Dissipativity of θ-methods and one-leg methods for nonlinear neutral delay integro-differential equations

Guanghua He
Zhejiang Yuexiu University of Foreign Languages
Department of Mathematics
Kuaiji Road 428, Shaoxing
China
hegh2003@126.com

Liping Wen
Xiangtan University
School of Mathematics and Computational Science
Xiangtan, Hunan
China
lpwen@xtu.edu.cn

Abstract: In this paper we study the dissipativity of a special class of nonlinear neutral delay integro-differential equations. The dissipativity of three kinds of important numerical methods, the linear θ-methods, one-leg θ-methods, and the one-leg methods is obtained when they are applied to these problems. Numerical experiments are presented to check our findings.

Key–Words: Linear θ-methods, One-leg θ-methods, One-leg methods, Nonlinear neutral delay integro-differential equations, Dissipativity, Absorbing set

1 Introduction

Many dynamical systems possess a bounded absorbing set, which all trajectories enter in a finite time and thereafter remain inside. These dynamical systems are called dissipative [2]. When numerical methods are used to solve them, we hope that the methods inherit the dissiapativity.

Since the pioneer work [3] of Humphries and Stuart which studied the dissipativity of Runge-Kutta methods for initial value problems (IVPs) of ordinary differential equations (ODEs) in 1994, many results about dissipativity of numerical methods for ODEs have been found [4, 5, 6, 7]. Huang [8] gave a sufficient condition for the dissipativity of theoretical solution, and investigated the dissipativity of (k, l) algebraically stable of Runge-Kutta methods for the delay differential equations (DDEs) with constant delay. Then, some results about the dissipativity of linear θ-methods, $G(c, p, 0)$-algebraically stable one-leg methods and multistep Runge-Kutta methods were also obtained [9, 10, 11]. Tian [12] investigated the dissipativity of DDEs with a bounded variable lag and the dissipativity of θ- methods in 2004. And Wen [13, 14] studied the dissipativity of Volterra functional differential equations.

Recently, Gan [15, 16, 17] studies the dissipativity of θ-methods for integro-differential equations (IDEs), nonlinear Volterra delay-integro-differential equations (VDIDEs) and pantograph equations, respectively. In addition, Zhen and Huang [18], Wen [19] and Wang [20] consider the dissipativity for nonlinear neutral delay differential equations (NDDEs). Wu and Gan [21] consider the dissipativity for a class of nonlinear neutral delay integro-differential equations (NDIDEs).

In this paper, we study the numerical dissipativity of a special class NDIDEs which arise widely in the fields of applied sciences, such as physics, biology, medicine, economics and so on [1]. Different from the above equations, our NDIDEs have two different constant delay variables. To our best knowledge, there is not any result on the numerical dissipativity of this kind of equations.

The paper is organized as follows. In section 2, the description of the problem class is given and a sufficient condition is presented to ensure that the NDIDEs is dissipative. In section 3 and section 4, the dissipativity of θ-methods and one-leg methods is studied respectively. And in section 5, numerical examples are presented to verify our findings.

2 Description of the problem class

Let $\langle\cdot,\cdot\rangle$ be inner product in C^d and $\|\cdot\|$ be the corresponding norm. Consider the nonlinear neutral delay integro-differential equations NDIDEs

$$\begin{aligned} &\frac{d}{dt}[y(t) - Ny(t-\tau_1)] = \\ &\quad f(y(t), y(t-\tau_1), \int_{t-\tau_2}^{t} g(t,\xi,y(\xi))d\xi), \\ &\quad t \geq 0, \\ &y(t) = \varphi(t), -\tau \leq t \leq 0, \end{aligned} \tag{1}$$

where τ_1, τ_2 are positive constants, $\tau = \max\{\tau_1, \tau_2\}$. $N \in C^{d\times d}$ is a constant matrix satisfying $\|N\| < 1$, $\varphi : [-\tau, 0] \to C^d$ is a continuous function and $f : C^d \times C^d \times C^d \to C^d$ is a locally Lipschitz continuous function, $g : [0, +\infty) \times [-\tau_2, +\infty) \times C^d \to C^d$ is a continuous function, f and g satisfy the following conditions,

$$\begin{aligned} &Re\langle u - Nv, f(u, v, w)\rangle \leq \\ &\quad \gamma + \alpha\|u\|^2 + \beta\|v\|^2 + \omega\|w\|^2, \\ &\qquad u, v, w \in C^d, \qquad (2) \\ &\|g(t, \theta, s)\| \leq c\|s\|, \\ &\quad \text{for all } t \geq 0, t - \tau_2 \leq \theta \leq t, \ \ s \in C^d, \ (3) \end{aligned}$$

where $\gamma, \alpha, \beta, \omega, c$ are real constants and $\beta \geq 0, \gamma \geq 0, \omega \geq 0$.

Definition 1. *The problem (1)-(3) is said to be dissipative in C^d, if there exists a bounded set $B \subset C^d$, such that for any given bounded set $\Phi \subset C^d$, there is a time $t^* = t^*(\Phi)$, such that for any given initial function $\varphi \in C[-\tau, 0]$ with $\varphi(t)$ contained in Φ for all $t \in [-\tau, 0]$, the values of the corresponding solution of the problem are contained in B for all $t \geq t^*$. Here B is called an absorbing set of the problem.*

In order to prove the dissipativity of (1)-(3), we introduce the Generalized Halanay inequality [14].

Lemma 2. *(Generalized Halanay inequality[14]) If $u(t), w(t) \geq 0$ for $t \in (-\infty, +\infty)$,*

$$\begin{cases} u'(t) \leq R(t) + A(t)u(t) + B(t) \max\limits_{t-\tau\leq\xi\leq t} w(\xi), \\ t \geq t_0, \\ w(t) \leq G(t)u(t) + H(t) \max\limits_{t-\tau\leq\xi\leq t} w(\xi), \\ t \geq t_0, \end{cases}$$

and

$$\max_{-\infty<\xi\leq t_0} w(\xi) \leq \frac{G_0}{1 - H_0} \max_{-\infty<\xi\leq t_0} u(\xi),$$

where $A(t)$ is a continuous function satisfying $A(t) \leq A_0$ with constant $A_0 < 0$. $A(t), B(t), G(t)$ and H(t) are nonnegative continuous functions satisfying $G(t) \leq G_0, H(t) \leq H_0$ with constants $G_0 \geq 0, 0 \leq H_0 < 1$ for $t \in [t_0, \infty)$. $\tau \geq 0$ is a constant, and if there exists $0 < p < 1$ such that

$$pA(t) + \frac{G_0}{1 - H_0} B(t) \leq 0, \forall\, t \geq t_0,$$

then we have

$$\begin{aligned} u(t) &\leq \frac{-\gamma}{(1 - p)A_0} + \phi \exp(-\mu^*(t - t_0)), t \geq t_0 \\ w(t) &\leq \frac{G_0}{1 - H_0} \frac{-\gamma}{(1 - p)A_0} \\ &\quad + \frac{G_0}{1 - H_0 e^{\mu^*\tau}} \phi \exp(-\mu^*(t - t_0)), \ t \geq t_0. \end{aligned}$$

where

$$\phi = \max_{-\infty<\xi\leq t_0} u(\xi), \qquad \gamma = \max_{t_0\leq\xi\leq\infty} R(\xi)$$

and $\mu^* > 0$ is defined as

$$\mu^* = \inf\{\mu(t) : \mu(t) + A(t) + B(t) \frac{G_0 e^{\mu(t)\tau}}{1 - H_0 e^{\mu(t)\tau}} = 0\}$$

Theorem 3. *Suppose $y(t)$ is the solution of (1)-(3) with $\alpha \leq 0$. If there exists $0 < p < 1$ such that*

$$p\alpha + \frac{4}{1 - 2\|N\|^2} (\beta - \alpha\|N\|^2 + \omega\tau_2^2 c^2) \leq 0,$$

and $t \geq t_0$, we have following two results.

