A Numerical Investigation of Blow-up in The Moving Heat Source Problems in Two-dimensions

Hancan Zhu
Shaoxing University
Department of Mathematics
West Ring Road 508, Shaoxing
China
Corresponding author: hczhu@usx.edu.cn

Kewei Liang
Zhejiang University
Department of Mathematics
Zheda Road 38, Hangzhou
China
matlkw@zju.edu.cn

Abstract: The temperature of a combustible material will rise or even blow up when a heat source moves across it. In this paper, we study the blow-up phenomenon in this kind of moving heat source problems in two-dimensions. First, a two-dimensional heat equation with a nonlinear source term is introduced to model the problem. The nonlinear source is localized around a circle which is allowed to move. By using the coordinate transformation, the equation is simplified to a one-dimensional one. Then it is solved by the moving collocation method. The numerical results show that the blow-up occurs if the speed of the heat source is slow, and the blow-up is avoided when the heat source moves fast enough.

Key–Words: Moving heat source, Blow-up, Moving mesh method, Reaction-diffusion equation, Moving collocation method, Local absorbing boundary conditions

1 Introduction

When a laster beam moves across a combustible material, the temperature will rise or even blow up. This kind of moving heat source problems in one-dimension have been well studied both by analytical methods and numerical methods [1, 2, 3, 4]. These studies show that the speed of the heat source will influence the blow-up. When the heat source moves fast enough, the blow-up will be prevented. In [4], the influences of the distance of two heat sources in occurrence of blow-up are also investigated.

In one-dimensional problems, the nonlinear heat source is modeled by a delta-function which means the source is extremely intense. While the point source delta-function model is appropriate in one-dimensional problems, it is not for high-dimensional problems [5, 6]. The blow-up will always happen if the point source is used in high-dimensions. This is meaningless from the physical viewpoint. Kirk [6] developed a new mathematical modeling in which the source is localized in a small bounded and convex domain. Then the influence of the velocity, the size and the strength of the source is investigated by using the analytical method.

The objective of the present work is to investigate the influence of the velocity of the heat source on the blow-up in two-dimensions by using the numerical method. The problem is modeled by the following reaction-diffusion equation

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)T(x,y,t) = \\
&\qquad D(x,y|x_0,y_0)F[T(x_0,y_0,t)], \\
&\qquad -\infty < x,y < \infty,\ t > 0, && (1)\\
&T(x,y,0) = T_0(x,y) \geq 0,\ x,y \in \mathbb{R}, && (2)\\
&T(x,y,t) \to 0 \ \text{ as } x^2+y^2 \to \infty,\ t>0, && (3)
\end{aligned}$$

with

$$\begin{aligned}
&D(x,y|x_0,y_0)\\
= \ &\frac{1}{\epsilon}\max\left(0, \frac{1}{\epsilon}[\epsilon - \left|\sqrt{x^2+y^2} - \sqrt{x_0^2+y_0^2}\right|]\right), \quad (4)
\end{aligned}$$

where ϵ is a parameter which indicates the width of the source. In the above model, $T(x,y,t)$ denotes the temperature at any point (x,y), and (x_0,y_0) is the location of the traveling heat source satisfied $\sqrt{x_0(t)^2 + y_0(t)^2} = r_0(t)$. The initial temperature $T_0(x,y)$ is continuous and bounded with $T_0(x,y) \to 0$ as $x^2+y^2 \to \infty$. The nonlinear source function $F(T)$ is smooth and satisfy,

$$F^{(k)}(T) > 0, \quad k = 0,1,2, \text{ for } T > 0.$$

The support for the localization function $D(x,y|x_0,y_0)$ which defines the shape and magnitude of the localized source is not convex. This is different from the model in paper [6]. To our best knowledge, there are not any theoretical or numerical results for this kind of moving heat source problems. However, the numerical simulation of the model (1)-(4) is difficult because of the moving singularity, the blowup phenomenon and the unbounded spatial domain in $\mathbb{R}^2$.

