Fuzzy Measures of a System's Effectiveness - An Application to Problem Solving [1]

MICHAEL GR. VOSKOGLOU
School of Technological Applications
Graduate Technological Educational Institute (T. E. I.)
Meg. Alexandrou 1, 263 34 Patras
GREECE
voskoglou@teipat.gr , http://eclass.teipat.gr/eclass/courses/523102

Abstract: They appear often situations in a system's operation characterized by a degree of vagueness and/or uncertainty. In the present paper we use principles of fuzzy logic to develop a general model representing such kind of situations. We also present 3 alternative methods for measuring a fuzzy system's effectiveness. These methods include the measurement of the system's total possibilistic uncertainty, the Shannon's entropy properly modified for use in a fuzzy environment and the "centroid" method in which the coordinates of the center of mass of the graph of the membership function involved provide an alternative measure of the system's performance An application of the above results is also developed concerning the Problem Solving process and two classroom experiments are presented illustrating the use of our results in practice.

Key Words: Systems Theory, Fuzzy Sets and Logic, Possibility, Uncertainty Theory, Problem solving.

1. Introduction: Systems' Modelling and Fuzzy Logic

A *system* is a set of interacting or interdependent components forming an integrated whole. A system comprises multiple views such as planning, analysis, design, implementation, deployment, structure, behavior, input and output data, etc. As an interdisciplinary and multi- perspective domain systems' theory brings together principles and concepts from ontology, philosophy of science, information and computer science, mathematics, as well as physics, biology, engineering, social and cognitive sciences, management and economics, strategic thinking, fuzziness and uncertainty, etc. Thus, it serves as a bridge for an interdisciplinary dialogue between autonomous areas of study. The emphasis with systems' theory shifts from parts to the organization of parts, recognizing that interactions of the parts are not static and constant, but dynamic processes.

Most systems share common characteristics including structure, behavior, interconnectivity (the various parts of a system have functional and structural relations to each other), sets of functions, etc. We scope a system by defining its boundary; this means choosing which entities are inside the system and which are outside, part of the environment.

The *systems' modelling* is a basic principle in engineering, in natural and in social sciences. When we face a problem concerning a system's operation (e.g. maximizing the productivity of an organization, minimizing the functional costs of a company, etc) a model is required to describe and represent the system's multiple views. The model is a simplified representation of the basic characteristics of the real system including only its entities and features under concern. In this sense, no model of a complex system could include all features and/or all entities belonging to the system. In fact, in this way the model's structure could become very complicated and therefore its use in practice could be very difficult and sometimes impossible. Therefore the construction of the model usually involves a deep abstracting process on identifying the system's dominant variables and the relationships governing them. The resulting structure of this action is known as the *assumed real system* (see Figure 1). The model, being an abstraction of the assumed real system, identifies and simplifies the relationships among these variables in a form amenable to analysis.

[1] The main body of this paper was presented in the form of a plenary lecture in the 16th WSEAS International Conference on Systems, Kos Island, Greece, July 14-16, 2012

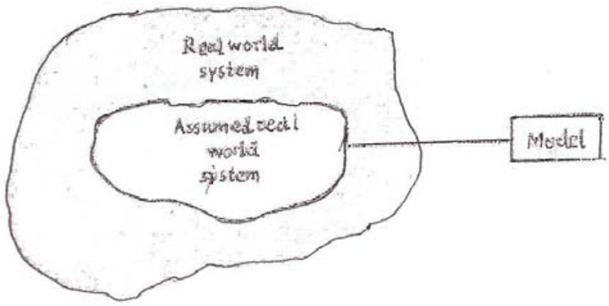

Figure 1: A graphical representation of the modelling process

A system can be viewed as a bounded transformation, i.e. as a process or a collection of processes that transforms inputs into outputs with the very broad meaning of the concept. For example, an output of a passengers' bus is the movement of people from departure to destination.

Many of these processes are frequently characterized by a degree of vagueness and/or uncertainty. For example, during the processes of learning, of reasoning, of problem-solving, of modelling, etc, the human cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teacher's point of view there usually exists an uncertainty about the degree of students' success in each of the stages of the corresponding didactic situation.

There used to be a tradition in science and engineering of turning to probability theory when one is faced with a problem in which uncertainty plays a significant role. This transition was justified when there were no alternative tools for dealing with the uncertainty. Today this is no longer the case. *Fuzzy logic,* which is based on fuzzy sets theory introduced by Zadeh [32] in 1965, provides a rich and meaningful addition to standard logic. The applications which may be generated from or adapted to fuzzy logic are wide-ranging and provide the opportunity for modelling under conditions which are inherently imprecisely defined, despite the concerns of classical logicians. Many systems may be modelled, simulated and even replicated with the help of fuzzy logic, not the least of which is human reasoning itself (e.g. [4], [8], [13], [21], [28], [31], etc) A real test of the effectiveness of an approach to uncertainty is the capability to solve problems, which involves different facets of uncertainty. Fuzzy logic has a much higher problem solving capability than standard probability theory. Most importantly, it opens the door to construction of mathematical solutions of computational problems which are stated in a natural language. In contrast, standard probability theory does not have this capability, a fact which is one of its principal limitations.

Another advantage of the fuzzy logic is that, apart of the quantitative information, it gives also the opportunity of a qualitative study by searching the behaviour of all possible profiles of the system's entities involved in the corresponding process.

All these gave us the impulsion to introduce principles of fuzzy logic to describe in a more effective way a system's operation in situations characterized by a degree of vagueness and/or uncertainty.

For general facts on fuzzy sets and logic and on uncertainty theory we refer freely to the book of Klir and Folger [5].