(i)

$$\begin{aligned} \|y(t)\|^2 \ &\leq \ \frac{2}{1 - 2\|N\|^2} \frac{-\gamma}{(1 - p)\alpha} \\ &\quad + \frac{1 - 2\|N\|^2}{1 - 2\|N\|^2 e^{\mu^*\tau}} \phi \exp(-\mu^*(t - t_0)) \end{aligned}$$

with $\phi = \max\limits_{t_0-\tau\leq\xi\leq t_0} \|\varphi(t)\|^2$ and

$$\begin{aligned} \mu^* \ &= \ \inf_{t\geq t_0} \{\mu(t) : \mu(t) + \alpha + (\beta - \alpha\|N\|^2 \\ &\quad + \omega\tau_2^2 c^2) \frac{4e^{\mu(t)\tau}}{1 - 2\|N\|^2 e^{\mu(t)\tau}} = 0\} \end{aligned}$$

where $t \geq t_0$ and $\mu^* > 0$.

(ii) For any $\varepsilon > 0$, the problem (1)-(3) is dissipative and there exist a absorbing set

$$B = B(0, \sqrt{\frac{2}{1 - 2\|N\|^2} \frac{-\gamma}{(1 - p)\alpha} + \varepsilon}).$$

Proof: Define functions

$$\begin{aligned} u(t) &= \begin{cases} \|y(t) - Ny(t - \tau_1)\|^2, & t \geq t_0, \\ \frac{1}{2}(1 - 2\|N\|^2)\|\varphi(t)\|^2, & t_0 - \tau \leq t \leq t_0, \end{cases} \\ w(t) &= \|y(t)\|^2, \ \ t \geq t_0 - \tau. \end{aligned}$$

We have

$$\begin{aligned} u'(t) \ &= \ \tfrac{d}{dt} \langle y(t) - Ny(t - \tau_1), y(t) - Ny(t - \tau_1)\rangle \\ &= \ 2Re\langle y(t) - Ny(t - \tau_1), f(t, y(t), y(t - \tau_1), \\ &\qquad \int_{t-\tau_2}^{t} g(t, \xi, y(\xi))d\xi)\rangle \\ &\leq \ 2(\gamma + \alpha\|y(t)\|^2 + \beta\|y(t - \tau_1)\|^2 \\ &\qquad + \omega\|\int_{t-\tau_2}^{t} g(t, \xi, y(\xi))d\xi\|^2) \\ &\leq \ 2(\gamma + \alpha w(t) + \beta w(t - \tau_1) \\ &\qquad + \omega c^2\tau_2^2 \max_{t-\tau_2\leq\xi\leq t} w(\xi)) \end{aligned}$$

and

$$\begin{aligned} u(t) &\leq 2(\|y(t)\|^2 + \|Ny(t-\tau_1)\|^2) \\ &\leq 2(w(t) + \|N\|^2 w(t-\tau_1)), \end{aligned}$$

when $t \geq t_0$. Therefore

$$2w(t) \geq u(t) - 2\|N\|^2 w(t-\tau_1).$$

For $\alpha \leq 0$, we have

$$\begin{aligned} u'(t) &\leq 2\gamma + \alpha u(t) + 2(\beta - \alpha\|N\|^2 \\ &\quad + \omega c^2 \tau_2^2) \max_{t-\tau\leq\xi\leq t} w(\xi), t \geq t_0. \end{aligned} \quad (4)$$

On the other side, we can get

$$\begin{aligned} \|y(t)\| &= \|y(t) - Ny(t-\tau_1) + Ny(t-\tau_1)\| \\ &\leq \|y(t) - Ny(t-\tau_1)\| + \|Ny(t-\tau_1)\|, \end{aligned}$$

which follows that

$$w(t) \leq 2u(t) + 2\|N\|^2 w(t-\tau_1). \quad (5)$$

From (4) and (5), we have

$$\begin{aligned} u'(t) &\leq 2\gamma + \alpha u(t) + B \sup_{t-\tau\leq\xi\leq t} w(\xi), \\ w(t) &\leq 2u(t) + 2\|N\|^2 \sup_{t-\tau\leq\xi\leq t} w(\xi), t \geq t_0, \end{aligned}$$

where $B(t) = 2(\beta - \alpha\|N\|^2 + \omega c^2 \tau_2^2)$. Then we get the conclusion immediately from Generalized Halanay inequality [14].

3 Dissipativity of the θ- methods

3.1 Dissipativity of the one-leg θ- methods

Consider the following ODE

$$\begin{aligned} y'(t) &= f(t, y(t)), \\ y(t_0) &= y_0, t \geq t_0. \end{aligned}$$

It can be solved by the one-leg θ- method leading to the following form

$$\begin{aligned} y_{n+1} = \; & y_n + hf(\theta t_{n+1} + (1-\theta)t_n, \\ & \theta y_{n+1} + (1-\theta)y_n) \end{aligned}$$

Or equivalently,

$$\begin{aligned} Y^{(n)} &= y_n + h\theta f(t_n + \theta h, Y^{(n)}), \\ y_{n+1} &= y_n + hf(t_n + \theta h, Y^{(n)}), \end{aligned} \quad (6)$$

where $h > 0$ is the integration step, $t_n = t_0 + nh$, and y_n, $Y^{(n)}$ approximate to $y(t_n)$, $y(t_n + \theta h)$ respectively.

Applying (6) to (1)-(3), we have

$$\begin{aligned} & Y^{(n)} - N\bar{Y}^{(n)} \\ &= y_n - N\bar{y}_n + h\theta f(Y^{(n)}, \bar{Y}^{(n)}, G^{(n)}), \\ & y_{n+1} - N\bar{y}_{n+1} \\ &= y_n - N\bar{y}_n + hf(Y^{(n)}, \bar{Y}^{(n)}, G^{(n)}), \end{aligned} \quad (7)$$

where y_n, $Y^{(n)}$, $\bar{y}_n$, $\bar{Y}^{(n)}$, $G^{(n)}$ approximate to $y(t_n)$, $y(t_n + \theta h)$, $y(t_n - \tau_1)$, $y(t_n + \theta h - \tau_1)$, $\int_{t_n+\theta h-\tau_2}^{t_n+\theta h} g(t_n + \theta h, \xi, y(\xi))d\xi$ respectively. When $n \leq 0$, we have $y_n = \varphi(t_n)$. And when $t_n + \theta h \leq 0$, we get $Y^{(n)} = \varphi(t_n + \theta h)$. If we let $\tau_1 = (m_1 - \delta_1)h$, $\tau_2 = (m_2 - \delta_2)h$, then $\bar{Y}^{(n)}$, $\bar{y}_n$ and $G^{(n)}$ can be described by interpolation as

$$\bar{Y}^{(n)} = \delta_1 Y^{(n-m_1+1)} + (1-\delta_1)Y^{(n-m_1)}, \quad (8)$$
$$\bar{y}_n = \delta_1 y_{n-m_1+1} + (1-\delta_1)y_{n-m_1}, \quad (9)$$

where m_1, m_2 are integers and $m_1, m_2 \geq 1$, $\delta_1, \delta_2 \in [0, 1)$. When $m_2 \geq 2$, we can obtain

$$\begin{aligned} & G^{(n)} \\ = \; & \frac{h(1-\delta_2)^2}{2} g(t_n + \theta h, t_{n-m_2} + \theta h, Y^{(n-m_2)}) \\ & + \frac{h(2-\delta_2^2)}{2} g(t_n+\theta h, t_{n-m_2+1} + \theta h, Y^{(n-m_2+1)}) \\ & + h\sum_{k=1}^{m_2-2} g(t_n + \theta h, t_{n-k} + \theta h, Y^{(n-k)}) \\ & + \frac{h}{2} g(t_n + \theta h, t_n + \theta h, Y^{(n)}). \end{aligned} \quad (10)$$

While $m_2 = 1$, we get

$$\begin{aligned} G^{(n)} = \; & \frac{\tau_2}{2}[(1-\delta_2)g(t_n + \theta h, t_{n-1} + \theta h, Y^{(n-1)}) \\ & + (1+\delta_2)g(t_n + \theta h, t_n + \theta h, Y^{(n)})]. \end{aligned} \quad (11)$$

Definition 4. *When a method is used to solve the problem (1)-(3) with step h, there is a constant r such that, for any function $\varphi(t)$, there exist an $n_0(\bar{\varphi}, h)$, $\bar{\varphi} = \sup_{-\tau\leq t\leq 0} \|\varphi(t)\|$ satisfying*

$$\|y_n\| \leq r, \quad n \geq n_0.$$

This method is said to be dissipative.