The moving mesh partial differential equation (MMPDE) method is very efficient to solve these singular equations [7, 8]. It has been applied in a few blow-up problems in bounded one-dimensional domain [3, 4, 9, 10, 11, 12, 14, 15]. In paper [3], Ma etc. use one of the moving mesh partial differential equations, MMPDE6, to compute a one-dimensional reaction-diffusion equation with a heat source which is modeled by a delta function. Based on the idea of the immersed interface method [16], an accurate approximate scheme is constructed. And five different approximations are derived depending on the location of the heat source. Then an accurate moving mesh algorithm is developed. In [4], the authors developed a new moving mesh method for the same equation with two or more heat sources by adding auxiliary mesh points exactly at the heat sources to capture the singularity. This method simplify the moving method algorithm of [3].

In papers [9, 10, 11], the monitor function is studied for the blow-up problems. Dimension analysis and dominance of equidistribution are proposed for choosing appropriate monitor functions. In [12], the authors investigate the parameter τ in MMPDE. They find that the constant value of the parameter τ sometimes may not be appropriate. Then they suggest an adaptive strategy for choosing a time-dependent parameter τ. In papers [14, 15], the moving collocation method is used in solving kinds of blow-up problems.

Recently, the blow-up problems in unbounded spatial domain are investigated [18, 22]. A novel local absorbing boundary conditions (LABCs) method is introduced and analyzed. Based on these works, Qiang [17] combine the LABCs method and the MMPDE method to capture the qualitative behavior of the blow-up singularities in the one-dimensional unbounded domain. Compared to the LABCs method on the fixed mesh, the combination of LABCs and MMPDE method is more efficient. There is no doubt that it is also very efficient to solve the blow-up problems in two-dimensions by combining the LABCs method and the MMPDE method. But general speaking, the blow-up problems in high-dimensions are difficult and very few adaptive numerical methods have been developed [19, 20, 21].

In this paper, the moving collocation method [23, 13] and the local absorbing boundary method [22] will be combined to solve the model (1)-(4). First, we choose a circular artificial boundary Γ_R for equation (1). Then the artificial boundary conditions are conducted and equation (1) is transformed to the polar coordinate form. Thus, the reaction-diffusion equation (1) in unbounded domain is changed to a new one which is defined in the moving bounded domain. By using a coordinate transformation, it will be easily changed to a fixed boundary problem. This new problem is then reduced to a one-dimensional equation considering its symmetry around the arc. At last, we simulate the blow-up by the moving collocation method. The numerical results show that the blow-up occurs if the speed of the heat source is slow, and the blow-up is avoided when the heat source moves fast enough.

The remaining parts of the paper are organized as follows. In section 2, the modeling is simplified to a one-dimensional reaction-diffusion equation by using local absorbing boundary conditions method and several coordinate transformations. In section 3, the numerical scheme is derived. In section 4, a number of numerical experiments are presented. Finally, conclusions are given in section 5.

2 Local absorbing boundary conditions and coordinate transformation

Introducing the coordinate transformation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \tag{5}$$

we denote

$$T(x,y,t) = T(r\cos\theta, r\sin\theta, t) \equiv u(r,\theta,t).$$

Following the artificial boundary condition method [22], we choose a circular artificial boundary Γ_R,

$$\Gamma_R = \{(r,\theta)\big| r = R,\ 0 \le \theta \le 2\pi\}, \tag{6}$$

where $R = 2r_0(t)$. Then the model (1)-(4) is changed to the following form

$$\begin{aligned} u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \\ +D(r|r_0)F[u(r_0,\theta,t)], \\ 0 \le r < R,\ 0 \le \theta < 2\pi,\ t > 0, \end{aligned} \tag{7}$$

$$\begin{aligned} u(r,\theta,0) = u_0(r,\theta) \ge 0, \\ 0 \le r \le R,\ 0 \le \theta < 2\pi, \end{aligned} \tag{8}$$