2. The general fuzzy model

Assume that we want to study the behavior of a system's n entities (objects), $n \geq 2$, during a process involving vagueness and/or uncertainty. Denote by S_i, $i=1,2,3$ the main stages of this process and by a, b, c, d, and e the linguistic labels of very low, low, intermediate, high and very high success respectively of a system's entity in each of the S_i's. Set

$$U = \{a, b, c, d, e\}.$$

We are going to attach to each stage S_i a fuzzy subset, A_i of U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of entities that faced very low, low, intermediate, high and very high success at stage S_i respectively, $i=1,2,3$, we define the *membership function* m_{A_i} for each x in U, as follows:

$$m_{A_i}(x)=\begin{cases}1, & if \quad \frac{4n}{5}<n_{ix}\le n\\ 0,75, & if \quad \frac{3n}{5}<n_{ix}\le \frac{4n}{5}\\ 0,5, & if \quad \frac{2n}{5}<n_{ix}\le \frac{3n}{5}\\ 0,25, & if \quad \frac{n}{5}<n_{ix}\le \frac{2n}{5}\\ 0, & if \quad 0\le n_{ix}\le \frac{n}{5}\end{cases}$$

Then the fuzzy subset A_i of U corresponding to S_i has the form:

$$A_i = \{(x, m_{Ai}(x)): \ x\in U\}, i=1, 2, 3.$$

In order to represent all possible *profiles (overall states)* of the system's entities during the corresponding process we consider a *fuzzy relation*, say R, in U^3 of the form:

$$R= \{(s, m_R(s)): s=(x, y, z) \in U^3\}.$$

We assume that the stages of the process that we study are depended to each other. This means that the degree of system's success in a certain stage depends upon the degree of its success in the previous stages, as it usually happens in practice. Under this hypothesis and in order to determine properly the membership function m_R we give the following definition:

Definition: *A profile $s=(x, y, z)$, with x, y, z in U, is said to be well ordered if x corresponds to a degree of success equal or greater than y and y corresponds to a degree of success equal or greater than z.*
For example, (c, c, a) is a well ordered profile, while (b, a, c) is not.
We define now the membership degree of a profile s to be

$$m_R(s) = m_{A_1}(x)m_{A_2}(y)m_{A_3}(z)$$

if s is well ordered, and 0 otherwise.

In fact, if for example the profile *(b, a, c)* possessed a nonzero membership degree, how it could be possible for an object that has failed during the middle stage, to perform satisfactorily at the next stage?
Next, for reasons of brevity, we shall write m_s instead of $m_R(s)$.
Then the *probability* p_s of the profile s is defined in a way analogous to crisp data, i.e. by

$$P_s = \frac{m_s}{\sum_{s\in U^3} m_s} \quad .$$

We define also the *possibility* r_s of s by

$$r_s=\frac{m_s}{\max\{m_s\}},$$

where $max\{m_s\}$ denotes the maximal value of m_s, for all s in U^3. In other words the possibility of s expresses the "relative membership degree" of s with respect to $max\{m_s\}$.

Assume further that one wants to study the *combined results* of behaviour of k different groups of a system's entities, $k\geq 2$, during the same process. For this we introduce the *fuzzy variables* $A_1(t)$, $A_2(t)$ and $A_3(t)$ with $t=1, 2,\ldots, k$. The values of these variables represent fuzzy subsets of U corresponding to the stages of the process for each of the k groups; e.g. $A_1(2)$ represents the fuzzy subset of U corresponding to the first stage of the process for the second group ($t=2$). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the probability $p(s)$ and the possibility $r(s)$ of each profile s with respect to the membership degrees of s for all groups. For this reason we introduce the *pseudo-frequencies*

$$f(s) = \sum_{t=1}^{k} m_s(t)$$

and we define the probability of a profile s by

$$p(s) = \frac{f(s)}{\sum_{s\in U^3} f(s)}.$$

We also define the possibility of s by

$$r(s) = \frac{f(s)}{\max\{f(s)\}},$$

where $max\{f(s)\}$ denotes the maximal pseudo-frequency.
Obviously the same method could be applied when one wants to study the combined results of behaviour of a group during k different situations.

3. Fuzzy measures of a system's effectiveness

There are *natural* and *human-designed* systems. Natural systems may not have an apparent objective, but their outputs can be interpreted as purposes. On the contrary, human-designed systems are made with purposes that are achieved by the delivery of outputs. Their parts must be related, i.e. they must be designed to work as a coherent entity.

The most important part of a human-designed system's study is probably the assessment, through the model representing it, of its performance. In fact, this could help the system's designer to make all the necessary modifications/improvements to the system's structure in order to increase its effectiveness.

In this paper we'll present three fuzzy measures of a system's effectiveness connected to the general fuzzy model developed above. The advantages and disadvantages of these measures will be also discussed and an application for the problem solving process will be presented illustrating our results.

The amount of information obtained by an action can be measured by the reduction of uncertainty resulting from this action. Accordingly a system's uncertainty is connected to its capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of a system's effectiveness in solving related problems.

Within the domain of possibility theory uncertainty consists of *strife (or discord)*, which expresses conflicts among the various sets of alternatives, and *non-specificity (or imprecision)*, which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives ([6]; p.28).

Strife is measured by the function $ST(r)$ on the ordered possibility distribution

$$r: \; r_1=1 \geq r_2 \geq \ldots\ldots \geq r_n \geq r_{n+1}$$

of a group of a system's entities defined by

$$ST(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\right],$$

while non-specificity is measured by the function

$$N(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log i\right].$$

The sum $T(r) = ST(r) + N(r)$ is a measure of the *total possibilistic uncertainty* for ordered possibility distributions. The lower is the value of $T(r)$, which means greater reduction of the initially existing uncertainty, the better the system's performance.

Another fuzzy measure for assessing a system's performance is the well known from classical probability and information theory *Shannon's entropy* [15]. For use in a fuzzy environment, this measure is expressed in terms of the Dempster-Shafer mathematical theory of evidence in the form:

$$H = -\frac{1}{\ln n}\sum_{s=1}^{n} m_s \ln m_s$$

([6], p. 20).

In the above formula n denotes the total number of the system's entities involved in the corresponding process. The sum is divided by ln n (the natural logarithm of n) in order to be normalized. Thus H takes values in the real interval [0, 1]. The value of H measures the system's total *probabilistic uncertainty* and the associated to it information. Similarly with the total possibilistic uncertainty, the lower is the final value of H, the better the system's performance. An advantage of adopting H as a measure instead of $T(r)$ is that H is calculated directly from the membership degrees of all profiles s without being necessary to calculate their probabilities p_s. In contrast, the calculation of $T(r)$ presupposes the calculation of the possibilities r_s of all profiles first. However, according to Shackle [14] human reasoning can be formalized more adequately by possibility rather, than by probability theory. But, as we have seen in the previous section, the possibility is a kind of "relative probability". In other words, the "philosophy" of possibility is not exactly the same with that of probability theory. Therefore, on comparing the effectiveness of two or more systems by these two measures, one may find non compatible results in boundary cases, where the systems' performances are almost the same.

