Viscosity Iterative Approximating the Common Fixed Points of Non-expansive Semigroups in Banach Spaces

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Abstract: Let C be a closed convex subset of a reflexive and strictly convex Banach space E and $\mathcal{F} = \{T(t); t > 0\}$ be a non-expansive semigroup on the C with the nonempty set of their common fixed points. The purpose of this paper is to study a new viscosity iterative method for a non-expansive semigroup and weakly contraction mappings. And it is proved that the new iterative approximate sequences converge strongly to the solution of a certain variational inequality. These results improve and extend some recent results of the other authors.

Key–Words: Non-expansive semigroup, Common fixed point, Uniformly Gâteaux differentiable norm, Weakly contraction, Iterative approximation, Strong convergence

1 Introduction

Let C be a closed convex subset of Hilbert space H and T be a nonexpansive mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Let $F(T)$ be nonempty and u be an element of C. In 1967, Halpern [1] firstly introduced the following explicit iterative scheme (1) in Hilbert space,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \tag{1}$$

where $\{\alpha_n\}$ is a real sequence and $\alpha_n \in [0, 1]$. He pointed out that the control conditions

$$(C_1) \qquad \lim_{n\to\infty} \alpha_n = 0$$

and

$$(C_2) \qquad \sum_{n=1}^{\infty} \alpha_n = \infty$$

are necessary for the convergence of the iterative scheme (1) to a fixed point of T.

In 1992, Wittman [2] showed that the strong convergence of the iteration scheme (1) under the control conditions $(C_1), (C_2)$ and

$$(C_3) \qquad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$$

in the Hilbert space. After that, Shioji and Takahashi [3] extended Wittman's results to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. In 2004, H. K. Xu [4] proposed the following viscosity iterative process $\{x_n\}$:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{2}$$

where $0 \le \alpha_n \le 1$, $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : C \to C$ is a fixed contractive mapping. He showed that $\{x_n\}$ strongly converges to a fixed point q of T in a uniformly smooth Banach space.

Recently, Chen and Song [5] introduced the following implicit and explicit viscosity iteration processes defined by (3) and (4) to nonexpansive semigroup case,

$$\begin{aligned} x_n &= \alpha_n f(x_n) \\ &\quad + (1 - \alpha_n)\frac{1}{t}\int_0^t T(s)x ds, n \ge 1, \end{aligned} \tag{3}$$

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) \\ &\quad + (1 - \alpha_n)\frac{1}{t}\int_0^t T(s)x ds, n \ge 1, \end{aligned} \tag{4}$$

and showed that $\{x_n\}$ converges to a same point of $\bigcap_{t>0} Fix(T(t))$ in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval (0 , t). Suzuki [6] was the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, n \ge 1, \tag{5}$$

for the nonexpansive semigroup case.

Benavides, Aceda and Xu [7] proved that if

$$\mathcal{F} = \{T(t) : t > 0\}$$

satisfies an asymptotic regularity condition and α_n fulfills the control conditions (C_1) and (C_2) and

$$(C_4) \qquad \lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$$

in a uniformly smooth Banach space, then both the implicit iteration process (5) and the explicit iteration (6) converge to a same point of $Fix(\mathcal{F})$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n, n \geq 1. \tag{6}$$

Song and Xu [8] introduced the following implicit and explicit viscosity iterative schemes, respectively:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, n \geq 1, \tag{7}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, n \geq 1. \tag{8}$$

They proved that the two iteration processes strongly converges to a same point q of $Fix(\mathcal{F})$ which is a solution of certain variational inequality in a reflexive and strictly convex Banach space with a uniformly G$\hat{a}$teaux differentiable norm.

Motivated and inspired by the above results, in this paper, we study the strong convergence of the viscosity iterative processes $\{z_m\}$ and $\{x_n\}$ by respectively equations (9) and (10). We consider the case $T(t)(t > 0)$ is a noexpansive semigroup with $\bigcap_{t>0} F(T(t)) \neq \emptyset, f : C \to C$ is a weakly contractive self-mapping, and define the implicit viscosity iterative method and explicit viscosity iterative method as follows

$$z_m = \alpha_m f(x_m) + (1-\alpha_m)T(t_m)z_m, m \geq 1, \tag{9}$$

and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n$$
$$+(1-\alpha_n-\beta_n)T(t_n)y_n,$$

$$y_n = \gamma_n x_n + (1-\gamma_n)T(t_n)x_n, n \geq 1. \tag{10}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in (0,1) with

$$\alpha_n + \beta_n \leq 1(n \geq 1),$$

and $\{\alpha_m\}, \{\gamma_n\}$ are two sequences in [0,1]. In a reflexive and strictly convex Banach space with a uniformly G$\hat{a}$teaux differentiable norm, we will prove that $\{z_m\}$ and $\{x_n\}$ strongly converge to some point

$$p \in \bigcap_{t>0} F(T(t)),$$

where p is a solution to the following variational inequality:

$$\langle (f-I)p, j(x-p)\rangle \leq 0, \forall x \in \bigcap_{t>0} F(T(t)).$$

So, our results extend and improve some related results considered by Song and Xu [8], Xu [9], Wu, Chang and Yuan[10] and the other authors.

