Variable-coefficient Simplest Equation Method For Solving Nonlinear Evolution Equations In Mathematical Physics

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Abstract: In this paper, a variable-coefficient simplest equation method is proposed to establish exact solutions for nonlinear evolution equations. For illustrating the validity of this method, we apply it to the asymmetric (2+1)-dimensional NNV system, the (2+1)-dimensional dispersive long wave equations and the (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations. As a result, some new exact solutions and solitary wave solutions involving arbitrary function as coefficients are obtained for them.

Key–Words: Simplest equation method; Variable-coefficient; Exact solutions; Nonlinear evolution equations; Traveling wave solutions; Solitary wave solutions
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1 Introduction

Nonlinear evolution equations (NLEEs) can be used to describe many nonlinear phenomena such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. Recently, research for seeking exact analytical solutions of NLEEs is a hot topic, and many powerful and efficient methods to find analytic solutions have been presented so far. For example, these methods include the known homogeneous balance method [1,2], the tanh-method [3-5], the inverse scattering transform [6], the Backlund transform [7,8], the Hirotas bilinear method [9,10], the generalized Riccati equation [11,12], the sine-cosine method [13], the Jacobi elliptic function expansion [14,15], the F-expansion method [16], the exp-function expansion method [17], the (G'/G)-expansion method [18,19]. However, we notice that most of the existing methods are dealing with constant coefficients, while very few methods are concerned of variable-coefficients.

In this paper, by introducing a new ansatz, we develop a variable-coefficient simplest equation method for solving NLEEs, which is the extension of the simplest equation method [20-23]. Then we apply this method to establish exact solutions of NLEEs.

We organize the rest of this paper as follows. In Section 2, we give the description of the variable-coefficient simplest equation method. Then in Section 3 we apply the method to solve the asymmetric (2+1)-dimensional asymmetric NNV system, the (2+1)-dimensional dispersive long wave equations and the (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations. Some conclusions are presented at the end of the paper.

2 Description of the variable-coefficient simplest equation method

Suppose that a nonlinear evolution equation, say in two or three independent variables x, y, t, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{xy}...) = 0, \qquad (1)$$

where $u = u(x, y, t)$ is an unknown function, P is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.
Step 1. Suppose that

$$u(x, y, t) = U(\xi), \quad \xi = \xi(x, y, t), \qquad (2)$$

and then Eq. (2) can be turned into the following form

$$\widetilde{P}(U,\ U',\ U'',\ ...) = 0. \qquad (3)$$

Step 2. Suppose that the solution of (3) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows:

$$U(\xi)=\sum_{i=0}^{m}a_i(x,y,t)(\frac{\phi'}{\phi})^i, \qquad (4)$$

where $a_m(x,y,t),\ a_{m-1}(x,y,t),\ ...,\ a_0(x,y,t)$ are all unknown functions to be determined later with $a_m(x,y,t)\neq 0$, and $\phi=\phi(\xi)$ satisfies some certain simplest equation with the following form

$$F(\phi,\phi',\phi'',...)=0. \qquad (5)$$

The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (3).

Step 3. Substituting (4) into (3) and using the relation between $\phi'(\xi)$ and $\phi(\xi)$ deduced by (5), collecting all terms with the same order of $\phi(\xi)$ together, the left-hand side of (3) is converted to another polynomial in $\phi(\xi)$. Equating each coefficient of this polynomial to zero, yields a set of partial differential equations for $a_m(x,y,t),\ a_{m-1}(x,y,t)...,\ a_0(x,y,t),\ \xi(x,y,t)$.

Step 4. Solving the equations in Step 3, and by using the solutions of Eq. (5), we can obtain exact solutions for Eq. (1).

Remark 1 *If we take Eq. (5) for different forms such as the Riccati equation, Bernoulli equation and so on, then different exact solutions for Eq. (1) can be obtained. Especially, if we substitute ϕ with G, and Eq. (5) takes the form $G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0$, then the described method above becomes the variable-coefficient (G'/G) method. So the (G'/G) method is a special case of the simplest equation method.*

Remark 2 *As the partial differential equations in Step 3 are usually over-determined, we may choose some special forms of $a_m,\ a_{m-1},\ ...,\ a_0$ as did in the following.*

3 Application of the variable-coefficient simplest equation method to some NLEEs

3.1 Asymmetric (2+1) dimensional NNV system

First we consider the asymmetric (2+1)-dimensional NNV system [24-26]:

$$\begin{cases} u_t-u_{xxx}+\alpha(uv)_x=0,\\ u_x+\beta v_y=0, \end{cases} \qquad (6)$$

where α and β are arbitrary nonzero constants.