Theorem 5. *Assume that method (7) satisfies $\theta \in [\frac{1}{2}, 1]$, and problem (1)-(3) satisfies $\alpha+\beta+\omega\tau_2^2 c^2 < 0$. Then the method is dissipative.*

Proof: Taking the inner products of each hand side of (7) with themselves, and noting that $\theta \in [\frac{1}{2}, 1]$ and the condition (2), we obtain

$$\begin{aligned}&\|y_{n+1}-N\bar{y}_{n+1}\|^2\\ \leq\ &\|y_n-N\bar{y}_n\|^2+2h[\gamma+\alpha\|Y^{(n)}\|^2\\ &+\beta\|\bar{Y}^{(n)}\|^2+\omega\|G^{(n)}\|^2].\end{aligned}$$

By deducing, we can easily obtain

$$\begin{aligned}&\|y_n-N\bar{y}_n\|^2\\ \leq\ &\|y_0-N\bar{y}_0\|^2+2hn\gamma+2h\alpha\sum_{j=0}^{n-1}\|Y^{(j)}\|^2\\ &+2h\beta\sum_{j=0}^{n-1}\|\bar{Y}^{(j)}\|^2+2h\omega\sum_{j=0}^{n-1}\|G^{(j)}\|^2.\end{aligned} \tag{12}$$

Based on equation (8), we have

$$\begin{aligned}&\|\bar{Y}^{(j)}\|^2\\ \leq\ &\delta_1^2\|Y^{(j-m_1+1)}\|^2+(1-\delta_1)^2\|Y^{(j-m_1)}\|^2\\ &+\delta_1(1-\delta_1)(\|Y^{(j-m_1+1)}\|^2+\|Y^{(j-m_1)}\|^2)\\ =\ &\delta_1\|Y^{(j-m_1+1)}\|^2+(1-\delta_1)\|Y^{(j-m_1)}\|^2.\end{aligned} \tag{13}$$

Notice that

$$\frac{h(1-\delta_2)^2}{2}+\frac{h(2-\delta_2^2)}{2}+(m_2-2)h+\frac{h}{2}=\tau_2.$$

Then, we have

$$\begin{aligned}&\|G^{(n)}\|^2\\ \leq\ &\tau_2(\frac{h(1-\delta_2)^2}{2}\|g(t_n+\theta h,t_{n-m_2}+\theta h,Y^{(n-m_2)}\|^2)\\ +\ &\frac{h(2-\delta_2^2)}{2}\|g(t_n+\theta h,t_{n-m_2+1}+\theta h,Y^{(n-m_2+1)})\|^2\\ +\ &h\sum_{k=1}^{m_2-2}\|g(t_n+\theta h,t_{n-k}+\theta h,Y^{(n-k)})\|^2\\ +\ &\frac{h}{2}\|g(t_n+\theta h,t_n+\theta h.Y^{(n)})\|^2).\end{aligned}$$

when $m_2\geq 2$, by taking the inner product of (14) with itself and using Cauchy-Schwarz inequality. By using (3), we obtain

$$\begin{aligned}&\|G^{(j)}\|^2\\ \leq\ &\tau_2c^2(\frac{h(1-\delta_2)^2}{2}\|Y^{(j-m_2)}\|^2\\ &+\frac{h(2-\delta_2^2)}{2}\|Y^{(j-m_2+1)}\|^2\\ &+h\sum_{k=1}^{m_2-2}\|Y^{(j-k)}\|^2+\frac{h}{2}\|Y^{(j)}\|^2).\end{aligned} \tag{14}$$

Putting (13) and (14) into (12), we have

$$\begin{aligned}&\|y_n-N\bar{y}_n\|^2\\ \leq\ &(1+\|N\|)^2\bar{\varphi}^2+2nh\gamma+(2\tau_1\beta+\omega\tau_2^3c^2)\bar{\varphi}^2\\ &+2h(\alpha+\beta+\omega\tau_2^2c^2)\sum_{j=0}^{n-1}\|Y^{(j)}\|^2.\end{aligned} \tag{15}$$

When $m_2=1$, we can get the same result as (15).

When $\gamma=0$, by using (10) and $\alpha+\beta+\omega\tau_2^2c^2<0$, we have

$$\lim_{n\to\infty}\|Y^{(n)}\|=0,\qquad \lim_{n\to\infty}\|\bar{Y}^{(n)}\|=0,$$

which means for all $\varepsilon>0$, there exists $n_0(\bar{\varphi},\varepsilon)>0$, such that

$$\|Y^{(n)}\|\leq\varepsilon,\|\bar{Y}^{(n)}\|\leq\varepsilon,\|G^{(n)}\|\leq c\tau_2\varepsilon,n\geq n_0.$$

Let

$$L=\sup_{\substack{\|u\|\leq\varepsilon\\ \|v\|\leq\varepsilon\\ \|w\|<c\tau_2\varepsilon}}\|f(u,v,w)\|,\ \ u,v,w\in C^d.$$

We have

$$\begin{aligned}\|y_n-N\bar{y}_n\|\ &\leq\ hL\theta+(1+\|N\|)\varepsilon\\ &:=R_0,n>n_0,\end{aligned}$$

from (7). Then, we can deduce that

$$\begin{aligned}\|y_n\|\ \leq\ &\frac{R_0}{1-\|N\|\delta_1}\sum_{i=0}^{n-n_0-1}(\frac{(1-\delta_1)\|N\|}{1-\|N\|\delta_1})^i\\ &+(\frac{(1-\delta_1)\|N\|}{1-\|N\|\delta_1})^{n-n_0}\varphi_0,\ n\geq n_0.\end{aligned} \tag{16}$$

When $\gamma>0$, using the method of [8, 10], we obtain

$$\begin{aligned}\|y_n-N\bar{y}_n\|^2\ &\leq\ 2(1+(2\tau_1\beta+\omega\tau_2^3c^2)R_0\\ &\quad+4(m_1+1)h\gamma\\ &:=\ R.n\geq n_1\end{aligned}$$

where $n_1=\frac{((1+\|N\|)^2+(2\tau_1\beta+\omega\tau_2^3c^2))\bar{\varphi}^2}{2h\gamma}+2(m_1+1)$. By deducing, we can easily get

$$\begin{aligned}\|y_n\|\ \leq\ &\frac{R}{1-\|N\|\delta_1}\sum_{i=0}^{n-n_0-1}(\frac{(1-\delta_1)\|N\|}{1-\|N\|\delta_1})^i\\ &+(\frac{(1-\delta_1)\|N\|}{1-\|N\|\delta_1})^{n-n_0}\varphi_0,\ n\geq n_0.\end{aligned} \tag{17}$$

Notice that $\|N\|<1$, $\frac{(1-\delta)\|N\|}{1-\|N\|\delta_1}<1$ and $\theta\in[\frac{1}{2},1]$. Obviously, one-leg θ- method is dissipative. This completes the proof of Theorem 4.