$$\begin{aligned} \gamma u_{tr} - \alpha u_t = -\delta u_r + \beta u + \gamma(-\frac{2}{R^3}u_{\theta\theta} \\ +\frac{1}{R^2}u_{\theta\theta r}) - \alpha\frac{1}{R^2}u_{\theta\theta},\ \text{ on } \Gamma_R, \end{aligned} \tag{9}$$

with

$$D(r|r_0) = \frac{1}{\epsilon}\max\big(0, \frac{1}{\epsilon}[\epsilon - |r - r_0|]\big). \quad (10)$$

The above equations (7)-(9) are defined in the two-dimensional domain that may be growing or decreasing in time. This problem can be called the moving boundary problem which is difficult to solve. The stability and convergence of numerical methods for the linear reaction-diffusion problem on a one-dimensional growing domain have been investigated [24].

In order to change the above moving boundary problem to a fixed one, we make the following coordinate transformation

$$\begin{cases} \bar{r} = r/R(t), \\ \bar{\theta} = \theta, \\ \bar{t} = t, \end{cases}$$

and denote

$$u(r,\theta,t) = u(\bar{r}R(\bar{t}),\bar{\theta},\bar{t}) \equiv \bar{u}(\bar{r},\bar{\theta},\bar{t}).$$

Then,

$$\frac{\partial \bar{u}}{\partial \bar{\theta}} = \frac{\partial u}{\partial \theta}, \quad (11)$$

$$\frac{\partial \bar{u}}{\partial \bar{r}} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial \bar{r}} = \frac{\partial u}{\partial r}R(t), \quad (12)$$

$$\begin{aligned}\frac{\partial \bar{u}}{\partial \bar{t}} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial t} \\ &= \bar{r}R'(t)\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}, \end{aligned} \quad (13)$$

and

$$\begin{aligned}\frac{\partial^2 \bar{u}}{\partial \bar{r}\bar{t}} &= \big(\frac{\partial u}{\partial r}R(t)\big)_{\bar{t}} \\ &= \frac{\partial^2 u}{\partial r^2}\bar{r}R'(t)R(t) + \frac{\partial^2 u}{\partial r \partial t}R(t) + \frac{\partial u}{\partial r}R'(t). \end{aligned} \quad (14)$$

According to the above relations (11)-(14), the moving boundary problem (7)-(9) are transformed to the following fixed boundary problem

$$\begin{aligned}\bar{u}_{\bar{t}} &= \frac{1}{R^2}\bar{u}_{\bar{r}\bar{r}} + \big(\frac{1}{\bar{r}R^2} + \frac{\bar{r}R'(\bar{t})}{R}\big)\bar{u}_{\bar{r}} + \frac{1}{\bar{r}^2R^2}\bar{u}_{\bar{\theta}\bar{\theta}} \\ &\quad + \bar{D}(\bar{r}|0.5)F[\bar{u}(0.5,\bar{\theta},\bar{t})], \quad (15) \\ &\quad 0 \le \bar{r} < 1,\ 0 \le \bar{\theta} < 2\pi, \bar{t} > 0, \\ \bar{u}(\bar{r},\bar{\theta},0) &= \bar{u}_0(\bar{r},\bar{\theta}) \ge 0, \\ &\quad 0 \le \bar{r} \le 1,\ 0 \le \bar{\theta} < 2\pi, \quad (16) \\ \frac{\gamma}{R}\bar{u}_{\bar{t}\bar{r}} - \alpha\bar{u}_{\bar{t}} &= \frac{\gamma\bar{r}R'(\bar{t})}{R^2}\bar{u}_{\bar{r}\bar{r}} + \big(\frac{\gamma R'(t)}{R^2} \\ &\quad -\alpha\frac{\bar{r}R'(\bar{t})}{R} - \frac{\delta}{R}\big)\bar{u}_{\bar{r}} + \beta\bar{u} - \big(\frac{2\gamma}{R^3} \\ &\quad + \frac{\alpha}{R^2}\big)\bar{u}_{\bar{\theta}\bar{\theta}} + \frac{\gamma}{R^3}\bar{u}_{\bar{\theta}\bar{\theta}\bar{r}}, \text{ on } \bar{\Gamma}, \quad (17)\end{aligned}$$