2 Preliminaries

Throughout this paper, let E be a reflexive and strictly convex Banach space and C be a closed convex subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|,$$
$$\|x\| = \|f\|\}, \forall x \in E,$$

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality mapping by j. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x, x_n \rightharpoondown x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for

$$\|x\| = \|y\| = 1, x \neq y;$$

the function $\delta : [0,2] \to [0,1]$ is said to be the modulus of convexity of Banach space E, where

$$\begin{aligned}\delta_\varepsilon \ &= \inf\{1 - \|x-y\|/2 : \\ &\|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\}.\end{aligned}$$

E is said to be uniformly convex if for each $\delta_\varepsilon > 0$. Let

$$S(E) = \{x \in E : \|x\| = 1\}.$$

The norm of Banach space E is said to be G$\hat{a}$teaux differentiable, if the

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. Moreover, if for each $y \in S(E)$, the limit exists uniformly for $x \in S(E)$, we say that the norm of E is uniformly G$\hat{a}$teaux differentiable. It is well known that each uniformly convex Banach space E is reflexive and strictly convex and if E is reflexive and smooth, then the duality mapping J is single valued (see [11-13]).

Definition 1 *Let C be a nonempty subset of a Banach space E and $T : C \to C$ a mapping. T is called a $Lipschitzian$ mapping if there exists a constant $L > 0$ such that*

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$, and L is called $Lipschitz$ constant of T. T is called nonexpansive mapping if $L = 1$, T is called contraction mapping if $L \in [0, 1)$.

Definition 2 *[12] An operator T with domain $D(T)$ and rang $R(T)$ in a Banach space E is said to be weakly contraction, if*

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|), \forall x, y \in C,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\psi(0) = 0, \psi(t) > 0$ for all $t > 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Remark 3 *If $\psi(t) = kt$ for all $t \geq 0$, where $k \in (0, 1)$, then T is a contraction with Lipschitz constant $1 - k$. It is obvious that the class of contraction mappings is a subclass of the class of weakly contraction.*

Definition 4 *A family $\mathcal{F} = \{T(t) : t \geq 0\}$ of mapping of C into itself is called nonexpansive semigroup of C, if it satisfies the following conditions:*

(1) $T(t_1+t_2)x = T(t_1)T(t_2)x$, for each $t_1, t_2 \geq 0$ and $x \in C$;

(2) $T(0)x = x$, for each $x \in C$;

(3) $\lim_{t\to 0} T(t)x = x$, for $x \in C$;

(4) for each $t > 0$, $T(t)$ is nonexpansive, that is,

$$\|T(t)x - T(t)y\| \leq \|x - y\|, \forall x, y \in C.$$

We shall denote by F the common fixed point set of $\mathcal{F}$, that is,

$$F := Fix(\mathcal{F}) = \{x \in C : T(t)x = x, t > 0\}$$
$$= \bigcap_{t>0} Fix(T(t)),$$

where $Fix(T) = \{x \in C : Tx = x\}$ is the set of fixed points of a mapping T.

Definition 5 *$\mathcal{F}$ is said to be uniformly asymptotically regular (in short,u.a.r) on C if for all $h \geq 0$ and any bounded subset K of C,*

$$\limsup_{t\to\infty, x\in K} \|T(h)(T(t)x) - T(t)x\| = 0.$$

Let μ be a continuous linear functional on l^∞ and let $(a_0, a_1, \cdots) \in l^\infty$, we use $\mu_m(a_m)$ instead of $\mu((a_0, a_1, \cdots))$, we call μ a $Banach\ limit$ when μ satisfies $\|\mu\| = \mu_m(1) = 1$ and $\mu_m(a_{m+1}) = \mu_m(a_m)$ for each $(a_0, a_1, \cdots) \in l^\infty$.

Lemma 6 *[14] Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, and $\{x_m\}$ a bounded sequence of E, let z_0 be a element of C and μ be a Banach limit. Then*

$$\mu_m\|x_m - z_0\|^2 = \min_{y\in C} \mu_m\|x_m - y\|^2,$$

if and only if

$$\mu_m\langle y - z_0, j(x_m - z_0)\rangle \leq 0, \forall y \in C.$$

Lemma 7 *[15] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$$

for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 8 *[12] Let $\{u_n\}$ and $\{v_n\}$ be two sequences of nonnegative real numbers such that $\lim_{n\to\infty} \frac{v_n}{u_n} = 0$ and $\sum u_n = \infty$. Let $\{\lambda_n\}$ be a sequence of nonnegative real numbers satisfying the recursive inequality:*

$$\lambda_{n+1} \leq \lambda_n - u_n\phi(\lambda_n) + v_n, \forall n \in N,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then $\{\lambda_n\}$ converges to zero.

3 Main Results

Theorem 9 *Let E be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $\{T(t)\}$ a $u.a.r$ nonexpansive semigroup from C into itself such that*

$$F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset,$$

and $f : C \to C$ a weakly contractive mapping with function ψ. Suppose $\lim_{m\to\infty} t_m = \infty$ and $\alpha_m \in [0, 1]$ such that $\lim_{m\to\infty} \alpha_m = 0$. If $\{z_m\}$ is defined by

$$z_m = \alpha_m f(z_m) + (1 - \alpha_m)T(t_m)z_m, m \geq 1.$$

Let $z_1 \in C$. Then as $m \to \infty$, $\{z_m\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the following variational inequality:

$$\langle f(p) - p, j(x-p)\rangle \leq 0, \forall x \in F. \tag{11}$$

Proof: We first show that the uniqueness of solution to the variational inequality (11) in F. In fact, suppose $p, q \in F$ satisfy (11), we have that

$$\langle f(p) - p, j(q-p)\rangle \leq 0$$

and

$$\langle f(q) - q, j(p-q)\rangle \leq 0.$$

Combining the above two inequalities, we have

$$\begin{aligned} \|q-p\|^2 &\leq \|f(p)-f(q)\|\|q-p\| \\ &\leq (\|p-q\| - \psi(\|p-q\|))\|q-p\|. \end{aligned}$$

Thus,

$$\|q-p\| \leq \|p-q\| - \psi(\|p-q\|),$$

we can obtain that $p-q=0$, or $p=q$.