Assume that $u(x,y,t)=U(\xi),\ v(x,y,t)=V(\xi)$, where $\xi=\xi(x,y,t)$. Then Eqs. (6) can be turned into

$$\begin{cases} \xi_t'U'-(\xi_x^3U'''+3\xi_x\xi_{xx}U''+\xi_{xxx}U')\\ \quad +\alpha\xi_x(UV)'=0,\\ \xi_x'U'+\beta\xi_yV'=0. \end{cases} \qquad (7)$$

Suppose that the solutions of Eqs. (7) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows:

$$U(\xi)=\sum_{i=0}^{m}a_i(y,t)(\frac{\phi'}{\phi})^i,\ V(\xi)=\sum_{i=0}^{n}b_i(y,t)(\frac{\phi'}{\phi})^i, \qquad (8)$$

where $a_i(y,t),b_i(y,t)$ are under-determined functions, and $\phi=\phi(\xi)$ satisfies Eq. (5). Balancing the order of U''' and $(UV)'$, U' and V' in Eqs. (7), we can obtain $m+3=m+n+1,\ m+1=n+1\Rightarrow\ m=2,\ n=2$. So we have

$$\begin{cases} U(\xi)=a_2(y,t)(\frac{\phi'}{\phi})^2+a_1(y,t)(\frac{\phi'}{\phi})+a_0(y,t),\\ V(\xi)=b_2(y,t)(\frac{\phi'}{\phi})^2+b_1(y,t)(\frac{\phi'}{\phi})+b_0(y,t). \end{cases} \qquad (9)$$

We will proceed the computation in two cases.

Case 1: $\phi=\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi)=a+\phi^2(\xi), \qquad (10)$$

Substituting (9) into (7), using Eq. (10) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_i(y,t),b_i(y,t),\ i=0,1,2$ and $\xi(x,y,t)$. Solving these equations, yields

$$\begin{aligned} \xi(x,y,t)=&-\sqrt{\tfrac{C_1}{6}\alpha}x+\int\tfrac{C_1\alpha^2}{\sqrt{6C_1\alpha}}F_1(t)dt\\ &+\int\sqrt{\tfrac{C_1\alpha}{6}\tfrac{F_2(y)}{\beta C_1}}dy,\\ a_2(y,t)=&F_2(y),\\ a_1(y,t)=&0,\\ a_0(y,t)=&-\tfrac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}C_2\sqrt{6\alpha C_1})}{3\alpha^2C_1^2},\\ b_2(y,t)=&C_1,\\ b_1(y,t)=&0,\\ b_0(y,t)=&F_1(t), \end{aligned}$$

where C_1 is an arbitrary nonzero constant, and $F_1(t),\ F_2(y)$ are two arbitrary functions with respect to the variable t and y respectively.

On the other hand, for Eq. (10), the following solutions are known to us.

When $\sigma < 0$,

$$\begin{cases} \phi_1(\xi) = -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi + c_0), \\ \phi_2(\xi) = -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi + c_0), \end{cases} \quad (11)$$

where c_0 is an arbitrary constant.

When $\sigma > 0$,

$$\begin{cases} \phi_3(\xi) = \sqrt{\sigma}\tan(\sqrt{\sigma}\xi + c_0), \\ \phi_4(\xi) = -\sqrt{\sigma}\cot(\sqrt{\sigma}\xi + c_0), \end{cases} \quad (12)$$

$$\phi_{5,6}(\xi) = \sqrt{\sigma}[\tan(2\sqrt{\sigma}\xi + c_0) \pm \sec(2\sqrt{\sigma}\xi + c_0)], \quad (13)$$

where c_0 is an arbitrary constant.

When $\sigma = 0$

$$\phi_7(\xi) = -\frac{1}{\xi + c_0}, \quad (14)$$

where c_0 is an arbitrary constant.

Combining the results above with (11)-(14), we can obtain the following exact solutions for the asymmetric (2+1)-dimensional NNV system.