3.2 Dissipativity of the linear θ- methods

In this subsection, we consider the dissipativity of the linear θ- method for solving problem (1)-(3). It can be written in the following form

$$\begin{aligned} & y_{n+1} - N\bar{y}_{n+1} \\ = \ & y_n - N\bar{y}_n + h\theta f(y_{n+1}, \bar{y}_{n+1}, G_{n+1}) \\ & +h(1-\theta) f(y_n, \bar{y}_n, G_n), \qquad (18) \end{aligned}$$

where y_n, $\bar{y}_n$, and G_n approximate to $y(t_n)$, $y(t_n - \tau_1)$, and $\int_{t_n-\tau_2}^{t_n} g(t_n, \xi, y(\xi))d\xi$ respectively. When $n \le 0$, we have $y_n = \varphi(t_n)$. If we let $\tau_1 = (m_1 - \delta_1)h, \tau_2 = (m_2 - \delta_2)h$, $\bar{y}_n, G_n$ can be described by interpolation as

$$\bar{y}_n = \delta_1 y_{n-m_1+1} + (1-\delta_1) y_{n-m_1}, \qquad (19)$$

where m_1, m_2 are integers and $m_1, m_2 \ge 1$; $\delta_1, \delta_2 \in [0, 1)$. When $m_2 \ge 2$

$$\begin{aligned} G_n \ = \ & \frac{h(1-\delta_2)^2}{2} g(t_n, t_{n-m_2}, y_{n-m_2}) \\ & + \frac{h(2-\delta_2^2)}{2} g(t_n, t_{n-m_2+1}, y_{n-m_2+1}) \\ & + h \sum_{k=1}^{m_2-2} g(t_n, t_{n-k}, y_{n-k}) \\ & + \frac{h}{2} g(t_n, t_n, y_n). \qquad (20) \end{aligned}$$

While $m_2 = 1$

$$\begin{aligned} G_n \ = \ & \frac{\tau_2}{2}[(1+\delta_2) g(t_n, t_n, y_n) \\ & +(1-\delta_2) g(t_n, t_{n-1}, y_{n-1})]. \qquad (21) \end{aligned}$$

Theorem 6. *Assume that the method (18)-(21) satisfies $\theta \in [\frac{1}{2}, 1]$. The problem (1)-(3) satisfies $\alpha + \beta + \omega\tau_2^2 c^2 < 0$. Then the method is dissipative.*

Proof: From (18), we have

$$\begin{aligned} y_n - N\bar{y}_n \quad & -h\theta f(y_n, \bar{y}_n, G_n) = y_{n-1} - N\bar{y}_{n-1} \\ & +h(1-\theta) f(y_{n-1}, \bar{y}_{n-1}, G_{n-1}). \ (22) \end{aligned}$$

Taking the inner products of each hand side of (22) with themselves, noting that $\theta \in [\frac{1}{2}, 1]$ and condition (2)-(3), we obtain

$$\begin{aligned} & \|y_n - N\bar{y}_n\|^2 + h^2\theta^2 \|f(y_n, \bar{y}_n, G_n)\|^2 \\ & \le \|y_{n-1} - N\bar{y}_{n-1}\|^2 \\ & \quad + h^2\theta^2 \|f(y_{n-1}, \bar{y}_{n-1}, G_{n-1})\|^2 \\ & \quad + 2h\gamma + 2h\theta(\alpha\|y_n\|^2 + \beta\|\bar{y}_n\|^2 + \omega\|G_n\|^2) \\ & \quad + 2h(1-\theta)(\alpha\|y_{n-1}\|^2 + \beta\|\bar{y}_{n-1}\|^2 \\ & \quad + \omega\|G_{n-1}\|^2). \end{aligned}$$

By deducing we can easily obtain

$$\begin{aligned} & \|y_n - N\bar{y}_n\|^2 + h^2\theta^2 \|f(y_n, \bar{y}_n, G_n)\|^2 \\ & \le \|y_0 - N\bar{y}_0\|^2 + h^2\theta^2 \|f(y_0, \bar{y}_0, G_0)\|^2 \\ & \quad + 2nh\gamma + 2h\alpha \sum_{j=1}^{n-1} \|y_j\|^2 \\ & \quad + 2h\beta \sum_{j=1}^{n-1} \|\bar{y}_j\|^2 + 2h\omega \sum_{j=1}^{n-1} \|G_j\|^2 \\ & \quad + 2h\theta\alpha\|y_n\|^2 + 2h(1-\theta)\alpha\|y_0\|^2 \\ & \quad + 2h\theta\beta\|\bar{y}_n\|^2 + 2h(1-\theta)\beta\|\bar{y}_0\|^2 \\ & \quad + 2h\theta\omega\|G_n\|^2 + 2h(1-\theta)\omega\|G_0\|^2. \quad (23) \end{aligned}$$

Next, from (19) and (20) we obtain

$$\|\bar{y}_j|^2 \le \delta_1 \|y_{j-m_1+1}\|^2 + (1-\delta_1)\|y_{j-m_1}\|^2, \quad (24)$$

$$\begin{aligned} & \|G_n\|^2 \\ \le \ & \tau_2 c^2 (\frac{h(1-\delta_2)^2}{2} \|y_{n-m_2}\|^2 \\ & + \frac{h(2-\delta_2^2)}{2} \|y_{n-m_2+1}\|^2 \\ & + h \sum_{k=1}^{m_2-2} \|y_{n-k}\|^2 + \frac{h}{2}\|y_n\|^2). \qquad (25) \end{aligned}$$

When $m_2 \ge 2$, from (23)-(25) we have

$$\begin{aligned} & \|y_n - N\bar{y}_n\|^2 + h^2\theta^2 \|f(y_n, \bar{y}_n, G_n)\|^2 \\ & \le 2h(\alpha + \beta + \omega\tau_2^2 c^2) \sum_{j=0}^{n-1} \|y_j\|^2 \\ & \quad + L_0 + 2hn\gamma + [2\beta\tau_1 + \omega\tau_2^3 c^2 \\ & \quad + 2h(1-\theta)(\beta + \omega\tau_2^2 c^2)]d_0, \qquad (26) \end{aligned}$$

where

$$L_0 = \sup_{\substack{\|u\| \le \sqrt{d_0} \\ \|v\| \le \sqrt{d_0} \\ \|w\| < \tau_2 c \sqrt{d_0}}} \left(\|u - Nv\|^2 + h^2\theta^2 \|f(u, v, w)\|^2\right),$$

$$d_0 = \bar{\varphi}^2.$$

We can easily prove that (26) is still valid for the case of $m_2 = 1$.