with

$$\bar{D}(\bar{r}|0.5) = \frac{1}{\epsilon}\max\big(0, \frac{R}{\epsilon}[\frac{\epsilon}{R} - (\bar{r} - 0.5)]\big), \quad (18)$$

and

$$\bar{\Gamma} = \big\{(\bar{r},\bar{\theta})\big|\bar{r} = 1,\ 0 \le \bar{\theta} \le 2\pi\big\}. \quad (19)$$

Considering the symmetry, the problem (15)-(19) is reduced to the following one-dimensional equations,

$$\begin{aligned}\bar{u}_t &= \frac{1}{R^2}\bar{u}_{\bar{r}\bar{r}} + \big(\frac{1}{\bar{r}R^2} + \frac{\bar{r}R'(t)}{R}\big)\bar{u}_{\bar{r}} + \bar{D}(\bar{r}|0.5) \\ &\quad F[\bar{u}(0.5,t)],\ 0 \le \bar{r} < 1,\ t > 0, \quad (20) \\ \bar{u}(\bar{r},0) &= \bar{u}_0(\bar{r}) \ge 0,\ \ 0 \le \bar{r} \le 1, \quad (21) \\ \frac{\gamma}{R}\bar{u}_{t\bar{r}} - \alpha\bar{u}_t &= \frac{\gamma\bar{r}R'(t)}{R^2}\bar{u}_{\bar{r}\bar{r}} + \big(\frac{\gamma R'(t)}{R^2} \\ &\quad -\alpha\frac{\bar{r}R'(t)}{R} - \frac{\delta}{R}\big)\bar{u}_{\bar{r}} + \beta\bar{u}, \quad \text{at } \bar{r} = 1, \quad (22)\end{aligned}$$

with

$$\bar{D}(\bar{r}|0.5) = \frac{1}{\epsilon}\max\big(0, \frac{R}{\epsilon}[\frac{\epsilon}{R} - (\bar{r} - 0.5)]\big). \quad (23)$$

3 Adaptive numerical method

We fix a mesh point at the location of heat source $\bar{r} = 0.5$, and divide the computational domain into two subdomains $[0, 0.5]$ and $[0.5, 1]$. Applying the MMPDE6

$$\frac{\partial^2 \dot{x}}{\partial \xi^2} = -\frac{1}{\tau}\frac{\partial}{\partial \xi}\big(M\frac{\partial x}{\partial \xi}\big) \quad (24)$$

in each subdomain, we can get a special time-dependent adaptive mesh $0 = \bar{r}_0 < \bar{r}_1(t) < ... < \bar{r}_{k-1}(t) < \bar{r}_k = 0.5 < \bar{r}_{k+1}(t) < ... < \bar{r}_N = 1$. In the numerical computing, the MMPDE6 is discretized by the centered finite differences method

$$\begin{aligned}\dot{x}_{i+1} - 2\dot{x}_i + \dot{x}_{i-1} &= -\frac{1}{\tau}\Big(M_{i+\frac{1}{2}}(x_{i+1} - x_i) \\ &\quad - M_{i-\frac{1}{2}}(x_i - x_{i-1})\Big). \quad (25)\end{aligned}$$