Another popular approach is the *"centroid" method*, in which the centre of mass of the graph of the membership function involved provides an alternative measure of the system's performance.

For this, given a fuzzy subset

$$A = \{(x, m(x)): x \in U\}$$

of the universal set U with membership function $m: U \rightarrow [0, 1]$, we correspond to each $x \in U$ an interval of values from a prefixed numerical distribution, which actually means that we replace U with a set of real intervals. Then, we construct the graph F of the membership function $y=m(x)$.

There is a commonly used in fuzzy logic approach to measure performance with the pair of numbers (x_c, y_c) as the coordinates of the *centre of mass*, say F_c, of the graph F, which we can calculate using the following well-known [22] formulas:

$$x_c = \frac{\iint_F x\,dx\,dy}{\iint_F dx\,dy}, \quad y_c = \frac{\iint_F y\,dx\,dy}{\iint_F dx\,dy} \qquad (1)$$

For example, assume that the set U of the linguistic labels of section 2 characterizes the performance of a group of students. When a student obtains a mark, say y, then his/her performance is characterized as very low (a) if $y \in [0, 1)$, as low (b) if $y \in [1, 2)$, as intermediate (c) if $y \in [2, 3)$, as high (d) if $y \in [3, 4)$ and as very high (e) if $y \in [4,5]$ respectively. In this case the graph F of the corresponding fuzzy subset of U is the bar graph of Figure 2

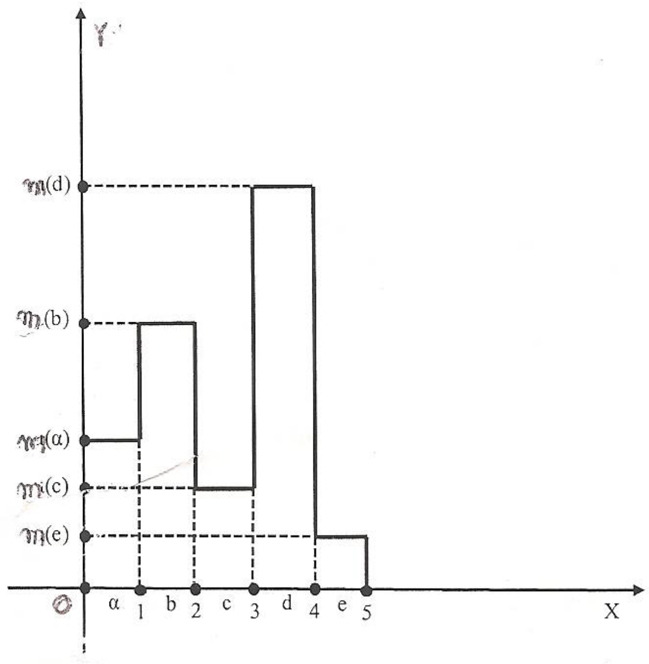

Figure 2: Bar graphical data representation

It is easy to check that, if the bar graph consists of *n* rectangles (in Figure 2 we have n=5), the formulas (1) can be reduced to the following formulas:

$$x_c = \frac{1}{2}\left(\frac{\sum_{i=1}^{n}(2i-1)y_i}{\sum_{i=1}^{n} y_i}\right), y_c = \frac{1}{2}\left(\frac{\sum_{i=1}^{n} y_i^{\,2}}{\sum_{i=1}^{n} y_i}\right) \tag{2}$$

Indeed, in this case $\iint_F dxdy$ is the total mass of the system which is equal to $\sum_{i=1}^{n} y_i$, $\iint_F xdxdy$ is the moment about the y-axis which is equal to

$$\sum_{i=1}^{n}\iint_{F_i} xdxdy = \sum_{i=1}^{n}\int_0^{y_i} dy \int_{i-1}^{i} xdx$$

$$= \sum_{i=1}^{n} y_i \int_{i-1}^{i} xdx = \frac{1}{2}\sum_{i=1}^{n}(2i-1)y_i ,$$

and $\iint_F ydxdy$ is the moment about the y-axis which is equal to

$$\sum_{i=1}^{n}\iint_{F_i} ydxdy = \sum_{i=1}^{n}\int_0^{y_i} ydy \int_{i-1}^{i} dx =$$

$$\sum_{i=1}^{n}\iint_{F_i} xdxdy = \sum_{i=1}^{n}\int_0^{y_i} dy \int_{i-1}^{i} xdx = \sum_{i=1}^{n}\int_0^{y_i} ydy = \frac{1}{2}\sum_{i=1}^{n} y_i^{\,2} .$$

From the above argument, where F_i, i=1,2,…,n , denote the n rectangles of the bar graph, it becomes evident that the transition from (1) to (2) is obtained under the assumption that all the intervals have length equal to 1 and that the first of them is the interval [0, 1].

In our case (n=5) formulas (2) are transformed into the following form:

$$x_c = \frac{1}{2}\left(\frac{y_1+3y_2+5y_3+7y_4+9y_5}{y_1+y_2+y_3+y_4+y_5}\right),$$

$$y_c = \frac{1}{2}\left(\frac{y_1^{\,2}+y_2^{\,2}+y_3^{\,2}+y_4^{\,2}+y_5^{\,2}}{y_1+y_2+y_3+y_4+y_5}\right).$$

Normalizing our fuzzy data by dividing each m(x), $x \in U$, with the sum of all membership degrees we can assume without loss of the generality that

$$y_1+y_2+y_3+y_4+y_5 = 1.$$

Therefore we can write:

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right),$$

$$y_c = \frac{1}{2}\left(y_1^{\,2} + y_2^{\,2} + y_3^{\,2} + y_4^{\,2} + y_5^{\,2}\right) \qquad (3)$$

with $y_i = \dfrac{m(x_i)}{\sum\limits_{x \in U} m(x)}$, where $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$ and $x_5 = e$.