When $\sigma < 0$:

$$\begin{cases} u_1(x,y,t) = -\sigma F_2(y)\left\{\frac{sech^2[\sqrt{-\sigma}\xi+c_0]}{\tanh[\sqrt{-\sigma}\xi+c_0]}\right\}^2 -\frac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}\sqrt{6}C_2\sqrt{\alpha C_1})}{3\alpha^2C_1^2}, \\ v_1(x,y,t) = -\sigma C_1\{\frac{sech^2[\sqrt{-\sigma}\xi+c_0]}{\tanh[\sqrt{-\sigma}\xi+c_0]}\}^2 + F_1(t). \end{cases} \quad (15)$$

$$\begin{cases} u_2(x,y,t) = -\sigma F_2(y)\{\frac{csch^2[\sqrt{-\sigma}\xi+c_0]}{coth[\sqrt{-\sigma}\xi+c_0]}\}^2 -\frac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}\sqrt{6}C_2\sqrt{\alpha C_1})}{3\alpha^2C_1^2}, \\ v_2(x,y,t) = -\sigma C_1\{\frac{csch^2[\sqrt{-\sigma}\xi+c_0]}{\coth[\sqrt{-\sigma}\xi+c_0]}\}^2 + F_1(t). \end{cases} \quad (16)$$

When $\sigma > 0$:

$$\begin{cases} u_3(x,y,t) = \sigma F_2(y)\{\frac{sec^2[\sqrt{\sigma}\xi+c_0]}{\tan[\sqrt{\sigma}\xi+c_0]}\}^2 -\frac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}\sqrt{6}C_2\sqrt{\alpha C_1})}{3\alpha^2C_1^2}, \\ v_3(x,y,t) = \sigma C_1\{\frac{sec^2[\sqrt{\sigma}\xi+c_0]}{\tan[\sqrt{\sigma}\xi+c_0]}\}^2 + F_1(t). \end{cases} \quad (17)$$

$$\begin{cases} u_4(x,y,t) = \sigma F_2(y)\{\frac{csc^2[\sqrt{\sigma}\xi+c_0]}{\cot[\sqrt{\sigma}\xi+c_0]}\}^2 -\frac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}\sqrt{6}C_2\sqrt{\alpha C_1})}{3\alpha^2C_1^2}, \\ v_4(x,y,t) = \sigma C_1\{\frac{csc^2[\sqrt{\sigma}\xi+c_0]}{\cot[\sqrt{\sigma}\xi+c_0]}\}^2 + F_1(t). \end{cases} \quad (18)$$

and

$$\begin{cases} u_{5,6}(x,y,t) = 4\sigma F_2(y)\times \\ \quad \{\frac{sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)\tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}^2 \\ \quad -\frac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}\sqrt{6}C_2\sqrt{\alpha C_1})}{3\alpha^2C_1^2}, \\ v_{5,6}(x,y,t) = 4\sigma C_1\times \\ \quad \{\frac{sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)\tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}^2 \\ \quad +F_1(t), \end{cases} \quad (19)$$

where

$$\xi = -\sqrt{\frac{C_1}{6}\alpha}x + \int \frac{C_1\alpha^2}{\sqrt{6C_1\alpha}}F_1(t)dt + \int \sqrt{\frac{C_1\alpha}{6}}\frac{F_2(y)}{\beta C_1}dy.$$

When $\sigma = 0$:

$$\begin{cases} u_7(x,y,t) = \frac{F_2(y)}{(\xi+c_0)^2} - \frac{8F_2(y)(\alpha^2C_1^2\sigma-\frac{3}{8}\sqrt{6}C_2\sqrt{\alpha C_1})}{3\alpha^2C_1^2}, \\ v_7(x,y,t) = \frac{C_1}{(\xi+c_0)^2} + F_1(t). \end{cases} \quad (20)$$

Case 2: $\phi = \phi(\xi)$ satisfies the following Bernoulli equation

$$\phi' + \lambda\phi = \mu\phi^3. \quad (21)$$

Substituting (9) into (7), using Eq. (21) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_i(y,t), b_i(y,t),\ i = 0,1,2$ and $\xi(x,y,t)$. Solving these equations, yields

$$a_2(y,t) = F_1(y),\ a_1(y,t) = F_1(y)\lambda,$$
$$a_0(y,t) = C_1F_1(y),\ b_2(y,t) = C_2,$$
$$b_1(y,t) = C_2\lambda,$$
$$b_0(y,t) = -\frac{\alpha C_2(C_1-\frac{\lambda^2}{6})\sqrt{6\alpha C_2}-12F_2'(t)}{\alpha\sqrt{6\alpha C_2}},$$
$$\xi(x,y,t) = -\frac{\sqrt{6\alpha C_2}}{12}x + \int \frac{\sqrt{6\alpha C_2}F_1(y)}{12\beta C_2}dy + F_2(t),$$

where $C_1,\ C_2$ are arbitrary constants with $C_2 \neq 0$, and $F_1(y),\ F_2(t)$ are two arbitrary functions with respect to the variable y and t respectively.

By the general solutions of Eq. (21), we have

$$\begin{cases} \phi(\xi) = \pm\frac{1}{\sqrt{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}}, \\ \frac{\phi'}{\phi} = -\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}, \end{cases} \quad (22)$$

where $\lambda,\ \mu,\ A$ are arbitrary constants with $\lambda \neq 0$, and $\mu^2 + A^2 \neq 0$.