Next we show the boundedness of $\{z_m\}$. Indeed, for any fixed $y \in F$, we have

$$\begin{aligned} &\|z_m - y\| = \|\alpha_m f(z_m) + (1-\alpha_m)T(t_m)z_m - y\| \\ &\leq \alpha_m\|f(z_m) - y\| + (1-\alpha_m)\|T(t_m)z_m - y\| \\ &\leq \alpha_m\|f(z_m) - f(y)\| + \alpha_m\|f(y) - y\| \\ &\quad +(1-\alpha_m)\|z_m - y\| \\ &\leq \alpha_m\|z_m - y\| - \alpha_m\psi(\|z_m - y\|) \\ &\quad +\alpha_m\|f(y) - y\| + (1-\alpha_m)\|z_m - y\| \\ &= \|z_m - y\| - \alpha_m\psi(\|z_m - y\|) \\ &\quad +\alpha_m\|f(y) - y\|. \end{aligned}$$

So, we obtain that

$$\psi(\|z_m - y\|) \leq \|f(y) - y\|.$$

Suppose $\{z_m - y\}$ is not bounded. Then there exists a sequence $\{m_k\}$ in $(0,\infty)$ with $m_k \to \infty$ as $k \to \infty$ such that

$$\|z_{m_k} - y\| > k, \forall k \in N. \tag{12}$$

Since ψ is nondecreasing and $\lim_{t\to\infty}\psi(t) = \infty$, it follows from (12) that

$$\psi(k) < \psi(\|z_{m_k} - y\|) \leq \|f(y) - y\|,$$

a contraction.

Thus $\{z_m\}$ is bounded, and so are $\{T(t_m)z_m\}$ and $\{f(z_m)\}$. This implies that

$$\begin{aligned} &\lim_{m\to\infty}\|z_m - T(t_m)z_m\| \\ &= \lim_{m\to\infty}\alpha_m\|T(t_m)z_m - f(z_m)\| = 0. \end{aligned}$$

Since $\{T(t)\}$ is $u.a.r$ nonexpansive semigroup and $\lim_{m\to\infty} t_m = \infty$, then for all $h > 0$,

$$\begin{aligned} &\lim_{m\to\infty}\|T(h)T(t_m)z_m - T(t_m)z_m\| \\ &\leq \limsup_{m\to\infty x\in K}\|T(h)T(t_m)x - T(t_m)x\| = 0, \end{aligned}$$

where K is any bounded subset of C containing $\{z_m\}$. Hence,

$$\begin{aligned} \|z_m &- T(h)z_m\| \leq \|z_m - T(t_m)z_m\| \\ &+\|T(t_m)z_m - T(h)T(t_m)z_m\| \\ &+\|T(h)T(t_m)z_m - T(h)z_m\| \\ &\leq 2\|z_m - T(t_m)z_m\| \\ &+\|T(t_m)z_m - T(h)T(t_m)z_m\| \to 0, \\ &m \to \infty. \end{aligned}$$

That is, for all $h > 0$,

$$\lim_{m\to\infty}\|z_m - T(h)z_m\| = 0. \tag{13}$$

We claim that the set $\{z_m\}$ is sequentially compact.

Define the function $\varphi : C \to R$ by

$$\varphi(x) := \mu_m\|z_m - x\|^2, x \in C.$$

Since E is reflexive,

$$\lim_{\|x\|\to\infty}\varphi(\|x\|) = \infty,$$

and φ is continuous convex function, we have that the set

$$M := \{y \in C : \varphi(y) = \inf_{x\in C}\varphi(x)\}, \tag{14}$$

which is nonempty closed convex and bounded. Furthermore, M is invariant under $T(t)$ (for all $t > 0$). In fact, for each $y \in M$, we have

$$\begin{aligned} \varphi(T(t)y) &= \mu_m\|z_m - T(t)y\|^2 \\ &\leq \mu_m\|T(t)z_m - T(t)y\|^2 \\ &\leq \mu_m\|z_m - y\|^2 = \varphi(y). \end{aligned}$$