When $\gamma = 0$, it follows from (28) and $\alpha + \beta + \omega\tau_2^2 c^2 < 0$ that

$$\|y_n\| < \varepsilon, n \ge n_0$$

which shows that for any $\varepsilon > 0$, there exists $n_0(\bar{\varphi}, \varepsilon)$, such that

$$\|y_n\| < \varepsilon, \ n \ge n_0. \qquad (27)$$

For the case of $\gamma > 0$, using the techniques similar to that presented in [8], we can conclude that there are $\widetilde{r} > 0$ and positive integer $\widetilde{n_0} > 0$ such that

$$\|y_n - N\bar{y}_n\|^2 \leq \widetilde{r}, n \geq \widetilde{n_0} \tag{28}$$

and

$$h^2\theta^2\|f(y_n, \bar{y}_n, G_n)\|^2 \leq \widetilde{r}, n \geq \widetilde{n_0},$$

where

$$\begin{cases} \widetilde{r} = 2\Big(L_1 + [2\beta\tau_1 + \omega\tau_2^3 c^2 \\ +2h(1-\theta)(\beta + \omega\tau_2^2 c^2)])d_1\Big) + 4\tau_1\gamma + 6h\gamma \\ \widetilde{n_0} = \frac{L_0 + [2\beta\tau_1 + \omega\tau_2^3 c^2 + 2h(1-\theta)(\beta + \omega\tau_2^2 c^2)])d_0}{2h\gamma} \\ +2m_1 + 1 \\ d_1 = \frac{4(m_1+1)\gamma}{-\alpha+\beta+\omega\tau_2^2 c^2} \\ L_1 = \sup\limits_{\substack{\|u\|\leq\sqrt{d_1} \\ \|v\|\leq\sqrt{d_1} \\ \|w\|<\tau_2 c\sqrt{d_1}}} \left(\|u-Nv\|^2 + h^2\theta^2\|f(u,v,w)\|^2\right). \end{cases}$$

When $m_1 = 1$,from (18) and (28) we obtain

$$\begin{aligned} \|y_n\| &\leq \sqrt{\widetilde{r}} + \|N(\delta_1 y_n + (1-\delta_1)y_{n-1})\|, \\ & \quad n \geq \widetilde{n_0}. \end{aligned} \tag{29}$$

From (29), we deduce that

$$\begin{aligned} \|y_n\| &\leq \frac{\sqrt{\widetilde{r}}}{1-\delta_1\|N\|} + \frac{(1-\delta_1)\|N\|}{1-\delta_1\|N\|}\|y_{n-1}\|, \\ & \quad n \geq \widetilde{n_0}. \end{aligned}$$

Then, we have

$$\|y_n\| \leq d_1 + \frac{\sqrt{\widetilde{r}}}{1-\|N\|}, n \geq n_0. \tag{30}$$

When $m_1 \geq 2$, from (18) and (28) we have

$$\begin{aligned} \|y_n\| &\leq \sqrt{\widetilde{r}} + \delta_1\|N\|\|y_{n-m_1+1}\| \\ & \quad +(1-\delta_1)\|N\|\|y_{n-m_1}\|, n \geq \widetilde{n_0}. \end{aligned}$$

By deducing, we can prove that (30) is still valid. A combination of (30) and (27) shows that the method is dissipative, which completes the proof of Theorem 5.

4 Dissipativity of the one-leg methods

4.1 Description of the one-leg methods

One-leg method for solving

$$\begin{aligned} y'(t) &= f(t, y(t)), \\ y(t_0) &= y_0, t \geq t_0. \end{aligned}$$

is

$$\rho(E)y_n = hf(\sigma(E)t_n, \sigma(E)y_n),$$

where E denotes the shift operator, $Ey_n = y_{n+1}$. Polynomial

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_i \xi^i, \ \ \sigma(\xi) = \sum_{j=0}^{k} \beta_i \xi^i$$

satisfy compatibility condition

$$\rho(1) = 0, \rho'(1) = \sigma(1) = 1, \rho(\xi), \sigma(\xi),$$

and they have no common factor.

In order to solve (1)-(3) ,we let $h = \frac{\tau_1}{m_1}, \tau_2 = (m_2 - \delta_2)h$ throughout of this section 4, where $\delta_2 \in [0, 1)$ and $m_1 \geq k, m_2 > 1$ are given positive constant. By complexification quadrature formula ,we obtain

$$\begin{aligned} & \rho(E)(y_n - Ny_{n-m_1}) \\ & = hf(\sigma(E)y_n, \sigma(E)y_{n-m_1}, G_n), \end{aligned} \tag{31}$$

where y_n, G_n approximate to $y(t_n)$ and $\int_{\sigma(E)t_n-\tau_2}^{\sigma(E)t_n} g(t_n, s, y(s))ds$ respectively. If $t_n \in [-\tau, 0]$, we can get $y_n = \varphi(t_n)$. Define

$$\begin{aligned} G_n = \ & \frac{h(1-\delta_2)^2}{2} g(\sigma(E)t_n, \sigma(E)t_{n-m_2}, \\ & \sigma(E)y_{n-m_2}) + \frac{h(2-\delta_2^2)}{2} g(\sigma(E)t_n, \\ & \sigma(E)t_{n-m_2+1}, \sigma(E)y_{n-m_2+1}) \\ & + h\sum_{k=1}^{m_2-2} g(\sigma(E)t_n, \sigma(E)t_{n-k}, \sigma(E)y_{n-k}) \\ & + \frac{h}{2} g(\sigma(E)t_n, \sigma(E)t_n, \sigma(E)y_n). \end{aligned}$$

Definition 7. *A method is said to be G-stable, if there exist a symmetric positive definite matrix* $G = (g_{ij})_{i,j=1}^k$, *for all* $x_0, \ldots, x_k \in R$ *, such that*

$$X_1^T G X_1 - X_0^T G X_0 \leq 2\sigma(E)x_0\rho(E)x_0,$$

where $X_i = (x_i, \ldots, x_{i+k-1})^T, i = 0, 1.$

By [22] ,we know G-stable is equivalent to A - stable.

4.2 Dissipativity of the one-leg methods

In order to prove our main conclusion , we introduce a lemma [23].

Lemma 8. *Suppose that the polynomial*

$$P_i(z) = \sum_{j=1}^{r} p_{ij} z^{j-1}, i = 1, \dots, r$$

form a basis for the space of polynomials of degree not more than $r-1$ and that $\delta = (\delta_1^T, \dots, \delta_r^T)^T \in R^{pr}$ is given. Then the system of equations

$$\sum_{j=1}^{r} p_{ij} u_j = \delta_i, i = 1, \dots, r$$

possesses a unique solution $u = (u_1^T, \dots, u_r^T)^T$. Furthermore, if P is the $r \times r$ matrix with entries p_{ij}, then there is a constant $C = C(P)$ such that any $u = (u_1^T, \dots, u_r^T)^T \in R^{pr}$ obeying the inequality

$$\sum_{j=1}^{r} \|p_{ij} u_j\| \le \|g_i\|$$

for all i, also satisfies

$$\|u\|_{*,\infty} \le C \|g\|_{*,\infty},$$

where $g = (g_1^T, \dots, g_r^T)^T \in R^{pr}$ and $\|u\|_{*,\infty} = \max_{1 \le i \le r} \|u_i\|$.

For simplicity, we suppose that (31) has unique solution for any sufficiently small step h.

Theorem 9. *Assume the method (31) is A-stable and the problem (1)-(3) satisfies $\alpha + \beta + \omega\tau_2^2 c^2 < 0$. Then the method is dissipative.*

Proof: Let $\{y_n\}_{n=0}^{\infty}$ are a sequence of solutions. Define G-norm of H_n is

$$\|H_n\|_G^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} g_{ij} \langle y_{n+i-1}, y_{n+j-1} \rangle$$

where $H_n = (y_n^T, \dots, y_{n+k-1}^T)^T$ and $Z_n = H_n - NH_{n-m_1}$. From the definition of G-stable we have

$$\begin{aligned} &\|Z_{n+1}\|_G^2 - \|Z_n\|_G^2 \\ &\le 2Re\langle \rho(E)(y_n - Ny_{n-m_1}), \\ &\quad \sigma(E)(y_n - Ny_{n-m_1}) \rangle \\ &= 2hRe\langle f(\sigma(E)y_n, \sigma(E)y_{n-m_1}, G_n), \\ &\quad \sigma(E)(y_n - Ny_{n-m_1}) \rangle \\ &\le 2h(\gamma + \alpha \|\sigma(E)y_n\|^2 \\ &\quad + \beta \|\sigma(E)y_{n-m_1}\|^2 + \omega \|G_n\|^2). \end{aligned} \tag{32}$$