3.1 The left boundary condition

Equation (20) is singular at point $\bar{r} = 0$ which is introduced by the coordinate transformation (5). To remove this singularity, we integrate equation (1) in a small disc [22]

$$\Omega_{\frac{\Delta r_1}{2}} = \big\{(r,\theta)\big|0 \le r \le \frac{\Delta r_1}{2},\ 0 \le \theta < 2\pi\big\},$$

where $\Delta r_1 := r_1 - r_0 = R(t)(\bar{r}_1 - \bar{r}_0)$. Since the heat source is outside of the disc, we get

$$\int_{\Omega_{\frac{\Delta r_1}{2}}} \frac{\partial T}{\partial t} \mathrm{d}\Omega = \int_{\Omega_{\frac{\Delta r_1}{2}}} \Delta T \mathrm{d}\Omega.$$

Then we obtain

$$\frac{\pi \Delta r_1^2}{4} \frac{\partial u}{\partial t}(0,0,t) = \frac{\Delta r_1}{2} \int_0^{2\pi} \frac{\partial u}{\partial r}(\frac{\Delta r_1}{2}, \theta, t) \mathrm{d}\theta,$$

resorting to the Gauss's theorem and the mid-point rule. Using the symmetry and the coordinate transformation, we have the following equation

$$\frac{R\Delta \bar{r}_1}{4} \bar{u}_t(0,t) = \frac{1}{R} \frac{\partial \bar{u}}{\partial \bar{r}}(\frac{\Delta \bar{r}_1}{2}, t), \tag{26}$$

which will be used as the left boundary condition for equation (20).

3.2 The moving collocation method

In this subsection, the problem (20)-(23) with its left boundary condition (26) is discretized by the moving collocation method [23]. Let $I_i = [\bar{r}_i, \bar{r}_{i+1}]$, $i = 0, ..., N-1$ and define the local coordinate $s^{(i)}$ by

$$s^{(i)} := (\bar{r} - \bar{r}_i(t))/H_i(t), \ \ H_i(t) := \bar{r}_{i+1}(t) - \bar{r}_i(t).$$

The solution $\bar{u}(\bar{r}, t)$ can be approximated by the following piecewise cubic Hermite polynomial

$$\begin{aligned} p(\bar{r}, t) &= v_i(t)\phi_1(s^{(i)}) + v_{\bar{r},i}(t)H_i(t)\phi_2(s^{(i)}) \\ &+ v_{i+1}(t)\phi_3(s^{(i)}) + v_{\bar{r},i+1}(t)H_i(t)\phi_4(s^{(i)}), \\ & \text{for } \bar{r} \in I_i, \ i = 0, ..., N-1, \end{aligned} \tag{27}$$

where $v_i(t)$ and $v_{\bar{r},i}(t)$ are the approximations to $\bar{u}(\bar{r}_i, t)$ and $\bar{u}_{\bar{r}}(\bar{r}_i, t)$. The shape functions are given by

$$\begin{aligned} \phi_1(s) &:= (1+2s)(1-s)^2, \quad \phi_2(s) := s(1-s)^2, \\ \phi_3(s) &:= (3-2s)s^2, \qquad \phi_4(s) := (s-1)s^2. \end{aligned}$$

For $\bar{r} \in I_i$, $i = 0, ..., N-1$, we can derive that

$$p_{\bar{r}}(\bar{r}, t) = \frac{1}{H_i(t)} \left(v_i(t) \frac{\mathrm{d}\phi_1}{\mathrm{d}s} + v_{\bar{r},i}(t)H_i(t) \frac{\mathrm{d}\phi_2}{\mathrm{d}s} \right. \\ \left. + v_{i+1}(t) \frac{\mathrm{d}\phi_3}{\mathrm{d}s} + v_{\bar{r},i+1}(t)H_i(t) \frac{\mathrm{d}\phi_4}{\mathrm{d}s} \right),$$