Hence, $T(t)y \in M$. As y is arbitrary, then $T(t)(M) \subset M$. Let $u \in F$, since every nonempty closed convex subset of a strictly convex and reflexive Banach space is a $Chebyshev$ set (see [13]), there exists an unique $p \in M$ such that

$$\|u - p\| = \inf_{x\in M}\|u - x\|,$$

since $T(t)u = u$ and $T(t)p \in M$,

$$\|u - T(t)p\| = \|T(t)u - T(t)p\| \leq \|u - p\|.$$

Hence $T(t)p = p$ by the uniqueness of $p \in M$. Since t is arbitrary, it follows that $p \in F$. Using Lemma 6 together with $p \in M$, we obtain that

$$\mu_m \langle z - p, j(z_m - p)\rangle \le 0, \forall z \in C.$$

In particular

$$\mu_m \langle f(p) - p, j(z_m - p)\rangle \le 0. \qquad (15)$$

Since f is weakly contraction, we have

$$\begin{aligned}
&\|z_m - p\|^2 = \langle z_m - f(z_m), j(z_m - p)\rangle \\
&+\langle f(z_m) - f(p), j(z_m - p)\rangle + \langle f(p) - p, j(z_m - p)\rangle \\
&\le \langle z_m - f(z_m), j(z_m - p)\rangle \\
&+\|f(z_m) - f(p)\|\|z_m - p\| + \langle f(p) - p, j(z_m - p)\rangle \\
&\le \langle z_m - f(z_m), j(z_m - p)\rangle + \|z_m - p\|^2 \\
&-\|z_m - p\|\psi(\|z_m - p\|) + \langle f(p) - p, j(z_m - p)\rangle,
\end{aligned}$$

$$\begin{aligned}
&\|z_m - p\|\psi(\|z_m - p\|) \\
&\le \langle z_m - f(z_m), j(z_m - p)\rangle + \langle f(p) - p, j(z_m - p)\rangle,
\end{aligned} \qquad (16)$$

and

$$\begin{aligned}
&\langle z_m - f(z_m), j(z_m - p)\rangle \\
&= (1 - \alpha_m)\langle T(t_m)z_m - f(z_m), j(z_m - p)\rangle \\
&= (1 - \alpha_m)\langle T(t_m)z_m - z_m + z_m - f(z_m), j(z_m - p)\rangle,
\end{aligned}$$

$$\begin{aligned}
&\langle z_m - f(z_m), j(z_m - p)\rangle \\
&\le \tfrac{1-\alpha_m}{\alpha_m}\langle T(t_m)z_m - z_m, j(z_m - p)\rangle \\
&\le \tfrac{1-\alpha_m}{\alpha_m}\langle T(t_m)z_m - T(t_m)p + p - z_m, j(z_m - p)\rangle \\
&\le 0.
\end{aligned}$$

Hence, we get

$$\langle z_m - f(z_m), j(z_m - p)\rangle \le 0, \forall m \in N. \qquad (17)$$

Together with above inequalities (15), (16), (17), we obtain that

$$\mu_m \|z_m - p\|\psi(\|z_m - p\|) \le 0,$$

therefore, there exists a subsequence $\{z_{m_i}\}$ of $\{z_m\}$ such that $z_{m_i} \to p (i \to \infty)$.

Next we show that p is a solution in F to the variational inequality(11).

Since the duality map j is a single-valued and norm topology to weak* topology uniformly continuous on bounded subset of E, and $z_{m_i} \to p, (i \to \infty)$, we have $\|(I - f)z_m - (I - f)p\| \to 0, (i \to \infty)$, and for all $x \in F$, we observe that

$$\begin{aligned}
&|\langle z_{m_i} - f(z_{m_i}), j(z_{m_i} - x)\rangle - \langle p - f(p), j(p - x)\rangle| \\
&= |\langle z_{m_i} - f(z_{m_i}) - (p - f(p)), j(z_{m_i} - x)\rangle \\
&\quad +\langle p - f(p), j(z_{m_i} - x) - j(p - x)\rangle| \\
&\le \|z_{m_i} - f(z_{m_i}) - (p - f(p))\|\|z_{m_i} - x\| \\
&\quad +|\langle p - f(p), j(z_{m_i} - x) - j(p - x)\rangle| \to 0, \\
&i \to \infty.
\end{aligned}$$

It follows from (17) that

$$\begin{aligned}
&\langle f(p) - p, j(x - p)\rangle \\
&= \lim_{i\to\infty} \langle f(z_{m_i}) - z_{m_i}, j(x - z_{m_i})\rangle \le 0.
\end{aligned}$$

That is, $p \in F$ is a solution of (11). Hence $p = q$ by uniqueness. In a similar way, it can be show that each cluster point of the sequence $\{z_m\}$ is equal to q. Therefore, $z_m \to p$ as $m \to \infty$.

Corollary 10 *Let E be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, and $\{T(t)\}$ a $u.a.r$ nonexpansive semigroup from C into itself such that $F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset$, and $f : C \to C$ a fixed contractive mapping with contractive coefficient $\beta \in [0, 1)$. Suppose $\lim_{m\to\infty} t_m = \infty$ and $\alpha_m \in [0, 1]$ such that $\lim_{m\to\infty} \alpha_m = 0$. If $\{z_m\}$ is defined by*

$$z_m = \alpha_m f(z_m) + (1 - \alpha_m)T(t_m)z_m, m \ge 1.$$

Let $z_1 \in C$. Then as $m \to \infty$, $\{z_m\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the variational inequality(11).

Theorem 11 *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space E with a uniformly Gâteaux differentiable norm, and $\{T(t)\}$ a $u.a.r$ nonexpansive semigroup from C into itself such that*

$$F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset,$$

and $f : C \to C$ a weakly contractive mapping with function ψ. Suppose $\lim_{n\to\infty} t_n = \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in (0,1) with $\alpha_n + \beta_n \le 1 (n \ge 1)$, and $\{\gamma_n\}$ a sequence in [0,1]. The sequence $\{x_n\}$ is given by (10). Let $x_1 \in C$ and assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \psi$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n$

$$\le \limsup_{n\to\infty} \beta_n < 1;$$

(iv) $\inf\{\psi(\|x_n - q\|)/\|x_n - q\| : x_n \neq q, n \in N\} = \delta > 0$ *for* $q \in F$;

(v) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$ *and* $\liminf_{n\to\infty} \gamma_n > 0$.

