Method, Computational Mathematics*, Springer Verlag, New York, 1991.

[5] P. A. Raviart, J. M. Thomas, A mixed finite element method for second order elliptic problems, in: *Mathematical Aspects of Finite Element Method*, Lecture Notes in Mathematics, Springer, New York, 606, 1977, pp. 292-315.

[6] G. Chavent and J. Jaffré, *Mathematical Models and Finite Elements for Reservoir Simulation*, Elsevier Science Publishers B.V, Netherlands, 1986.

[7] J. Roberts and J.M. Thomas, *Mixed and Hybrid methods, Handbook of numerical analysis II, Finite element methods 1*, P. Ciarlet and J. Lions, Amsterdam, 1989.

[8] R. E. Bank, A. Weiser, Some posteriori error estimators for elliptic partial differential equations, *Math, Comp.*, 44, 1985, pp.283-301.

[9] M. Ainsworth, J. Oden, A posteriori error estimates for Stokes' and Oseen's equations, *SIAM J. Numer. Anal.*, 34, 1997, pp. 228-245.

[10] R. E. Bank, B. Welfert, A posteriori error estimates for the Stokes problem, *SIAM J. Numer. Anal.*, 28, 1991, pp. 591-623.

[11] C. Carstensen, S. A. Funken, A posteriori error control in low-order finite element discretizations of incompressible stationary flow problems,*Math. Comp.*, 70, 2001, pp. 1353-1381.

[12] D. Kay, D. Silvester, A posteriori error estimation for stabilized mixed approximations of the Stokes equations, *SIAM J. Sci. Comput*, 21, 1999, pp.1321-1336.

[13] R. Verfurth, A posteriori error estimators for the Stokes equations, *Numer. Math.*, 55, 1989, pp. 309-325.

[14] H. Elman, D. Silvester, A. Wathen, Finite Elements and Fast Iterative Solvers: with Applications in *Incompressible Fluid Dynamics*, Oxford University Press, Oxford, 2005.

[15] R. Verfurth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Wiley-Teubner, Chichester, 1996.

[16] D.H. Wu, I.G. Currie, Analysis of a posteriori error indicator in viscous flows,*Int. J. Num. Meth. Heat Fluid Flow*, 12, 2000, pp. 306-327.

[17] Y. He, A. Wang, L. Mei, Stabilized finite-element method for the stationary Navier-Stokes equations, *J. Eng. Math.*, 51, 2005, pp. 367-380.

[18] V. John, Residual a posteriori error estimates for two-level finite element methods for the Navier-Stokes equations, *App. Numer, Math.*, 37, 2001, pp. 503-518.

[19] E. Creuse, G. Kunert, S. Nicaise, A posteriori error estimation for the Stokes problem: Anisotropic and isotropic discretizations,

Mathematical Models and Methods, 14, 2004, pp.1297-1341

[20] P. Clement, Approximation by finite element functions using local regularization, *RAIRO. Anal. Numer.*, 2, 1975, pp. 77-84.

[21] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, Springer-Verlag, Berlin Heiderlberg New York, 1981.