When $\gamma \neq 0$, we define

$$F = \sup_{\substack{\|u\|^2, \|v\|^2 \le 4k\gamma/-(\alpha+\beta+\omega\tau_2^2 c^2) \\ \|w\|^2 \le \tau_2^2 c^2 4k\gamma/-(\alpha+\beta+\omega\tau_2^2 c^2)}} \|f(u, v, w)\|$$

and

$$\bar{\varphi} = \sup_{-m \le i \le 0} \|\sigma(E) y_i\|.$$

Let

$$\begin{aligned} s = \quad & \lfloor \frac{-\tau_1\gamma + 2\beta\tau_1 + \omega\tau_2^3 c^2}{2k\tau_1\gamma} \\ & - \frac{\tau_1(\alpha + \beta + \omega\tau_2^2 c^2))\bar{\varphi}^2 + \|Z_0\|_G^2}{2k\tau_1\gamma} \rfloor + 1, \end{aligned}$$

and

$$M = km_1 s,$$

where $\lfloor x \rfloor$ means the largest integer part of x. The sum of (32) from $n = 1$ to $n = M - 1$ is

$$\begin{aligned} & \|Z_M\|_G^2 - \|Z_0\|_G^2 \\ \le \; & 2Mh\gamma + 2h\alpha \sum_{i=0}^{M-1} \|\sigma(E)y_i\|^2 \\ + \; & 2h\beta \sum_{i=0}^{M-1} \|\sigma(E)y_{i-m_1}\|^2 + 2h\omega \sum_{i=0}^{M-1} \|G_i\|^2 \end{aligned} \tag{33}$$

According to (3) and (31), we can get

$$\begin{aligned} & \|G_n\|^2 \\ \le \; & \tau_2 c^2 (\frac{h(1-\delta_2)^2}{2} \|\sigma(E)y_{n-m_2}\|^2 \\ & + \frac{h(2-\delta_2^2)}{2} \|\sigma(E)y_{n-m_2+1}\|^2 \\ + \; & h \sum_{k=1}^{m_2-2} \|\sigma(E)y_{n-k}\|^2 + \frac{h}{2} \|\sigma(E)y_n\|^2). \end{aligned} \tag{34}$$

Putting (34) into (33), we have

$$\begin{aligned} & \|Z_M\|_G^2 - \|Z_0\|_G^2 \\ \le \; & 2Mh\gamma + (2\beta\tau_1 + \omega\tau_2^3 c^2)\bar{\varphi}^2 \\ & + 2h(\alpha + \beta + \omega\tau_2^2 c^2) \sum_{i=0}^{M-1} \|\sigma(E)y_i\|^2. \end{aligned} \tag{35}$$

Similarly, the sum of (32) from 0 to $M - m_1$ is

$$\begin{aligned} & \|Z_{M-m_1}\|_G^2 - \|Z_0\|_G^2 \\ & \le [2\beta\tau_1 + \omega\tau_2^3 c^2 - 2\tau_1(\alpha + \beta \\ & + \omega\tau_2^2 c^2)]\bar{\varphi}^2 + 2(M - m_1)h\gamma \\ & + 2h(\alpha + \beta + \omega\tau_2^2 c^2) \sum_{i=-m_1}^{M-m_1-1} \|\sigma(E)y_i\|^2. \end{aligned} \tag{36}$$

Therefore, by (35) and (36), we obtain

$$\begin{aligned}
&\|Z_{M-m_1}\|_G^2 + \|Z_M\|_G^2 \\
&\le 2\|Z_0\|_G^2 + (4M - 2m_1)h\gamma \\
&\quad +2[2\beta\tau_1 + \omega\tau_2^3 c^2 \\
&\quad -\tau_1(\alpha+\beta+\omega\tau_2^2c^2)]\bar{\varphi}^2 \\
&\quad +2h(\alpha+\beta+\omega\tau_2^2c^2) \\
&\times \sum_{i=0}^{M-1}(\|\sigma(E)y_i\|^2 + \|\sigma(E)y_{i-m_1}\|^2),
\end{aligned}$$

which contains

$$\begin{aligned}
&\sum_{i=0}^{M-1}(\|\sigma(E)y_i\|^2 + \|\sigma(E)y_{i-m_1}\|^2) \\
&\le \frac{4M\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)} \\
&= \frac{4km_1s\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}. \qquad (37)
\end{aligned}$$

We rewrite (37) to the following form

$$\begin{aligned}
&\sum_{i=0}^{s-1}\sum_{j=m_1i}^{m_1(i+1)-1}\sum_{l=kj}^{k(j+1)-1}(\|\sigma(E)y_l\|^2+\|\sigma(E)y_{l-m_1}\|^2) \\
&\le \frac{4km_1s\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}. \qquad (38)
\end{aligned}$$

So there exist an $i_0 \in [0, s-1]$, such that

$$\begin{aligned}
&\sum_{j=m_1i_0}^{m_1(i_0+1)-1}\sum_{l=kj}^{k(j+1)-1}(\|\sigma(E)y_l\|^2 + \|\sigma(E)y_{l-m_1}\|^2) \\
&\le \frac{4km_1\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}.
\end{aligned}$$

Furthermore, there exist a $j_0 \in [m_1i_0, m_1(i_0+1)-1]$, such that

$$\begin{aligned}
&\sum_{l=kj_0}^{k(j_0+1)-1}(\|\sigma(E)y_l\|^2 + \|\sigma(E)y_{l-m_1}\|^2) \\
&\le \frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}.
\end{aligned}$$

Therefore , we have

$$\|\sigma(E)y_n\|^2 + \|\sigma(E)y_{n-m_1}\|^2 \le \frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}$$

and

$$\begin{aligned}
&\|\rho(E)(y_n - Ny_{n-m_1})\|^2 \\
&= h^2\|f(\sigma(E)y_n, \sigma(E)y_{n-m_1}, G_n,)\|^2 \\
&\le h^2F^2
\end{aligned}$$

for all $n \in [kj_0, k(j_0+1)-1]$. Based on the fact that the method is A-stable and $\rho(E)$, $\sigma(E)$ have no common factor, we conclude that $\sigma(E)$ is a polynomial with degree k. Then, $\{z^i\rho(z), z^i\sigma(z), i = 0,\ldots,k-1\}$ constitute a basis for the space of polynomials of degree not more than $2k-1$. Let $r = 2k$, $y_j = (y_{kj_0},\ldots,y_{k(j_0+2)-1})^T$ and

$$\|g_i\| \le \eta, i = 0,\ldots,2k-1,$$

where

$$\eta = \max\{\sqrt{\frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}}, hF\}.$$

By using lemma 7 on (36) and (37), we find out there exist a constant C_1 and a coefficient which depends only on $\rho(z)$, $\sigma(z)$, such that

$$\|y_n - Ny_{n-m_1}\| \le C_1\eta.$$

Furthermore , we can prove that $\|H_{ki_0}\| \le R_0$ when $kj_0 \le M$.