Then as $n \to \infty$, $\{x_n\}$ defined by (10) converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the variational inequality (11).

Proof: The proof is divided into five steps.

Step 1. We show that $\{x_n\}$ is bounded. Take $q \in F$. It follows that

$$\begin{aligned}\|x_{n+1} &-q\| \le \alpha_n\|f(x_n)-q\|+\beta_n\|x_n-q\| \\ &+(1-\alpha_n-\beta_n)\|T(t_n)y_n-q\| \\ &\le \alpha_n\|f(x_n)-f(q)\|+\alpha_n\|f(q)-q\| \\ &+\beta_n\|x_n-q\|+(1-\alpha_n-\beta_n)\|y_n-q\| \\ &\le \alpha_n\|x_n-q\|-\alpha_n\psi(\|x_n-q\|) \\ &+\alpha_n\|f(q)-q\|+\beta_n\|x_n-q\| \\ &+(1-\alpha_n-\beta_n)\|y_n-q\| \\ &=(\alpha_n+\beta_n)\|x_n-q\|-\alpha_n\psi(\|x_n-q\|) \\ &+\alpha_n\|f(q)-q\|+(1-\alpha_n-\beta_n)\|y_n-q\|,\end{aligned}$$

and

$$\begin{aligned}\|y_n-q\| &= \|\gamma_n(x_n-q) \\ &+(1-\gamma_n)(T(t_n)x_n-q)\| \\ &\le \gamma_n\|x_n-q\|+(1-\gamma_n)\|x_n-q\| \\ &= \|x_n-q\|.\end{aligned}$$

Since $0<\delta=\inf\{\psi(\|x_n-q\|)/\|x_n-q\|:$

$$x_n \ne q, n\in N\},$$

and together with the above two inequalities, we have

$$\begin{aligned}\|x_{n+1} &-q\| \le \|x_n-q\|-\alpha_n\delta\|x_n-q\| \\ &+\alpha_n\|f(q)-q\| \\ &=(1-\alpha_n\delta)\|x_n-q\|+\alpha_n\|f(q)-q\|.\end{aligned}$$

By induction,

$$\|x_n-q\| \le \max\{\|x_1-q\|, \frac{1}{\delta}\|f(q)-q\|\}, n\ge 1,$$

consequently, $\{x_n\}$ is bounded, and so are $\{y_n\}, \{T(t_n)x_n\}, \{T(t_n)y_n\}$ and $\{f(x_n)\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Indeed, define a sequence $\{z_n\}$ by

$$x_{n+1}=\beta_n x_n+(1-\beta_n)z_n, n\ge 1,$$

and we have

$$\begin{aligned}&z_{n+1}-z_n \\ &=\tfrac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\tfrac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &=\tfrac{\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1}-\beta_{n+1})T(t_{n+1})y_{n+1}}{1-\beta_{n+1}} \\ &-\tfrac{\alpha_n f(x_n)+(1-\alpha_n-\beta_n)T(t_n)y_n}{1-\beta_n} \\ &=\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\tfrac{\alpha_n}{1-\beta_n}f(x_n) \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}T(t_{n+1})y_{n+1} \\ &-\tfrac{1-\alpha_n-\beta_n}{1-\beta_n}T(t_n)y_n \\ &=\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\tfrac{\alpha_n}{1-\beta_n}f(x_n) \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}(T(t_{n+1})y_{n+1}-T(t_{n+1})y_n) \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}(T(t_{n+1})y_n-T(t_n)y_n) \\ &+(\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}-\tfrac{1-\alpha_n-\beta_n}{1-\beta_n})T(t_n)y_n,\end{aligned}$$

and

$$\begin{aligned}&\|y_{n+1}-y_n\| \\ &=\|\gamma_{n+1}x_{n+1}+(1-\gamma_{n+1})T(t_{n+1})x_{n+1} \\ &-\gamma_n x_n-(1-\gamma_n)T(t_n)x_n\| \\ &\le \gamma_{n+1}\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|x_n\| \\ &+(1-\gamma_{n+1})\|T(t_{n+1})x_{n+1}-T(t_{n+1})x_n\| \\ &+(1-\gamma_{n+1})\|T(t_{n+1})x_n-T(t_n)x_n\| \\ &+|\gamma_{n+1}-\gamma_n|\|T(t_n)x_n\| \\ &\le \|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|x_n\| \\ &+(1-\gamma_{n+1})\|T(t_{n+1})x_n-T(t_n)x_n\| \\ &+|\gamma_{n+1}-\gamma_n|\|T(t_n)x_n\|.\end{aligned}$$