But

$$0 \le (y_1 - y_2)^2 = y_1^2 + y_2^2 - 2y_1y_2,$$

therefore

$$y_1^2 + y_2^2 \ge 2y_1y_2$$

with the equality holding if, and only if, $y_1 = y_2$.
In the same way one finds that

$$y_1^2 + y_3^2 \ge 2y_1y_3,$$

and so on. Hence it is easy to check that

$$(y_1+y_2+y_3+y_4+y_5)^2 \le 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2),$$

with the equality holding if, and only if $y_1=y_2=y_3=y_4=y_5$.

But $y_1+y_2+y_3+y_4+y_5 = 1$, therefore

$$1 \le 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2) \qquad (4),$$

with the equality holding if, and only if $y_1=y_2=y_3=y_4=y_5=\frac{1}{5}$.

Then the first of formulas (3) gives that $x_c = \frac{5}{2}$.

Further, combining the inequality (4) with the second of formulas (3) one finds that

$$1 \le 10y_c, \qquad \text{or} \qquad y_c \ge \frac{1}{10}.$$

Therefore the unique minimum for y_c corresponds to the centre of mass $F_m(\frac{5}{2}, \frac{1}{10})$.

The ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from formulas (3) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the centre of mass in this case is the point $F_i(\frac{9}{2}, \frac{1}{2})$.

On the other hand the worst case is when $y_1=1$ and $y_2=y_3=y_4=y_5=0$. Then for formulas (3) we find that the centre of mass is the point $F_w(\frac{1}{2}, \frac{1}{2})$.

Therefore the "area" where the centre of mass F_c lies is represented by the triangle $F_w F_m F_i$ of Figure 3.

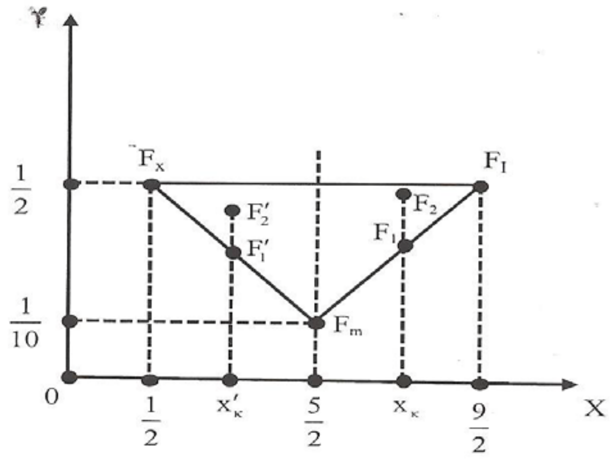

Figure 3: Graphical representation of the "area" of the centre of mass

Then from elementary geometric considerations it follows that for two groups of a system's objects with the same $x_c \ge 2{,}5$ the group having the centre of mass which is situated closer to F_i is the group with the higher y_c; and for two groups with the same $x_c < 2.5$ the group having the centre of mass which is situated farther to F_w is the group with the lower y_c.
Based on the above considerations it is logical to formulate our criterion for comparing the groups' performances in the following form:

- *Among two or more groups the group with the biggest x_c performs better.*
- *If two or more groups have the same $x_c \ge 2.5$, then the group with the higher y_c performs better.*

- *If two or more groups have the same $x_c < 2.5$, then the group with the lower y_c performs better.*

From the above description it becomes clear that the application of the "centroid" method in practice is simple and evident and needs no complicated calculations in its final step. However, we must emphasize that this method *treats differently the idea of a system's performance*, than the two measures of uncertainty presented above do. In fact, the *weighted average* plays the main role in this method, i.e. the result of the system's performance close to its ideal performance has much more weight than the one close to the lower end. In other words, while the measures of uncertainty are dealing with the *average system's performance*, the "centroid" method is mostly looking at the *quality of the performance*. Consequently, some differences could appear in evaluating a system's performance by these different approaches. Therefore, it is argued that a combined use of all these (3 in total) measures could help the user in finding the ideal profile of the system's performance according to his/her personal criteria of goals.

4. Modelling the process of Problem Solving (PS)

In earlier papers we have developed models similar to the general fuzzy model developed above for a more effective description of several situations involving fuzziness and uncertainty in the areas of Education (for the processes of Learning and of Mathematical modelling), of Artificial Intelligence (for Case-Based and Analogical Reasoning) and of Management (for the evaluation of the fuzzy data obtained by a market's research and for Decision Making); see for example [30] and its references. Notice also, that Subbotin et al., based on our fuzzy model for the process of learning [25], have applied the "centroid" method on comparing students' mathematical learning abilities [19] and for measuring the scaffolding (assistance) effectiveness provided by the teacher to students [20]. Also Perdikaris has used the total possiblistic uncertainty [10] and the Shannon's entropy [11] for assessing students' geometrical reasoning skills in terms of the corresponding van Hieles' levels.

In this paper we shall apply our general fuzzy model developed above for representing the Problem Solving (PS) process.

As the world economy moved from an industrial to a knowledge economy, it can be argued that the nature of many problems also changed and new problems have arisen which may require a different approach to overcome them. Educational institutions and governments have recognized long ago the importance of PS and volumes of research have been written about PS (see [3], [9], etc). Universities and other higher learning institutions are entrusted with the task of producing graduates that have such higher order thinking skills among other skills (e.g. see [1], etc).

Mathematics by its nature is a subject whereby PS forms its essence. According to *Schoenfeld* [17] a problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no "surprises" in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) it is an exercise.

In an earlier paper [29] we have examined the role of problem in learning mathematics and we have attempted a review of the evolution of research on PS in mathematics education from its emergency as a self sufficient science at the end of the 1960's until today. Here is a rough chronology of that progress:

1950's – 1960's: *Polya's* theories on the use of heuristic strategies in PS ([12], etc)

1970's: Emergency of mathematics education as a self – sufficient science (research methods were almost exclusively statistical). Research on PS was mainly based on Polya's ideas.

1980's: A framework describing the PS process, and reasons for success or failure in PS (e.g. [7], [16], etc.)

1990's: Models of teaching using PS, e.g. constructivist view of learning (see [27] and its relevant references), Mathematical modelling and applications (see [26] and its references), etc.