Next we will prove that $\{H_n\}_{n=0}^{\infty}$ enter an open ball $B(0, R_0)$ after at least $\widetilde{M}$ steps, where

$$\widetilde{s} = \lfloor\frac{-\tau_1\gamma + R^2 + 2\tau_1(\beta+\omega\tau_2^2c^2)\frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}}{2k\tau_1\gamma}\rfloor + 1$$

and

$$\widetilde{M} = km_1\widetilde{s}.$$

Let $N = kj_0$. From (38), we have

$$\begin{aligned}
&\sum_{i=0}^{\widetilde{s}-1}\sum_{j=m_1i}^{m_1(i+1)-1}\sum_{l=kj}^{k(j+1)-1}(\|\sigma(E)y_{N+l}\|^2 \\
&+\|\sigma(E)y_{N+l-m_1}\|^2) \\
&\le \frac{4km_1\widetilde{s}\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}.
\end{aligned}$$

So there exist an $j_1 \in [m_1i_1, m_1(i_1+1)-1]$, such that

$$\begin{aligned}
&\sum_{l=kj_1}^{k(j_1+1)-1}(\|\sigma(E)y_{N+l}\|^2+\|\sigma(E)y_{N+l-m_1}\|^2) \\
&\le \frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}.
\end{aligned}$$

Furthermore, there exist a $j_1 \in [m_1i_1, m_1(i_1+1)-1]$ such that

$$\begin{aligned}
&\sum_{l=kj_1}^{k(j_1+1)-1}(\|\sigma(E)y_{N+l}\|^2 + \|\sigma(E)y_{N+l-m_1}\|^2) \\
&\le \frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}.
\end{aligned}$$

Therefore , we have

$$\|\sigma(E)y_n\|^2 + \|\sigma(E)y_{n-m_1}\|^2 \le \frac{4k\gamma}{-(\alpha+\beta+\omega\tau_2^2c^2)}$$

and

$$\begin{aligned}&\|\rho(E)(y_n - Ny_{n-m_1})\|^2\\ &= h^2\|f(\sigma(E)y_n, \sigma(E)y_{n-m_1}, G_n)\|^2\\ &\le h^2F^2\end{aligned}$$

for all $n \in [N + kj_1, N + k(j_1 - 1)]$. Therefore $H_n - NH_{n-m_1}$ will enter an open ball $B(0, \widetilde{R})$ after at least $N + kj_1$ steps, where

$$\|H_n - NH_{n-m_1}\|_G \le \widetilde{R}$$

for all $n \in [N, N + kj_1]$.

By the similar method in proving the Theorem 4 and 5, we obtain

$$\|H_n\| \le \widetilde{R_1}. \tag{39}$$

When $\gamma = 0$, we have

$$\lim_{n\to\infty} \|\sigma(E)y_n\| = 0$$

according to (34) and $\alpha+\beta+\omega\tau_2^2c^2 < 0$. This reveals that there exist an $n_0(\bar{\varphi}, \varepsilon) > 0$, such that

$$\|\sigma(E)y_n\| \le \varepsilon.$$

For $\varepsilon > 0$, if we let

$$\widetilde{F}_0 = \sup_{\substack{\|u\|,\|v\|\le\varepsilon\\ \|w\|\le\tau_2c\varepsilon}} \|f(u, v, w)\|$$

then

$$\begin{aligned}&\|\rho(E)(y_n - Ny_{n-m_1})\|\\ &= h\|f(\sigma(E)y_n, \sigma(E)y_{n-m_1}, G_n)\|\\ &\le h\widetilde{F}_0.\end{aligned}$$

By deducing, we know that there exist a constant $\widetilde{C}_1$ which depends only on the method itself, such that

$$\|y_n - Ny_{n-m_1}\| \le \widetilde{C}_1\varepsilon$$

for all $n \ge n_0$. Hence, according to the equivalence principle of the norm, there exist a constant $\widetilde{C}_2$, such that

$$\|H_n - NH_{n-m_1}\|_G \le \widetilde{C}_2\varepsilon.$$

Furthermore, we obtain

$$\|H_n\|_G \le \widetilde{C}'_2\varepsilon. \tag{40}$$

In summary, (39) and (40) show that the one-leg method is dissipative, which completes the proof of Theorem 7.

Remark Problem (1)-(3) is a very general one. (i) Gan [16] obtained the dissipativity of θ-methods for (VDIDEs) which is a special case of problem (1)-(3) by taking $N = 0$ and $\tau_1 = \tau_2$. Theorem 5 and 6 can be directly applied to such problems. (ii) When equation (1) does not contain integral item, problem (1)-(3) is reduced to NDDEs which was considered in [19]. Theorem 5 and 6 can also be directly applied to such problems. (iii) When $N = 0$ and equation (1) does not contain integral item, problem (1)-(3) is equal to (DDEs). From the result of our paper, we can get the same conclusion as Huang[8].

5 Numerical examples

In this section ,we will consider a given nonlinear neutral delay integro-differential equation

$$\begin{aligned}&\frac{d}{dt}[y_1(t) - 0.1y_2(t-1)]\\ &= -4y_1(t) + \sin y_2(t)\sin y_1(t-1)\\ &\quad +0.6\int_{t-2}^{t} \sin ty_1(\theta)d\theta + \sin t, t \ge 0,\\ &\frac{d}{dt}[y_2(t) - 0.2y_1(t-1)]\\ &= -5y_2(t) - \cos y_1(t)\cos y_2(t-1)\\ &\quad +0.4\int_{t-2}^{t} \cos ty_2(\theta)d\theta + \cos t, t \ge 0,\\ &y_1(t) = \sin t,\\ &y_2(t) = \cos t, -2 \le t \le 0.\end{aligned} \tag{41}$$

Here and later, we let $\gamma = 2.15$, $\alpha = -3$, $\beta = 0.64$, $\omega = 0.33$, $c = 1$, $\tau_2 = 2$, so the equation (41) satisfies (2) and (3) with $\alpha + \beta + \omega\tau_2^2c^2 < 0$.

First, we use one-leg θ-method (11) to solve (41) where we take θ=1, $h = 0.05$. From figures 1 and 2, we can see that the solution is oscillating, but it enters into an absorbing set after some time. This reveals the method is dissipative.

Second, we use linear θ-method (18) to solve (41) where we take θ=0.5, $h = 0.05$. From figures 3 and 4, we can see that the method is also dissipative.

At last, we use one-leg method as follows

$$\begin{aligned}&\frac{3}{2}(y_{n+2} - N\bar{y}_{n+2})\\ =\ &2(y_{n+1} - N\bar{y}_{n+1}) - \frac{1}{2}(y_n - N\bar{y}_n)\\ &+hf(y_{n+2}, \bar{y}_{n+2}, G_n)\end{aligned}$$

to solve (41) where $h = 0.05$. Figures 5 and 6 show the dissipative of the method again.

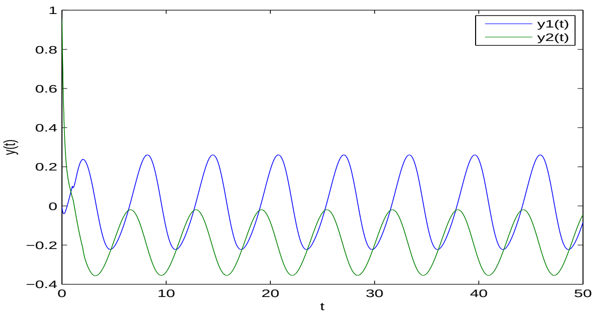

Figure 1: The numerical solution curve of one-leg θ m method while $h = 0.05$ and $\theta = 1$

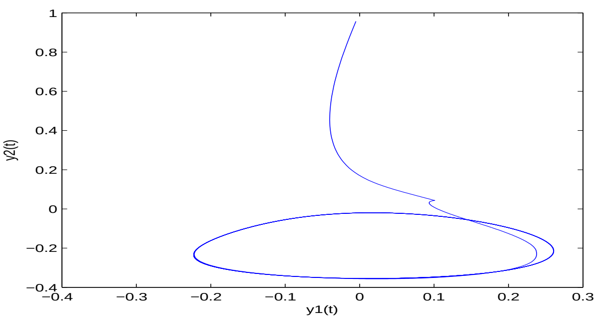

Figure 2: The phase space curve of one-leg θ- m method while $h = 0.05$ and $\theta = 1$