Together with the above two inequalities, we obtain that

$$\begin{aligned}&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\tfrac{\alpha_n}{1-\beta_n}\|f(x_n)\| \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}\|T(t_{n+1})y_{n+1}-T(t_{n+1})y_n\| \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}\|T(t_{n+1})y_n-T(t_n)y_n\| \\ &+|\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}-\tfrac{1-\alpha_n-\beta_n}{1-\beta_n}|\|T(t_n)y_n\| \\ &-\|x_{n+1}-x_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\tfrac{\alpha_n}{1-\beta_n}\|f(x_n)\| \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}[\|x_{n+1}-x_n\| \\ &+|\gamma_{n+1}-\gamma_n|(\|x_n\|+\|T(t_n)x_n\|) \\ &+(1-\gamma_{n+1})\|T(t_{n+1})x_n-T(t_n)x_n\|] \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}\|T(t_{n+1})y_n-T(t_n)y_n\| \\ &+|\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}-\tfrac{1-\alpha_n-\beta_n}{1-\beta_n}|\|T(t_n)y_n\| \\ &-\|x_{n+1}-x_n\| \\ &=\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\|+\tfrac{\alpha_n}{1-\beta_n}\|f(x_n)\| \\ &+(\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}-1)\|x_{n+1}-x_n\| \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}[|\gamma_{n+1}-\gamma_n|(\|x_n\|+\|T(t_n)x_n\|) \\ &+(1-\gamma_{n+1})\|T(t_{n+1})x_n-T(t_n)x_n\|] \\ &+\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}\|T(t_{n+1})y_n-T(t_n)y_n\| \\ &+|\tfrac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}-\tfrac{1-\alpha_n-\beta_n}{1-\beta_n}|\|T(t_n)y_n\|.\end{aligned}$$

If $t_{n+1}>t_n$, by $(u.a.r)$, we have

$$\begin{aligned}&\|T(t_{n+1})x_n-T(t_n)x_n\| \\ &=\|T(t_{n+1}-t_n)T(t_n)x_n-T(t_n)x_n\|\to 0. \quad (18)\end{aligned}$$

If $t_{n+1}<t_n$,interchange t_{n+1} and t_n, we also can obtain

$$\|T(t_{n+1})x_n-T(t_n)x_n\|\to 0,$$

and similarly, we get

$$\|T(t_{n+1})y_n-T(t_n)y_n\|\to 0. \qquad (19)$$

Thus it follows from the conditions (i), (iii), (v) and (18), (19), we obtain that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, by Lemma 7, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0,$$

which imply that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{20}$$

Step 3. For each $t\in(0,\infty)$, $\|T(t)x_n-x_n\|\to 0$. Indeed, we have that

$$\begin{aligned}&\|x_{n+1}-T(t_n)x_n\|\\&\le\|x_{n+1}-T(t_n)y_n+T(t_n)y_n-T(t_n)x_n\|\\&\le\|x_{n+1}-T(t_n)y_n\|+\|y_n-x_n\|\\&=\|x_{n+1}-T(t_n)y_n\|+(1-\gamma_n)\|x_n-T(t_n)x_n\|\\&\le\|x_{n+1}-T(t_n)y_n\|+(1-\gamma_n)\|x_{n+1}-x_n\|\\&+(1-\gamma_n)\|x_{n+1}-T(t_n)x_n\|.\end{aligned}$$

So

$$\|x_{n+1}-T(t_n)x_n\|\le\frac{1}{\gamma_n}\|x_{n+1}-T(t_n)y_n\|$$
$$+\frac{1-\gamma_n}{\gamma_n}\|x_{n+1}-x_n\|.$$

And as

$$\begin{aligned}&\|x_{n+1}-T(t_n)y_n\|=\|\alpha_nf(x_n)+\beta_nx_n\\&+(1-\alpha_n-\beta_n)T(t_n)y_n-T(t_n)y_n\|\\&\le\alpha_n\|f(x_n)-T(t_n)y_n\|+\beta_n\|x_n-T(t_n)y_n\|\\&\le\alpha_n\|f(x_n)-T(t_n)y_n\|\\&+\beta_n\|x_n-x_{n+1}\|\\&+\beta_n\|x_{n+1}-T(t_n)y_n\|,\end{aligned}$$

$$\begin{aligned}&(1-\beta_n)\|x_{n+1}-T(t_n)y_n\|\\&\le\alpha_n\|f(x_n)-T(t_n)y_n\|+\beta_n\|x_n-x_{n+1}\|\end{aligned}$$

$$\begin{aligned}&\|x_{n+1}-T(t_n)y_n\|\\&\le\tfrac{\alpha_n}{1-\beta_n}\|f(x_n)-T(t_n)y_n\|+\tfrac{\beta_n}{1-\beta_n}\|x_n-x_{n+1}\|,\end{aligned}$$

by (i), (iii), (v), (20) and together with above inequalities, we get

$$\|x_n-T(t_n)x_n\|\to 0,(n\to\infty). \tag{21}$$

Let K be any bounded subset of C which contains the sequence $\{x_n\}$. It follows that

$$\begin{aligned}&\|T(t)x_n-x_n\|\le\|T(t)x_n-T(t)T(t_n)x_n\|\\&+\|T(t)T(t_n)x_n-T(t_n)x_n\|+\|T(t_n)x_n-x_n\|\\&\le 2\|x_n-T(t_n)x_n\|+\sup_{x\in K}\|T(t)T(t_n)x-T(t_n)x\|.\end{aligned}$$