$$p_{\bar{r}\bar{r}}(\bar{r}, t) = \frac{1}{H_i^2(t)} \left(v_i(t) \frac{\mathrm{d}^2\phi_1}{\mathrm{d}s^2} + v_{\bar{r},i}(t)H_i(t) \frac{\mathrm{d}^2\phi_2}{\mathrm{d}s^2} \right. \\ \left. + v_{i+1}(t) \frac{\mathrm{d}^2\phi_3}{\mathrm{d}s^2} + v_{\bar{r},i+1}(t)H_i(t) \frac{\mathrm{d}^2\phi_4}{\mathrm{d}s^2} \right),$$

$$p_t(\bar{r}, t) = \frac{\mathrm{d}v_i(t)}{\mathrm{d}t}\phi_1 + \left(\frac{\mathrm{d}v_{\bar{r},i}(t)}{\mathrm{d}t} H_i(t) \right. \\ \left. + v_{\bar{r},i}(t) \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} \right) \phi_2 + \frac{\mathrm{d}v_{i+1}(t)}{\mathrm{d}t}\phi_3 \\ + \left(\frac{\mathrm{d}v_{\bar{r},i+1}(t)}{\mathrm{d}t} H_i(t) + v_{\bar{r},i+1}(t) \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} \right) \phi_4 \\ - p_{\bar{r}}(\bar{r}, t) \left(\frac{\mathrm{d}\bar{r}_i}{\mathrm{d}t} + s^{(i)} \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} \right),$$

$$p_{t\bar{r}}(\bar{r}, t) = \frac{1}{H_i(t)} \left[\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} \frac{\mathrm{d}\phi_1}{\mathrm{d}s} + \left(\frac{\mathrm{d}v_{\bar{r},i}(t)}{\mathrm{d}t} H_i(t) \right. \right. \\ \left. + v_{\bar{r},i}(t) \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} \right) \frac{\mathrm{d}\phi_2}{\mathrm{d}s} + \frac{\mathrm{d}v_{i+1}(t)}{\mathrm{d}t} \frac{\mathrm{d}\phi_3}{\mathrm{d}s} \\ \left. + \left(v_{\bar{r},i+1}(t)H_i(t) + v_{\bar{r},i+1}(t) \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} \right) \frac{\mathrm{d}\phi_4}{\mathrm{d}s} \right] \\ - \frac{p_{\bar{r}}(\bar{r}, t)}{H_i(t)} \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} - p_{\bar{r}\bar{r}}(\bar{r}, t) \left(\frac{\mathrm{d}\bar{r}_i}{\mathrm{d}t} + s^{(i)} \frac{\mathrm{d}H_i(t)}{\mathrm{d}t} \right),$$

where ϕ_j, $\frac{\mathrm{d}\phi_j}{\mathrm{d}s}$, $\frac{\mathrm{d}^2\phi_j}{\mathrm{d}s^2}$, $j = 1, \cdots, 4$ are functions of $s^{(i)}$.

Following the processes in [23], the equation (20) is rewritten into the general divergence form

$$\bar{u}_t - (\frac{1}{\bar{r}R^2} + \frac{\bar{r}R'(t)}{R})\bar{u}_{\bar{r}} - \bar{D}(\bar{r}|0.5)F[\bar{u}(0.5, t)] \\ = \frac{\partial}{\partial \bar{r}} \left(\frac{1}{R^2} \bar{u}_{\bar{r}} \right). \tag{28}$$

Using the idea of cell average for each half of I_i ($i = 0, \cdots, N-1$), the piecewise linear approximation and the moving collocation points $r_{ij} = \bar{r}_i(t) + c_j H_i(t)$ ($j = 1, 2$), we obtain

$$\begin{aligned} & p_t(r_{i1}, t) - (\frac{1}{r_{i1}R^2} + \frac{r_{i1}R'(t)}{R})p_{\bar{r}}(r_{i1}, t) \\ & \quad - \bar{D}(r_{i1}|0.5)F[p(0.5, t)] \\ & = \frac{1}{R^2 H_i} \left[-(1 + 2/\sqrt{3})p_{\bar{r}}(\bar{r}_i, t) \right. \\ & \quad + (4/\sqrt{3})p_{\bar{r}}(\frac{\bar{r}_i + \bar{r}_{i+1}}{2}, t) \\ & \quad \left. + (1 - 2/\sqrt{3})p_{\bar{r}}(\bar{r}_{i+1}, t) \right], \end{aligned} \tag{29}$$