Substituting the results above into Eqs. (9), and combining with (22), we can obtain the following exact solutions:

$$\begin{cases} u_8(x,y,t) = C_1F_1(y) + F_1(y)(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}) \\ \qquad +F_1(y)\lambda(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}})^2, \\ v_8(x,y,t) = -\frac{\alpha C_2(C_1-\frac{\lambda^2}{6})\sqrt{6\alpha C_2}-12F_2'(t)}{\alpha\sqrt{6\alpha C_2}} \\ \qquad +C_2\lambda(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}) \\ \qquad +C_2(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}})^2, \end{cases} \quad (23)$$

where $\xi = -\frac{\sqrt{6\alpha C_2}}{12}x + \int \frac{\sqrt{6\alpha C_2}F_1(y)}{12\beta C_2}dy + F_2(t)$.

Especially, if we set $\mu = \lambda A$ in Eq. (23), then we obtain the following solitary wave solutions:

$$\begin{cases} u_9(x,y,t) = C_1F_1(y) - \frac{\lambda F_1(y)}{2}[(1+\tanh(\lambda\xi))] \\ \qquad +\frac{\lambda^3F_1(y)}{4}[1+\tanh(\lambda\xi)]^2, \\ v_9(x,y,t) = -\frac{\alpha C_2(C_1-\frac{\lambda^2}{6})\sqrt{6\alpha C_2}-12F_2'(t)}{\alpha\sqrt{6\alpha C_2}} \\ \qquad -\frac{C_2\lambda^2}{2}[(1+\tanh(\lambda\xi))] \\ \qquad +\frac{C_2\lambda^2}{4}[1+\tanh(\lambda\xi)]^2. \end{cases}$$

Remark 3 *In [24-26], some exact solutions for the asymmetric (2+1) dimensional NNV system are established using different methods. We note that the established solutions involving arbitrary functions as coefficients above are different from them essentially as we have used a new variable-coefficient method here, and are new exact solutions which have been reported by other authors in the literature.*

3.2 (2+1)-dimensional dispersive long wave equations

We consider the known (2+1)-dimensional dispersive long wave equations [27-40]:

$$\begin{cases} u_{yt} + v_{xx} + (uu_x)_y = 0, \\ v_t + u_x + (uv)_x + u_{xxy} = 0. \end{cases} \quad (24)$$

Some types of exact solutions for Eqs. (27)-(28) have been obtained in [27-40] by use of various methods including the Riccati sub-equation method [27, 28, 33], the nonlinear transformation method [29], Jacobi function method [31, 32, 40], (G'/G)-expansion method [30], modified CK's direct method [34], EXP-function method [32], Hopf-Cole transformation method [36], modified extended Fan's sub-equation method [37, 38], generalized algebraic method [39].

To apply the proposed method, similar as the process above, we assume that $u(x,y,t) = U(\xi)$, $v(x,y,t) = V(\xi)$, $\xi = \xi(x,y,t)$. Then Eqs. (24) can be turned into

$$\begin{cases} \xi''_{yt}U' + \xi'_y\xi'_tU'' + \xi''_{xx}V' + (\xi'_x)^2V'' \\ \qquad +\xi'_x\xi'_y(UU'' + U'^2) + \xi''_{yx}UU' = 0, \\ \xi'_tV' + \xi'_xU' + \xi'_x(U'V + UV') + \xi'''_{yxx}U' \\ \qquad +(\xi'_y\xi''_{xx} + 2\xi'_x\xi''_{yx})U'' + \xi'_y(\xi'_x)^2U''' = 0. \end{cases} \quad (25)$$

Suppose that the solutions of Eqs. (25) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows:

$$\begin{cases} U(\xi) = \sum_{i=0}^{m} a_i(y,t)(\frac{\phi'}{\phi})^i, \\ V(\xi) = \sum_{i=0}^{n} b_i(y,t)(\frac{\phi'}{\phi})^i. \end{cases} \quad (26)$$

By balancing the highest order derivatives and nonlinear terms in Eqs. (25) we obtain $m = 1$, $n = 2$. So

$$\begin{cases} U(\xi) = a_1(y,t)(\frac{\phi'}{\phi}) + a_0(y,t), \\ V(\xi) = b_2(y,t)(\frac{\phi'}{\phi})^2 + b_1(y,t)(\frac{\phi'}{\phi}) + b_0(y,t). \end{cases} \quad (27)$$

Similarly, we will proceed the computation in two cases.