2000's: While early work on PS focused mainly on analyzing the PS process and on describing the proper heuristic strategies to be used in each of its stages, more recent investigations have focused mainly on solvers' behavior and required attributes during the PS process; e. g. [2], [18], etc.

Carlson και Bloom [2] drawing from the large amount of literature related to PS developed a broad taxonomy to characterize major PS attributes that have been identifying as relevant to PS success. This taxonomy gave genesis to their "*Multidimensional Problem-Solving Framework" (MPSF)*, which includes the following 4 phases: *Orientation*, *Planning*, *Executing* and *Checking*.

It has been observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated through out the remainder of the solution process; only in a few cases a solver obtained linearly the solution of a problem (i.e. he/she made this cycle only once). Thus embedded in the framework are two cycles (one cycling back and one cycling forward), each of which includes the three out of the four phases, that is planning, executing and checking. It has been also observed that, when contemplating various solution approaches during the planning phase of the PS process, the solvers were at times engaged in a *conjecture-imagine-evaluate (accept/reject)* sub-cycle. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected to the phase of planning (see Figure 4, taken from [2]).

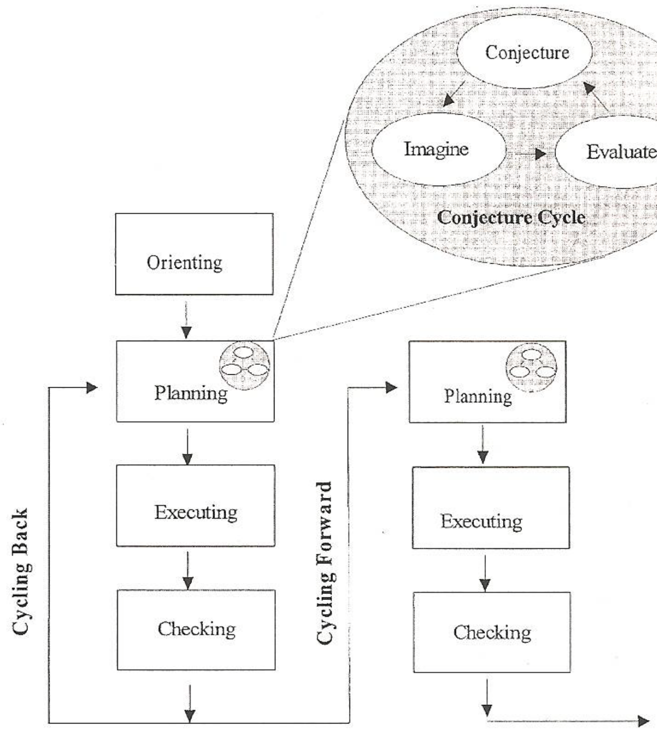

Figure 4: A graphical representation of Carlson's and Bloom's MPSF

The construction of our fuzzy model for the PS process is based on MPSF. For this, we consider a group of n solvers, n ≥ 2, working (each one individually) on the solution of the same problems. We denote by S_i , $i=1,2,3$ the phases of orientation/planning, executing and checking. The phase of orientation being a preliminary step of the PS process can be considered without loss of the generality as a sub-phase of planning.

To each of the S_i' s we attach a fuzzy subset, say A_i, of the set U of the linguistic labels considered in section 2 defining also the membership function m_{A_i} as we did in section 2. The development of the rest of our model for PS relies then upon the lines of our general fuzzy model presented in detail in sections 2 and 3.

In order to illustrate the use of our results in practice, we performed the experiments presented in the next section.

5. Applications of the model for PS

The following two experiments took place recently at the Graduate Technological Educational Institute (T.E.I.) of Patras in Greece. In the first of them our subjects were 35 students of the School of Technological Applications, i.e. future engineers, and our basic tool was a list of 10 problems (see Appendix) given to students for solution (time allowed 3 hours). Before starting the experiment we gave the proper instructions to students emphasizing among the others that we are interested for <u>all their efforts</u> (successful or not) during the PS process, and therefore they must keep records on their papers for all of them, at <u>all stages</u> of the PS process. This manipulation enabled as in obtaining realistic data from our experiment for each stage of the PS process and not only those based on students' final results that could be obtained in the usual way of graduating their papers.

Our characterizations of students' performance at <u>each stage</u> of the PS process involved:

- Negligible success, if they obtained (at the particular stage) positive results for less than 2 problems.

- Low success, if they obtained positive results for 2, 3, or 4 problems.
- Intermediate success, if they obtained positive results for 5, 6, or 7 problems.
- High success, if they obtained positive results for 8, or 9 problems.
- Complete success, if they obtained positive results for all problems.

Examining students' papers we found that 15, 12 and 8 students had intermediate, high and complete success respectively at stage of planning. Therefore we obtained that $n_{1a}=n_{1b}=0$, $n_{1c}=15$, $n_{1d}=12$ and $n_{1e}=8$. Thus, by the definition of $m_{A_i}(x)$ planning corresponds to a fuzzy subset of U of the form:

$$A_1 = \{(a,0),(b,0),(c, 0{,}5),(d, 0{,}25),(e,0,.25)\},$$

In the same way we represented the stages of executing and checking as fuzzy sets in U by

$$A_2 = \{(a,0),(b,0),(c, 0{,}5),(d, 0{,}25),(e,0)\} \text{ and}$$

$$A_3 = \{(a, 0{,}25),(b, 0{,}25),(c, 0{,}25),(d,0),(e,0)\}$$

respectively.

Next we calculated the membership degrees of the 5^3 (ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles as it is described in section 2 (column of $m_s(1)$ in Table 1). For example, for the profile $s=(c, c, a)$ one finds that $m_s=m_{A_1}(c).m_{A_2}(c).m_{A_3}(a) = 0{,}5.0{,}5.0{,}25) =0{,}06225$.

It is a straightforward process then to calculate in terms of the membership degrees the Shannon's entropy for the student group, which is $H\approx 0{,}289$.