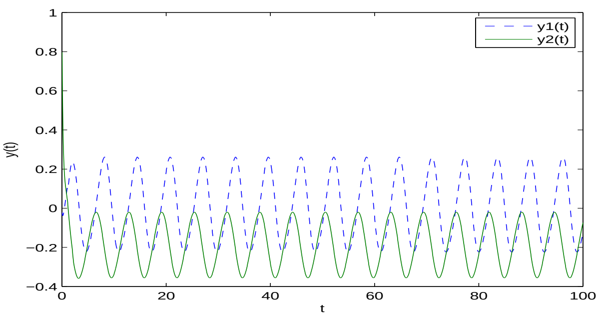

Figure 3: The numerical solution curve of linear θ- m method while $h = 0.01$ and $\theta = 0.5$

6 Conclusion

In this paper, we discuss a special class of nonlinear neutral delay integro-differential equations which has

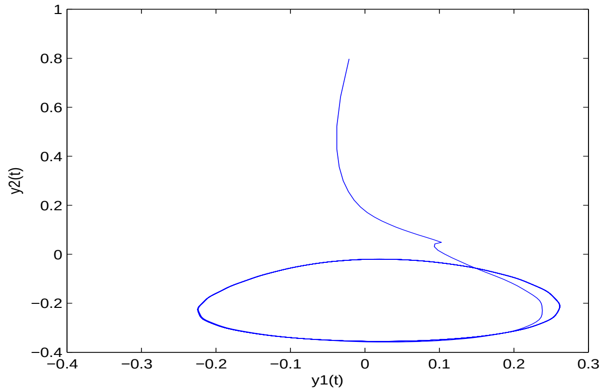

Figure 4: The phase space curve of linear θ- method while $h = 0.01$ and $\theta = 0.5$

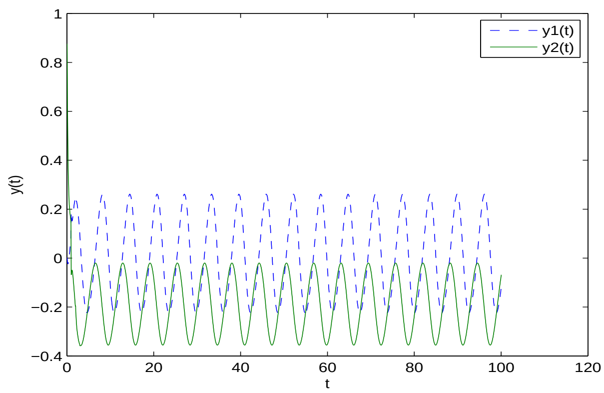

Figure 5: The numerical solution curve of one-leg method while $h = 0.05$

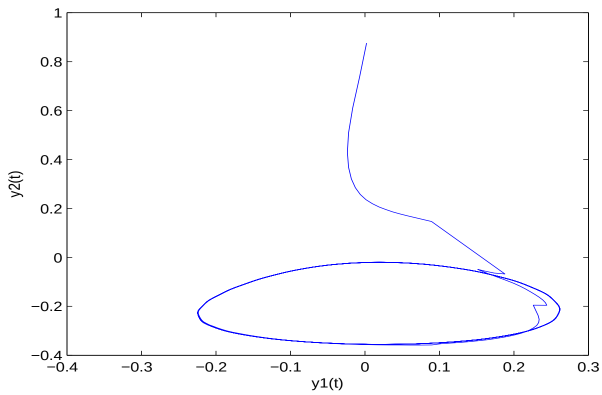

Figure 6: The phase space curve of one-leg method while $h = 0.05$

two different constant delay variables. At first, we prove the dissipativity of the problem itself. Then, we prove that one-leg θ-methods, linear θ-methods, one-leg methods are dissipative with some given conditions when they are used to solve our problem.

Our present work has mainly focused on these three methods. In fact, it could also be extend to other numerical methods, such as Runge-kutta methods, multi-step methods. This research is ongoing.

References:

[1] J. K. Hale, *Theory of functional differential equations*, Springer, New York, 1977.

[2] K. Lorenz, Deterministic nonperiodic flow, *J.Atmosphericb Sci.*, 20, 1963, pp. 130–141.

[3] A. R. Humphries, A. M. Stuart, Runge-Kutta methods for dissipative and gradient dynamical systems, *SIAM J.Numer.Anal.*, 31, 1994, pp. 1452–1485.

[4] A. T. Hill, Global dissipativity for A-stable methods, *SIAM J. Numer. Anal.*, 34, 1997, pp. 119–142.

[5] A. T. Hill, Dissipativity of Runge-Kutta methods in Hilbert spaces, *BIT*, 37, 1997, pp. 37–42.

[6] A. G. Xiao, Dissipativity of general linear methods, *Numer. Math. A: J. Chinese Univ.*, 18, 1996, pp. 183–189 (in Chinese).

[7] A. G. Xiao, Dissipativity of general linear methods for dissipative dynamical systems in Hilbert spaces, *Math. Numer. Sinica*, 22, 2000, pp. 429–436 (in Chinese).

[8] C. M. Huang, Dissipativity of Runge-Kutta methods for dynamical systems with delays, *IMA J. Numer. Anal.*, 20, 2000, pp. 153–166

[9] C. M. Huang, G. N. Chen, Dissipativity of θ-methods for delay dynamical systems, *Chinese J. Numer. Math. Appl.*, 23, 2001, pp. 108–114.

[10] C. M. Huang, Dissipativity of one-leg methods for dynamical systems with delays, *Appl. Numer. Math.*, 35, 2000, pp. 11–22.

[11] C. M. Huang, Q.S. Chang, Dissipativity of multistep RungeCKutta methods for dynamical systems with delays, *Math. Comput. Modell.*, 40 , 2004, pp. 1285–1296.

[12] H. J. Tian, Numerical and analytic dissipativity of the θ-method for delay differential equation with a bounded variable lag, *Int. J. Bifurcat. Chaos*, 14 ,2004, pp. 1839–1845.

[13] L. P. Wen, S. F. Li, Dissipativity of Volterra functional differential equations, *J. Math. Anal. Appl.*, 324 , 2006, pp. 696–706.

[14] L. P. Wen, W. S. Wang, Y. X. Yu, Dissipativity and asymptotic stability of nonlinear neutral delay-integro-differential equations, *Nonlinear Anal-Theory.*, 72, 2010, pp 1746-1754.

[15] S. Q. Gan, Dissipativity of linear θ-methods for integro-differential equations,*Comput. Math. Appl.*, 52, 2006 , pp 449–458.

[16] S. Q. Gan, Dissipativity of θ-methods for nonlinear Volterra delay-integro-differential equations, *Comput. Math. Appl.*, 206 , 2007, pp 898–907.

[17] S. Q. Gan, Exact and discretized dissipativity of the pantograph equation,*Comput. Math. Appl.*, 25 , 2007, pp 81–88.

[18] Z. Cheng, C. M. Huang, Dissipativity for nonlinear neutral delay differential equations,*J. Syst. Simul.*, 19 , 2007, pp 3184–3187 (in Chinese).

[19] L. P. Wen, W. S. Wang, Y. X. Yu, Dissipativity of θ-methods for a class of nonlinear neutral delay differential equations, *Appl. Math. Comput.*, 202 , 2008, pp 780–786.

[20] W. S. Wang, S. F. Li, Dissipativity of RungeCKutta methods for neutral delay differential equations with piecewise constant delay, *Appl. Math. Lett.* , 21 ,2008, pp 983–991.

[21] S. F. Wu, S. Q. Gan, Analytical and numerical stability of neutral delay integro-differential equations and neutral delay partial differential equations, *Comput. Math. Appl.*, 55 ,2008, pp 2426–2443.

[22] S. F. Li, *Algorithm theory of stiff differential equations*, Hunan Science and Technology, Chang sha, 1977.

[23] H. Tian, L. Fan, J. Xiang, Numerical dissipativity of multistep methods for delay differential equations, *Appl. Math. Comput.*, 188, 2007, pp 934–941.