Case 1: If $\phi = \phi(\xi)$ satisfies the Riccati equation Eq. (10), then substituting (27) into (25), using Eq. (10) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_0(y,t), a_1(y,t), b_0(y,t), b_1(y,t), b_2(y,t)$ and $\xi(x,y,t)$. Solving these equations, yields

$$\begin{aligned} &\xi(x,y,t) = C_1x + \int \frac{F_1(y)}{4C_1\sigma}dy + \frac{y}{4C_1\sigma} + F_2(t), \\ &a_1(y,t) = \pm 2C_1, \\ &a_0(y,t) = -\frac{F_2'(t)}{C_1}, \\ &b_2(y,t) = -\frac{F_1(y)+1}{2\sigma}, \\ &b_1(y,t) = 0, \\ &b_0(y,t) = F_1(y), \end{aligned}$$

where C_1 is an arbitrary nonzero constant, and $F_1(y)$, $F_2(t)$ are two arbitrary functions with respect to the variable y and t respectively.

By the general solutions of Eq. (10) denoted in (11)-(14) we can obtain the following exact solutions for Eqs. (24).

When $\sigma < 0$:

$$\begin{cases} u_1(x,y,t) = \pm 2C_1\sqrt{-\sigma}\{\frac{sech^2[\sqrt{-\sigma}\xi+c_0]}{\}} \\ \qquad -\frac{F_2'(t)}{C_1}, \\ v_1(x,y,t) = \frac{F_1(y)+1}{2}\{\frac{sech^2[\sqrt{-\sigma}\xi+c_0]}{\}}^2 \\ \qquad +F_1(y). \end{cases} \quad (28)$$

$$\begin{cases} u_2(x,y,t) = \mp 2C_1\sqrt{-\sigma}\{\frac{csch^2[\sqrt{-\sigma}\xi+c_0]}{\coth[\sqrt{-\sigma}\xi+c_0]}\} \\ \qquad -\frac{F_2'(t)}{C_1}, \\ v_2(x,y,t) = \frac{F_1(y)+1}{2}\{\frac{csch^2[\sqrt{-\sigma}\xi+c_0]}{\coth[\sqrt{-\sigma}\xi+c_0]}\}^2 \\ \qquad +F_1(y). \end{cases} \quad (29)$$

When $\sigma > 0$:

$$\begin{cases} u_3(x,y,t) = \pm 2C_1\sqrt{\sigma}\{\frac{sec^2[\sqrt{\sigma}\xi+c_0]}{\tan[\sqrt{\sigma}\xi+c_0]}\} \\ \qquad -\frac{F_2'(t)}{C_1}, \\ v_3(x,y,t) = -\frac{F_1(y)+1}{2}\{\frac{\sec^2[\sqrt{\sigma}\xi+c_0]}{\tan[\sqrt{\sigma}\xi+c_0]}\}^2 \\ \qquad +F_1(y). \end{cases} \quad (30)$$

$$\begin{cases} u_4(x,y,t) = \mp 2C_1\sqrt{\sigma}\{\frac{csc^2[\sqrt{\sigma}\xi+c_0]}{\cot[\sqrt{\sigma}\xi+c_0]}\} \\ \qquad -\frac{F_2'(t)}{C_1}, \\ v_4(x,y,t) = -\frac{F_1(y)+1}{2}\{\frac{csc^2[\sqrt{\sigma}\xi+c_0]}{\cot[\sqrt{\sigma}\xi+c_0]}\}^2 \\ \qquad +F_1(y). \end{cases} \quad (31)$$

$$\begin{cases} u_{5,6}(x,y,t) = \pm 4C_1\sqrt{\sigma} \\ \{\frac{sec^2[2\sqrt{\sigma}\xi+c_0]\pm sec(2\sqrt{\sigma}\xi+c_0)tan(2\sqrt{\sigma}\xi+c_0)}{tan(2\sqrt{\sigma}\xi+c_0)\pm sec(2\sqrt{\sigma}\xi+c_0)}\} \\ \qquad -\frac{F_2'(t)}{C_1}, \\ v_{5,6}(x,y,t) = -2(F_1(y)+1) \\ \{\frac{\sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)\tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}^2 \\ \qquad +F_1(y), \end{cases} \quad (32)$$

where $\xi = C_1x + \int \frac{F_1(y)}{4C_1\sigma}dy + \frac{y}{4C_1\sigma} + F_2(t)$.