Further, from the values of the column of $m_s(1)$ it turns out that the maximal membership degree of students' profiles is $0{,}06225$. Therefore the possibility of each s in U^3 is given by

$$r_s=\frac{m_s}{0{,}06225}.$$

Calculating the possibilities of all profiles (column of $r_s(1)$ in Table 1) one finds that the ordered possibility distribution for the student group is:

r: $r_1= r_2 = 1$, $r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0{,}5$, $r_9 = r_{10} = r_{11} = r_{12} = r_{13} = r_{14} = 0{,}258$, $r_{15} = r_{16} = \ldots\ldots \ldots\ldots= r_{125} = 0$.

Thus with the help of a calculator one finds that

$$\mathrm{ST}(r)=\frac{1}{\log 2}[\sum_{i=2}^{14}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}] \approx$$

$$\frac{1}{0{,}301}[0{,}5\log\frac{2}{2}+0{,}242\log\frac{8}{5}+0{,}258\log\frac{14}{6{,}548}] \approx$$

$$3{,}32 \,.\, 0{,}242 \,.\, 0{,}204 + 0{,}258 \,.\, 0{,}33 \approx 0.445 \text{ and}$$

$$\mathrm{N}(r)=\frac{1}{\log 2}[\sum_{i=2}^{n}(r_i - r_{i+1})\log i]$$

$$=\frac{1}{\log 2}(0{,}5\log 2+0{,}242\log 8+0{,}258\log 14)$$

$$\approx 0{,}5+3.0{,}242+0{,}857.1{,}146\approx 2{,}208 \,.$$

Therefore we finally have that $T(r)\approx 2{,}653$.

A few days later we performed the same experiment with a group of 30 students of the School of Management and Economics. Working as above we found that

$$A_1=\{(a, 0),(b, 0{,}25),(c, 0{,}5),(d, 0 ,25),(e, 0)\},$$

$$A_2=\{(a, 0{,}25),(b, 0{,}25),(c, 0{,}5),(d, 0),(e, 0)\}$$

$$A_3=\{(a, 0{,}25),(b, 0{,}25),(c,0{,}25),(d, 0),(e, 0)\}.$$

Then we calculated the membership degrees of all possible profiles of the student group (column of m_s (2) in Table 1) and the Shannon's entropy, which is $H\approx 0{,}312$.

Since the maximal membership degree is again 0,06225, the possibility of each s is given by the same formula as for the first group. Calculating the possibilities of all profiles (column of $r_s(2)$ in Table 1) one finds that the ordered possibility distribution of the second group is:

r: $r_1 = r_2 = 1$, $r_3 = r_4= r_5 = r_6 = r_7 = r_8 = 0{,}5$, $r_9 = r_{10} = r_{11}= r_{12} = r_{13} = 0{,}258$, $r_{14} = r_{15} =\ldots\ldots= r_{125} = 0$

Finally, working in the same way as above one finds that $T(r) = 0{,}432+2{,}179 = 2{,}611$.

Table 1: Profiles with non zero membership degrees
(The outcomes of the above Table were obtained with accuracy up to the third decimal point).

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	$f(s)$	$r(s)$
b	b	b	0	0	0.016	0.258	0.016	0.129
b	b	a	0	0	0.016	0.258	0.016	0.129
b	a	a	0	0	0.016	0.258	0.016	0.129
c	c	c	0.062	1	0.062	1	0.124	1
c	c	a	0.062	1	0.062	1	0.124	1
c	c	b	0	0	0.031	0.5	0.031	0.25
c	a	a	0	0	0.031	0.5	0.031	0.25
c	b	a	0	0	0.031	0.5	0.031	0.25
c	b	b	0	0	0.031	0.5	0.031	0.25
d	d	a	0.016	0.258	0	0	0.016	0.129
d	d	b	0.016	0.258	0	0	0.016	0.129
d	d	c	0.016	0.258	0	0	0.016	0.129
d	a	a	0	0	0.016	0.258	0.016	0.129
d	B	a	0	0	0.016	0.258	0.016	0.129
d	B	b	0	0	0.016	0.258	0.016	0.129
d	C	a	0.031	0.5	0.031	0.5	0.062	0.5
d	C	b	0.031	0.5	0.031	0.5	0.062	0.5
d	C	c	0.031	0.5	0.031	0.5	0.062	0.5
e	C	a	0.031	0.5	0	0	0.031	0.25
e	C	b	0.031	0.5	0	0	0.031	0.25
e	C	c	0.031	0.5	0	0	0.031	0.25
e	d	a	0.016	0.258	0	0	0.016	0.129
e	d	b	0.016	0.258	0	0	0.016	0.129
e	d	c	0.016	0.258	0	0	0.016	0.129

Therefore, since *2,611<2,653*, it turns out that the second group had in general a slightly better performance than the first one. Notice that the values of the Shannon's entropy lead to the opposite conclusion (since *0,312>0,289*), but this, as we have already explained in section 2, is not surprising in cases, where the difference between the performances of the two groups is very small. Further, using formulas (3) of section 3, one can compare the performances of the two groups by the "centroid" method in each of the listed above phases of the PS process as follows:
Denote by A_{ij} the fuzzy subset of U attached to the phase S_j , j=1,2,3 , of the PS process with respect to the student group i, i=1,2.
In the first phase of orientation/planning we have

$$A_{11} = \{(a, 0),(b, 0),(c, 0{,}5),(d, 0{,}25),(e, 0{,}25)$$

$$A_{21}= \{(a, 0),(b, 0{,}25),(c, 0{,}5),(d , 0{,}25),(e, 0)\}$$

and respectively

$$x_{c11} = \frac{1}{2}(5.0{,}5+7.0{,}25+9.0{,}25) = 3{,}25$$

$$x_{c21} = \frac{1}{2}(3.0{,}25+5.0{,}5+7.0{,}25) = 2{,}25 .$$

By our criterion the first group demonstrates better performance.
At the second stage of solution we have:

$A_{12} = \{(a, 0),(b, 0),(c, 0{,}5),(d, 0{,}25),(e, 0)\}$,

$A_{22}=\{(a, 0{,}25),(b, 0{,}25),(c, 0{,}5),(d, 0),(e, 0)\}$.