So we have

$$\|T(t)x_n-x_n\|\to 0,(n\to\infty). \tag{22}$$

Step 4. We show that

$$\limsup_{n\to\infty}\langle(I-f)p,j(p-x_{n+1})\rangle\le 0. \tag{23}$$

Let

$$z_m=\alpha_mf(z_m)+(1-\alpha_m)T(t_m)z_m,$$

where t_m and α_m satisfies the conditions of Theorem 9. Then we have that

$$\lim_{m\to\infty}z_m=p.$$

From the definition of ψ, we know that

$$\lim_{m\to\infty}\psi(\|z_m-p\|)=\psi(0)=0.$$

Since

$$\begin{aligned}&\|z_m-x_{n+1}\|^2\\&=\alpha_m\langle f(z_m)-x_{n+1},j(z_m-x_{n+1})\rangle\\&+(1-\alpha_m)\langle T(t_m)z_m-x_{n+1},j(z_m-x_{n+1})\rangle\\&=(1-\alpha_m)\langle T(t_m)z_m-T(t_m)x_{n+1},j(z_m-x_{n+1})\rangle\\&+(1-\alpha_m)\langle T(t_m)x_{n+1}-x_{n+1},j(z_m-x_{n+1})\rangle\\&+\alpha_m\langle f(z_m)-f(p)+f(p)+z_m\\&\quad-z_m+p-p-x_{n+1},j(z_m-x_{n+1})\rangle\\&\le(1-\alpha_m)\|z_m-x_{n+1}\|^2\\&+(1-\alpha_m)\langle T(t_m)x_{n+1}-x_{n+1},j(z_m-x_{n+1})\rangle\\&+\alpha_m\langle f(z_m)-f(p)-z_m+p,j(z_m-x_{n+1})\rangle\\&+\alpha_m\langle f(p)-p,j(z_m-x_{n+1})\rangle\\&+\alpha_m\langle z_m-x_{n+1},j(z_m-x_{n+1})\rangle\\&\le\|z_m-x_{n+1}\|^2\\&+(1-\alpha_m)\langle T(t_m)x_{n+1}-x_{n+1},j(z_m-x_{n+1})\rangle\\&+\alpha_m\langle f(p)-p,j(z_m-x_{n+1})\rangle\\&+\alpha_m(\|f(z_m)-f(p)\|+\|z_m-p\|)\|z_m-x_{n+1}\|,\end{aligned}$$

so, we can obtain that

$$\begin{aligned}&\langle f(p)-p,j(x_{n+1}-z_m)\rangle\\&\le\tfrac{1-\alpha_m}{\alpha_m}\|T(t_m)x_{n+1}-x_{n+1}\|\|z_m-x_{n+1}\|\\&+(2\|z_m-p\|-\psi(\|z_m-p\|))\|z_m-x_{n+1}\|\\&\le M_1\tfrac{1-\alpha_m}{\alpha_m}\|T(t_m)x_{n+1}-x_{n+1}\|\\&+2M_1\|z_m-p\|-\psi(\|z_m-p\|)M_1\\&\le\tfrac{M_1}{\alpha_m}\|T(t_m)x_{n+1}-x_{n+1}\|\\&+2M_1\|z_m-p\|-\psi(\|z_m-p\|)M_1,\end{aligned}$$

where M_1 is a constant such that

$$\|x_{n+1}-z_m\|\le M_1.$$

Firstly, we take limit as $n \to \infty$, and then as $m \to \infty$ in above inequality (using (22))

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle f(p) - p, j(x_{n+1} - z_m)\rangle \leq 0.$$

On the other hand, since J is single-valued and norm topology to weak* topology uniformly continuous on bounded set of E and

$$\lim_{m\to\infty} z_m = p,$$

we get

$$\lim_{m\to\infty} (x_{n+1} - z_m) = x_{n+1} - p.$$

Therefore, we have

$$\langle f(p)-p, j(x_{n+1}-z_m)\rangle \longrightarrow \langle f(p)-p, j(x_{n+1}-p)\rangle.$$

Thus, given $\varepsilon > 0$, there exists $N \geq 1$, such that if $m > N$, for all n, we have

$$\begin{aligned}&\langle f(p) - p, j(x_{n+1} - p)\rangle \\ &\qquad < \langle f(p) - p, j(x_{n+1} - z_m)\rangle + \varepsilon. \qquad (24)\end{aligned}$$

Therefore, by taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in both sides of (24)

$$\begin{aligned}&\limsup_{n\to\infty} \langle f(p) - p, j(x_{n+1} - p)\rangle \\ &\leq \limsup_{m\to\infty} \limsup_{n\to\infty} \langle f(p) - p, j(x_{n+1} - z_m)\rangle + \varepsilon.\end{aligned}$$

Since ε is arbitrary, we obtain(23).

Thus, there exists a sequence $\{\varepsilon_n\}$ in $(0, \infty)$ which $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$\langle (I - f)p, j(p - x_{n+1})\rangle \leq \varepsilon_n, \forall n \in N.$$