When $\sigma = 0$:

$$\begin{cases} u_7(x,y,t) = \frac{\mp 2C_1}{(\xi+c_0)} - \frac{F_2'(t)}{C_1}, \\ v_7(x,y,t) = \frac{-(F_1(y)+1)}{2\sigma(\xi+c_0)^2} + F_1(y). \end{cases} \quad (33)$$

Case 2: If $\phi = \phi(\xi)$ satisfies Eq. (21), then substituting (27) into (25), using Eq. (21) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_0(y,t), a_1(y,t), b_0(y,t), b_1(y,t), b_2(y,t)$ and $\xi(x,y,t)$. Solving these equations, yields

$$\begin{aligned} &\xi(x,y,t) = \frac{C_1x}{4} - \int \frac{C_1F_1(t)}{4}dt + \frac{\lambda C_1^2}{8}t \\ &\quad -\frac{C_1t}{4}F_2(y) + F_3(y), \\ &a_1(y,t) = C_1, \\ &a_0(y,t) = F_1(t) + F_2(y), \\ &b_2(y,t) = \frac{C_1}{2}tF_2'(y) - 2F_3'(y), \\ &b_1(y,t) = \frac{C_1^2\lambda}{2}tF_2'(y) - 2C_1\lambda F_3'(y), \\ &b_0(y,t) = -F_2'(y) - 1, \end{aligned}$$

where C_1 is an arbitrary nonzero constant, and $F_1(t),\ F_2(y),\ F_3(y)$ are arbitrary functions.

Substituting the results above into Eqs. (27), and combining with (23), we can obtain the following exact solutions:

$$\begin{cases} u_8(x,y,t) = F_1(t) + F_2(y) + C_1(-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}), \\ v_8(x,y,t) = -F_2'(y) - 1 \\ \qquad +[\frac{C_1^2\lambda}{2}tF_2'(y) - 2C_1\lambda F_3'(y)](-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}}) \\ \qquad +[\frac{C_1}{2}tF_2'(y) - 2F_3'(y)](-\frac{A\lambda e^{2\lambda\xi}}{\frac{\mu}{\lambda}+Ae^{2\lambda\xi}})^2. \end{cases} \quad (34)$$

Especially, If we set $\mu = \lambda A$ in Eq. (34), then we obtain the following solitary wave solutions:

$$\begin{cases} u_9(x,y,t) = F_1(t) + F_2(y) - \frac{C_1\lambda}{2}[(1+\tanh(\lambda\xi))], \\ v_9(x,y,t) = -F_2'(y) - 1 \\ \quad -[\frac{C_1^2\lambda^2}{4}tF_2'(y) - C_1\lambda^2F_3'(y)][(1+\tanh(\lambda\xi))] \\ \quad +[\frac{C_1\lambda^2}{8}tF_2'(y) - \frac{\lambda^2}{2}F_3'(y)][(1+\tanh(\lambda\xi))]^2. \end{cases}$$

Remark 4 *The established solutions in Eqs. (28)-(34) for the (2+1)-dimensional dispersive long wave equations can not be obtained by the methods in [27-40]. As involving arbitrary functions as coefficients, they are new exact solutions to our best knowledge.*

3.3 (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations

We consider the known (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations [41]:

$$\begin{cases} u_y = q_x, \\ v_x = q_y, \\ q_t = q_{xxx} + q_{yyy} + 6(qu)_x + 6(qv)_y. \end{cases} \quad (35)$$

Assume that $u(x,y,t) = U(\xi),\ v(x,y,t) = V(\xi),\ q(x,y,t) = Q(\xi),\ \xi = \xi(x,y,t)$. Then Eqs.

(35) can be turned into

$$\begin{cases} \xi'_y U' - \xi'_x Q' = 0 \\ \xi'_x V' - \xi'_y Q' = 0 \\ \xi'_t Q' - \xi'''_{xxx} Q' - 3\xi'_x \xi''_{xx} Q'' - (\xi'_x)^3 Q''' - \xi'''_{yyy} Q' \\ -3\xi'_y \xi''_{yy} Q'' - (\xi'_y)^3 Q''' - 6\xi'_x (QU)' - 6\xi'_y (QV)' = 0. \end{cases} \quad (36)$$

Suppose that the solutions of Eqs. (36) can be expressed by a polynomial in $(\frac{\phi'}{\phi})$ as follows: and suppose

$$\begin{cases} U(\xi) = \sum_{i=0}^{m} a_i(x,t)(\frac{\phi'}{\phi})^i, \\ V(\xi) = \sum_{i=0}^{n} b_i(x,t)(\frac{\phi'}{\phi})^i \\ Q(\xi) = \sum_{i=0}^{p} b_i(x,t)(\frac{\phi'}{\phi})^i. \end{cases} \quad (37)$$