Normalizing the membership degrees in the first of the above fuzzy subsets of U (0,5 : 0,75 $\approx$ 0,67 and 0,25 : 0,75 $\approx$ 0,33) we get

$A_{12} = \{(a, 0),(b, 0),(c, 0{,}67),(d, 0{,}33),(e, 0)\}$,

$A_{22}=\{(a, 0{,}25),(b, 0{,}25),(c, 0{,}5),(d, 0),(e, 0)\}$

and respectively

$$x_{c12} = \frac{1}{2}(5.0{,}67+7.0{,}33) = 5{,}66$$

$$x_{c22} = \frac{1}{2}(0{,}25+3.0{,}25+5.0{,}25) = 3{,}25 .$$

By our criterion the first group demonstrates again a significantly better performance.
Finally, at the third phase of checking we have

$A_{13}= A_{23} =$
$\{(a, 0{,}25),(b, 0{,}25),(c, 0{,}25),(d, 0),(e, 0)\}$,

which obviously means that in this phase the performances of both groups are identical. Based on our calculations we can conclude that the first group demonstrated a significantly better performanceat the phases of orientation/planning and of executing, but performed identically with the second one at the phase of checking.

Remark: In earlier papers we have also developed a stochastic model for the representation of the PS process by applying a Markov chain on the stages of *Schoenfeld's "Expert Performance Model for PS"* ([23], [24]). There are many similarities between Carlson's and Blum's MPSF [2] and Schoenfeld's model [16]. However, their main qualitative difference is that, while in the former case emphasis is given to the solver's behavior and required attributes rather, the latter is oriented towards the PS process itself (use of the proper heuristic strategies at each stage of the process).

Our stochastic model for the PS process is self restricted to give quantitative information only through the description of the *ideal behavior* of a group of solvers (i.e. how they must act for the solution of a problem and not how they really act in practice).

6. Discussion and conclusions

The following conclusions can be drawn from the discussion performed in this paper:

- In studying a system's operation a model is required to describe and represent all its multiple views. An essential part of a human-designed system's study is the assessment, through the model, of its performance. In fact, this could help the system's designer to make all the necessary modifications/improvements to the system's structure in order to increase its effectiveness.
- In this paper we developed a general fuzzy model for representing processes in a system's operation involving vagueness and/or uncertainty. We also presented 3 methods of measuring a system's effectiveness connected to the above model. The first of them concerns the measurement of the total possibilistic uncertainty defined on the system's profiles ordered possibility distribution and being equal to the sum of strife and non specificity. The second concerns the measurement of the system's probabilistic uncertainty expressed by a modified version of the Shannon's entropy for use in a fuzzy environment. Finally, the third one is the, so called, "centroid" method, in which the coordinates of the center of mass of the graph of the membership function involved provide an alternative measure of the system's performance. Each one of the above methods adheres its own advantages and disadvantages and a combined use of them could help the user in finding the ideal profile of the system's performance according to his/her personal criteria of goals.
- In earlier papers we have applied similar fuzzy models for a more effective description of several processes in the areas of Education, of Artificial Intelligence and of Management. In the present paper we applied our general fuzzy model for the description of the PS process The construction of the fuzzy model for the PS process was based on Carlson's and Blum's "Multidimensional PS Framework" (MPSF).
- Two classroom experiments were also presented illustrating the use of our results for problem solving and showing the usefulness and applicability of our model in practice.
- In contrast to our stochastic (Markov chain) model for the PS process developed in earlier papers, which is restricted to give quantitative information only, our fuzzy model has the advantage of giving also a qualitative/realistic view of the PS process through the calculation of the probabilities and/or possibilities of all possible solvers' profiles. Nevertheless, the characterization of the problem solvers' performance in terms of a set of linguistic labels, which are fuzzy themselves, is a disadvantage of the fuzzy model, because this characterization depends on the user's personal criteria. A "live" example about this is the different evaluations for the two groups of solvers obtained by using our fuzzy measures for the PS skills in our classroom experiments presented in section 5. Therefore the stochastic could be used as a tool for the validation of the fuzzy model in the effort of achieving a worthy of credit mathematical analysis of the PS process.

Our plans for future research on the subject involve:

- The attempt to extend our measuring method on an individual basis as well (not only for groups).
- The possible extension of our fuzzy model for the description of other real life situations involving fuzziness and/or uncertainty.
- Further experimental applications of our model in order to obtain more creditable statistical data.

References

[1] Astin, A. W., *What matters in college? For critical years revisited*, Jossey-Bass Inc., San Francisco, 1993.

[2] Carlson, M. P. & Bloom, I. , The cyclic nature of problem solving: An emergent multidimensional problem solving framework, *Educational Studies in Mathematics*, 58, 45-75, 2005.

[3] Halpern, D. F., *Critical thinking across the curriculum: A brief edition of thought and knowledge*, Lawrence Erlbaum associates, London, 1997

[4] Hawas, Y., E., A Simulation-Based Fuzzy Model for Traffic Signal Control, In Munteanu, V. et al (Eds), *Recent Advances in Neural Networks, Fuzzy Systems and Evolutionary Computing* (Proceedings of the 11th WSEAS Int. Conference on Fuzzy Systems), 123-128, Iasi, Romania, 2010

[5] Klir, G. J. & Folger, T. A., *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London, 1988

[6] Klir, J. G. , Principles of Uncertainty: What are they? Why do we mean them? *Fuzzy Sets and Systems*, 74, 15-31, 1995

[7] Lester, F. K., Garofalo, J. & Krol, D. L., Self-confidence, interest, beliefs and metacognition: Key influences on problem-solving behavior, in: D. B. Mcleod & V. M. Adams (Eds), *Affect and Mathematical Problem Solving: A New Perspective*, 75-88, Springer-Verlag, New York, 1989.

[8] Mosoiu, C. et al, A Fuzzy System Approach to Symptoms in Schizoprenia, In Munteanu, V. et al (Eds), *Recent Advances in Neural Networks, Fuzzy Systems and Evolutionary Computing* (Proceedings of the 11th WSEAS Int. Conference on Fuzzy Systems), 268-272, Iasi, Romania, 2010

[9] National Council of Teachers of Mathematics (NCTM), *Principles and standards of school mathematics*, available at http://standards.nctm.org , 2010

[10] Perdikaris, S., Measuring the student group capacity for obtaining geometric information in the van Hiele development though process: A fuzzy approach, *Fuzzy Sets and Mathematics*, 16 (3), 81-86, 2002.