Step 5. $\lim_{n\to\infty} \|x_n - p\| = 0$. Indeed, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \langle \alpha_n(f(x_n) - p) + \beta_n(x_n - p) \\
&+(1 - \alpha_n - \beta_n)(T(t_n)y_n - p), j(x_{n+1} - p)\rangle \\
&= \langle \alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) \\
&+(1 - \alpha_n - \beta_n)(T(t_n)y_n - p), j(x_{n+1} - p)\rangle \\
&+\alpha_n\langle f(p) - p, j(x_{n+1} - p)\rangle \\
&\leq \|\alpha_n(f(x_n) - f(p)) + \beta_n(x_n - p) \\
&+(1 - \alpha_n - \beta_n)(T(t_n)y_n - p)\|\|x_{n+1} - p\| \\
&+\alpha_n\varepsilon_n \\
&\leq [\alpha_n\|f(x_n) - f(p)\| + \beta_n\|x_n - p\| \\
&+(1 - \alpha_n - \beta_n)\|T(t_n)y_n - p\|]\|x_{n+1} - p\| \\
&+\alpha_n\varepsilon_n \\
&\leq [\alpha_n\|x_n - p\| - \alpha_n\psi(\|x_n - p\|) \\
&+\beta_n\|x_n - p\| \\
&+(1 - \alpha_n - \beta_n)\|y_n - p\|]\|x_{n+1} - p\| \\
&+\alpha_n\varepsilon_n \\
&\leq [(\alpha_n + \beta_n)\|x_n - p\| - \alpha_n\psi(\|x_n - p\|) \\
&+(1 - \alpha_n - \beta_n)\|\gamma_n(x_n - p) \\
&+(1 - \gamma_n)(T(t_n)x_n - p)\|]\|x_{n+1} - p\| \\
&+\alpha_n\varepsilon_n \\
&\leq [\|x_n - p\| - \alpha_n\psi(\|x_n - p\|)]\|x_{n+1} - p\| \\
&+\alpha_n\varepsilon_n \\
&\leq \tfrac{1}{2}[\|x_n - p\| - \alpha_n\psi(\|x_n - p\|)]^2 \\
&+\tfrac{1}{2}\|x_{n+1} - p\|^2 + \alpha_n\varepsilon_n.
\end{aligned}$$

So,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\leq \|x_n - p\|^2 - 2\alpha_n\psi(\|x_n - p\|)\|x_n - p\| \\
&+\alpha_n^2(\psi(\|x_n - p\|))^2 + 2\alpha_n\varepsilon_n \\
&\leq \|x_n - p\|^2 - 2\alpha_n\psi(\|x_n - p\|)\|x_n - p\| \\
&+\alpha_n^2(\psi(M))^2 + 2\alpha_n\varepsilon_n,
\end{aligned}$$

for some $M > 0$. Since $\{\|x_n - p\|\}$ is bounded, thus, for $\lambda_n = \|x_n - p\|^2$, we obtain the following recursive inequality:

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\phi(\lambda_n) + \omega_n,$$

where

$$\omega_n = \alpha_n[\alpha_n(\psi(M))^2 + 2\varepsilon_n]$$

and

$$\phi(t) = 2\sqrt{t}\psi(\sqrt{t}).$$

So $\{x_n\}$ converges strongly to p by Lemma 8. □

If $\gamma_n = 1$, the following result is clearly gained.

Corollary 12 *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space E with a uniformly Gâteaux differentiable norm, and $\{T(t)\}$ a u.a.r nonexpansive semigroup from C into itself such that*

$$F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset,$$

and $f : C \to C$ a weakly contractive mapping with function ψ. Suppose $\lim_{n\to\infty} t_n = \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in (0,1) with $\alpha_n + \beta_n \leq 1$. The sequence $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)T(t_n)x_n.$$

Let $x_1 \in C$ and assume that $\{\alpha_n\}, \{\beta_n\}, \psi$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\inf\{\psi(\|x_n - q\|)/\|x_n - q\| :$

$$x_n \neq q, n \in N\} = \delta > 0, \forall q \in F.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the variational inequality (11).

Corollary 13 *Let C be a nonempty closed convex subset of a real reflexive strictly convex Banach space E with a uniformly Gâteaux differentiable norm, and $\{T(t)\}$ a $u.a.r$ nonexpansive semigroup from C into itself such that*

$$F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset,$$

and $f : C \to C$ a fixed contractive mapping with contractive coefficient $\beta \in [0,1)$. Suppose $\lim_{n\to\infty} t_n = \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in (0,1) with $\alpha_n + \beta_n \leq 1 (n \geq 1)$, and $\{\gamma_n\}$ a sequence in [0,1]. The sequence $\{x_n\}$ is given by (10). Let $x_1 \in C$ and assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \psi$ satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\inf\{\psi(\|x_n - q\|)/\|x_n - q\| :$

$$x_n \neq q, n \in N\} = \delta > 0, forq \in F;$$

(v) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$ *and* $\liminf_{n\to\infty} \gamma_n > 0$.
Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the variational inequality (11).

Corollary 14 *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E with a uniformly Gâteaux differentiable norm, and $\{T(t)\}$ a $u.a.r$ nonexpansive semigroup from C into itself such that*

$$F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset,$$

and $f : C \to C$ a weakly contractive mapping with function ψ. Suppose $\lim_{n\to\infty} t_n = \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in (0,1) with $\alpha_n + \beta_n \leq 1 (n \geq 1)$, and $\{\gamma_n\}$ a sequence in [0,1]. The sequence $\{x_n\}$ is given by (10). Let $x_1 \in C$ and assume that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \psi$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iv) $\inf\{\psi(\|x_n - q\|)/\|x_n - q\| :$

$$x_n \neq q, n \in N\} = \delta > 0, forq \in F;$$

(v) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$ *and* $\liminf_{n\to\infty} \gamma_n > 0$.
Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the variational inequality (11).

Remark 15 *When $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \psi$ satisfy different conditions, the results in this paper extend and improve some related results considered by Song and Xu [8], Xu [9], Wu, Chang and Yuan[10] and the other authors.*

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