By balancing U' and Q', V' and Q', Q''' and $(QU)'$ in Eqs. (36) we obtain $m = n = p = 2$. So

$$\begin{cases} U(\xi) = a_2(x,t)(\frac{\phi'}{\phi})^2 + a_1(x,t)(\frac{\phi'}{\phi}) + a_0(x,t), \\ V(\xi) = b_2(x,t)(\frac{\phi'}{\phi})^2 + b_1(x,t)(\frac{\phi'}{\phi}) + b_0(x,t) \\ Q(\xi) = c_2(x,t)(\frac{\phi'}{\phi})^2 + c_1(x,t)(\frac{\phi'}{\phi}) + c_0(x,t). \end{cases} \quad (38)$$

For simplicity, we only assume that $\phi = \phi(\xi)$ satisfies the Riccati equation Eq.(10). Substituting (38) into (36), using Eq.(10) and collecting all the terms with the same power of ϕ together, equating each coefficient to zero, yields a set of under-determined partial differential equations for $a_i(x,t), b_i(x,t), c_i(x,t),\ i = 0, 1, 2$ and $\xi(x,y,t)$. Solving these equations, yields

$$\begin{aligned} &\xi(x,y,t) = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t), \\ &a_2(x,t) = \frac{F_1^2(t)}{C_1}, \\ &a_1(x,t) = 0, \\ &a_0(x,t) = \frac{C_! x F_1'(t) + (-C_1)^{\frac{3}{2}} F_2'(t)}{6C_1 F_1(t)} \\ &\qquad - \frac{8C_1^3\sigma + 8\sigma F_1^3(t) + 3C_1^2 F_3(t)}{3C_1 F_1(t)}, \\ &b_2(x,t) = C_1, \\ &b_1(x,t) = 0, \\ &b_0(x,t) = F_3(t), \\ &c_2(x,t) = F_1(t), \\ &c_1(x,t) = 0, \\ &c_0(x,t) = 0, \end{aligned}$$

where $C_1,\ C_2$ are arbitrary constants with $C_1 < 0$, and $F_1(t),\ F_2(t),\ F_3(t)$ are arbitrary functions with respect to the variable t.

By the general solutions of Eq. (10) denoted in (11)-(14) we can obtain the following exact solutions for Eqs. (35).

When $\sigma < 0$:

$$\begin{cases} u_1(x,y,t) = -\frac{\sigma F_1^2(t)}{C_1}\{\frac{sech^2[\sqrt{-\sigma}\xi + c_0]}{\tanh[\sqrt{-\sigma}\xi + c_0]}\}^2 \\ \qquad + \frac{C_! x F_1'(t) + (-C_1)^{\frac{3}{2}} F_2'(t)}{6C_1 F_1(t)} \\ \qquad - \frac{8C_1^3\sigma + 8\sigma F_1^3(t) + 3C_1^2 F_3(t)}{3C_1 F_1(t)}, \\ v_1(x,y,t) = -C_1\sigma\{\frac{sech^2[\sqrt{-\sigma}\xi + c_0]}{\tanh[\sqrt{-\sigma}\xi + c_0]}\}^2 + F_3(t) \\ q_1(x,y,t) = -\sigma F_1(t)\{\frac{sech^2[\sqrt{-\sigma}\xi + c_0]}{\tanh[\sqrt{-\sigma}\xi + c_0]}\}^2, \end{cases} \quad (39)$$

where $\xi = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t)$.

And

$$\begin{cases} u_2(x,y,t) = -\frac{\sigma F_1^2(t)}{C_1}\{\frac{csch^2[\sqrt{-\sigma}\xi + c_0]}{\coth[\sqrt{-\sigma}\xi + c_0]}\}^2 \\ \qquad + \frac{C_! x F_1'(t) + (-C_1)^{\frac{3}{2}} F_2'(t)}{6C_1 F_1(t)} \\ \qquad - \frac{8C_1^3\sigma + 8\sigma F_1^3(t) + 3C_1^2 F_3(t)}{3C_1 F_1(t)}, \\ v_2(x,y,t) = -C_1\sigma\{\frac{csch^2[\sqrt{-\sigma}\xi + c_0]}{\coth[\sqrt{-\sigma}\xi + c_0]}\}^2 + F_3(t) \\ q_2(x,y,t) = -\sigma F_1(t)\{\frac{csch^2[\sqrt{-\sigma}\xi + c_0]}{\coth[\sqrt{-\sigma}\xi + c_0]}\}^2, \end{cases} \quad (40)$$

where $\xi = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t)$.