[11] Perdikaris, S., Using Fuzzy Sets to Determine the Continuity of the van Hiele Levels, *Journal of Mathematical Sciences & Mathematics Education*, 6(1), 39-46, 2011.

[12] Polya, G., *How to solve it*, Princeton Univ. Press, Princeton, 1945

[13] Ramazi, H. and Rassaein, M., Fuzzy Logoc Application in Compiling Geohazard Macro-Zone Maps, In Mastorakis, N. et al (Eds), *Recent Advances in Fuzzy Systems* (Proceedings of the 10th WSEAS Int. Conference on Fuzzy Systems), 47-53, Prague, Czech Republic, 2009.

[14] Shackle, G. L. S., *Decision, Order and Time in Human Affairs*, Cambridge University Press, Cambridge, 1961

[15] Shannon, C. E. (1948), A mathematical theory of communications, *Bell Systems Technical Journal*, 27, 379-423 and 623-656.

[16] Schoenfeld, A. (1980), Teaching problem solving skills, *American Mathematical Monthly*, 87, 794-805.

[17] Schoenfeld, A. , The wild, wild, wild world of problem solving: A review of sorts, *For the Learning of Mathematics,* 3, 40-47, 1983.

[18] Schoenfeld, A., Problem solving, teaching, and more: Toward a theory of goal-directed behaviour, *Proceedings CIEAEM 59*, 48-52, Dobogoko, Hungary, 2007

[19] Subbotin, I., Badkoobehi, H. & Bilotskii, N. (2004), Application of Fuzzy Logic to Learning Assessment, *Didactics of Mathematics: Problems and Investigations*. Volume 22, 38-41.

[20] Subbotin, I., Mossovar-Rahmani, F. & Bilotskii, N., Fuzzy logic and the concept of the Zone of Proximate Development, *Didactics of Mathematics: Problems and Investigations*. Volume 36, 101-108, 2011.

[21] Sumarni, A. B., Tahir, A. & Sabariah, B., Evaluation of Steady Stae Vector of Fuzzy Autocatalytic Set of Fuzzy Graph Type-3 of an Incineration Process, WSEAS Transactions on Mathematics, Issue 8, Vol. 9, 658-667, 2010

[22] van Broekhoven, E. & De Baets, B. (2006), Fast and accurate center of gravity defuzzification of fuzzy system outputs defined on trapezoidal fuzzy partitions, *Fuzzy Sets and Systems*, 157, Issue 7, 904-918.

[23] Voskoglou, M. Gr. & Perdikaris, S., A Markov Chain Model in Problem Solving, *International Journal of Mathematics Education in science and Technology*, Vol. 22, No. 6, 909-914, 1991

[24] Voskoglou, M. Gr. & Perdikaris, S., Measuring Problem Solving Skills, *International Journal of Mathematics Education in Science and Technology*, Vol. 22, No. 6, 909-914, 1991 *International Journal of Mathematics Education in Science and Technology*, Vol. 24, No. 3, 443-447, 1993.

[25] Voskoglou, M. Gr., The process of learning mathematics: A fuzzy set approach, *Heuristic and Didactics of Exact Sciences* (Ukraine), 10, 9-13, 1999.

[26] Voskoglou, M. Gr., The use of mathematical modelling as a learning device of mathematics, *Quaderni di Ricerca in Didattica (Scienze Mathematiche)*, University of Palermo, 16, 53-60, 2006

[27] Voskoglou, M. Gr., The mathematics teacher in the modern society, *Quaderni di Ricerca in Didattica (Scienze Mathematiche)*, University of Palermo, 19, 24-30, 2009.
[28] Voskoglou, M. Gr., A fuzzy representation of CBR systems. In N. E. Mastorakis et al (Eds.), Latest advances on Systems, Vol. 1, 48-53, WSEAS 14th International Conference on Systems, Corfu Island, Greece, 2010
[29] Voskoglou, M. Gr., Problem Solving from Polya to Nowadays: A Review and Future Perspectives. In R. V. Nata (Ed.), *Progress in Education*, Vol. 22, Chapter 4, 65-82, Nova Publishers, N. Y., 2011
[30] Voskoglou, M. Gr., *Stochastic and fuzzy models in Mathematics Education, Artificial Intelligence and Management*, Lambert Academic Publishing, Saarbrucken, Germany, 2011 (for more details look at http://amzn.com./3846528218).
[31] Voskoglou, M. Gr. & Subbotin, I. Ya., Fuzzy Models for Analogical Reasoning, *International Journal of Applications of Fuzzy Sets*, 2, 19-38, 2012
[32] Zadeh, L. A. (1965), Fuzzy Sets, *Information and Control*, 8, 338-353.

Appendix

List of the problems given for solution to students in our classroom experiment

Problem 1: We want to construct a channel to run water by folding across its longer side the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water?

Problem 2: Given the matrix A= $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and a positive integer n, find the matrix A^n.

Problem 3: Calculate the integral $\int \frac{x}{x^2+4}dx$.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet (A=1, B=2, C=3 etc). Let us correspond also to each word consisting of 4 letters a 2X2 matrix in the obvious way; e.g. the matrix $\begin{bmatrix} 19 & 15 \\ 13 & 5 \end{bmatrix}$ corresponds to the word SOME. Using the matrix E= $\begin{bmatrix} 8 & 5 \\ 11 & 7 \end{bmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how the receiver could decode your message?

Problem 5: The demand function $P(Q_d)=25-Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s)=2Q_s+1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the market's equilibrium occurs at (Q_0, P_0), the producers who would supply at lower price than P_0 benefit. Find the total gain to producers'.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.

Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?

Problem 8: The rate of increase of the population of a country is analogous to the number of its inhabitants. If the population is doubled in 50 years, in how many years it will be tripled? (ANSWER: In $50\frac{\ln 3}{\ln 2} \approx 79$ years).

Problem 9: A company circulates for first time in the market a new product, say K. Market's research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will turn to K next week

.i) Find the market's share for K two weeks after its first circulation, provided that the market's conditions remain unchanged.

ii) Find the market's share for K in the long run, i.e. when the consumers' preferences will be stabilized.

Problem 10: Among all cylinders having a total surface of 180π m^2, which one has the maximal volume?