When $\sigma > 0$:

$$\begin{cases} u_3(x,y,t) = \frac{\sigma F_1^2(t)}{C_1}\{\frac{\sec^2[\sqrt{\sigma}\xi + c_0]}{\tan[\sqrt{\sigma}\xi + c_0]}\}^2 \\ \qquad + \frac{C_! x F_1'(t) + (-C_1)^{\frac{3}{2}} F_2'(t)}{6C_1 F_1(t)} \\ \qquad - \frac{8C_1^3\sigma + 8\sigma F_1^3(t) + 3C_1^2 F_3(t)}{3C_1 F_1(t)}, \\ v_3(x,y,t) = C_1\sigma\{\frac{\sec^2[\sqrt{\sigma}\xi + c_0]}{\tan[\sqrt{\sigma}\xi + c_0]}\}^2 + F_3(t) \\ q_3(x,y,t) = \sigma F_1(t)\{\frac{\sec^2[\sqrt{\sigma}\xi + c_0]}{\tan[\sqrt{\sigma}\xi + c_0]}\}^2, \end{cases} \quad (41)$$

where $\xi = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t)$.

$$\begin{cases} u_4(x,y,t) = \frac{\sigma F_1^2(t)}{C_1}\{\frac{\csc^2[\sqrt{\sigma}\xi + c_0]}{cot[\sqrt{\sigma}\xi + c_0]}\}^2 \\ \qquad + \frac{C_! x F_1'(t) + (-C_1)^{\frac{3}{2}} F_2'(t)}{6C_1 F_1(t)} \\ \qquad - \frac{8C_1^3\sigma + 8\sigma F_1^3(t) + 3C_1^2 F_3(t)}{3C_1 F_1(t)}, \\ v_4(x,y,t) = C_1\sigma\{\frac{\csc^2[\sqrt{\sigma}\xi + c_0]}{\cot[\sqrt{\sigma}\xi + c_0]}\}^2 + F_3(t) \\ q_4(x,y,t) = \sigma F_1(t)\{\frac{\csc^2[\sqrt{\sigma}\xi + c_0]}{\cot[\sqrt{\sigma}\xi + c_0]}\}^2, \end{cases} \quad (42)$$

where $\xi = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t)$.

And

$$\begin{cases} u_{5,6}(x,y,t) = \frac{4\sigma F_1^2(t)}{C_1} \\ \{\frac{\sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}^2 \\ \qquad +\frac{C_!xF_1'(t)+(-C_1)^{\frac{3}{2}}F_2'(t)}{6C_1F_1(t)} \\ \qquad -\frac{8C_1^3\sigma+8\sigma F_1^3(t)+3C_1^2F_3(t)}{3C_1F_1(t)}, \\ v_{5,6}(x,y,t) = 4C_1\sigma \\ \{\frac{\sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}^2 \\ \qquad +F_3(t) \\ q_{5,6}(x,y,t) = 4\sigma F_1(t) \\ \{\frac{\sec^2[2\sqrt{\sigma}\xi+c_0]\pm\sec(2\sqrt{\sigma}\xi+c_0)tan(2\sqrt{\sigma}\xi+c_0)}{\tan(2\sqrt{\sigma}\xi+c_0)\pm\sec(2\sqrt{\sigma}\xi+c_0)}\}^2, \end{cases} \tag{43}$$

where $\xi = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t)$.

When $\sigma = 0$:

$$\begin{cases} u_7(x,y,t) = \frac{F_1^2(t)}{C_1(\xi+c_0)}^2 \\ \qquad +\frac{C_!xF_1'(t)+(-C_1)^{\frac{3}{2}}F_2'(t)}{6C_1F_1(t)} \\ \qquad -\frac{8C_1^3\sigma+8\sigma F_1^3(t)+3C_1^2F_3(t)}{3C_1F_1(t)}, \\ v_7(x,y,t) = \frac{C_1}{(\xi+c_0)^2} + F_3(t) \\ q_7(x,y,t) = \frac{F_1(t)}{(\xi+c_0)^2}, \end{cases} \tag{44}$$

where $\xi = \sqrt{-C_1}y - \frac{F_1(t)x}{\sqrt{-C_1}} + F_2(t)$.

Remark 5 *The established solutions in Eqs. (39)-(44) for the (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations are new exact solutions so far in the literature.*

4 Conclusions

We have proposed a variable-coefficient simplest equation method for solving nonlinear evolution equations, and applied it to find exact solutions of the asymmetric (2+1)-dimensional NNV system, the (2+1)-dimensional dispersive long wave equations and the (2+1)-dimensional Boussinesq and Kadomtsev-Petviashvili equations. As a result, some new exact solutions and solitary wave solutions involving arbitrary function as coefficients for them have been obtained. These solutions may provide some references for the research in related physical phenomena. Finally, we note the proposed method in this paper can also be applied to other nonlinear evolution equations.

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