$$\begin{aligned} & p_t(r_{i2}, t) - (\frac{1}{r_{i2}R^2} + \frac{r_{i2}R'(t)}{R})p_{\bar{r}}(r_{i2}, t) \\ & \quad - \bar{D}(r_{i2}|0.5)F[p(0.5, t)] \\ & = \frac{1}{R^2 H_i} \left[-(1 - 2/\sqrt{3})p_{\bar{r}}(\bar{r}_i, t) \right. \\ & \quad - (4/\sqrt{3})p_{\bar{r}}(\frac{\bar{r}_i + \bar{r}_{i+1}}{2}, t) \\ & \quad \left. + (1 + 2/\sqrt{3})p_{\bar{r}}(\bar{r}_{i+1}, t) \right]. \end{aligned} \tag{30}$$

Similarly, the boundary conditions and the initial conditions can be approximated by

$$\left[\frac{\gamma}{R}p_{t\bar{r}} - \alpha p_t - \frac{\gamma\bar{r}R'(t)}{R^2}p_{\bar{r}\bar{r}} - (\frac{\gamma R'(t)}{R^2} - \alpha\frac{\bar{r}R'(t)}{R} - \frac{\delta}{R})\bar{u}_{\bar{r}} - \beta p\right]\bigg|_{\bar{r}=1} = 0, \quad (31)$$

$$\frac{R\Delta\bar{r}_1}{4}p_t(0,t) - \frac{1}{R}p_{\bar{r}}\left(\frac{\Delta\bar{r}_1}{2},t\right) = 0, \quad (32)$$

and

$$p(\bar{r}_i,0) = \bar{u}_0(\bar{r}_i), \quad (33)$$

$$p_{\bar{r}}(\bar{r}_i,0) = \frac{\mathrm{d}\bar{u}_0}{\mathrm{d}\bar{r}}(\bar{r}_i), \quad (34)$$

where $i = 0, 1, \cdots, N$.

After the above spatial discretization, the discrete mesh equation (25) of MMPDE6 and the discretized PDE system involving the collocation approximation (29)-(30) for the physical PDE, the corresponding boundary conditions (31)-(32) and initial condition (33)-(34) form an ODE system which is solved by the ODE solver ODE15s.

4 Numerical examples

In this section, some numerical experiments will be presented which includes both linear and curvilinear motions of the heat source. The influence of the velocity of heat source on the blow-up will be investigated. And the efficiency of the adaptive moving mesh method will be demonstrated.

The parameter τ in MMPDE6 is given by $\tau = 1\times 10^{-3}$. And the monitor function $M(\bar{r},t)$ gives [3]

$$M(\bar{r},t) = a\bar{u}^2 + b\left((\bar{r}-0.5)^2+\epsilon_1\right)^{-\frac{1}{4}} + (1-a-b)(\bar{r}^2+\epsilon_1),$$

with $a = 0.4, b = 0.3, \epsilon_1 = \frac{1}{(N-1)^2}$. Here N is the number of spatial mesh points which is taken by 65 in all following examples. In numerical computing, the monitor function M is always replaced by a smoothed one [25]

$$\widetilde{M}_i = \sqrt{\frac{\sum\limits_{k=i-p}^{i+p}(M_k)^2\left(\frac{\gamma}{1+\gamma}\right)^{|k-i|}}{\sum\limits_{k=i-p}^{i+p}\left(\frac{\gamma}{1+\gamma}\right)^{|k-i|}}},$$

where $\gamma > 0$ and $p \geq 0$ are two smoothing parameters, given by $\gamma = 2$ and $p = 2$. The parameters in the artificial boundary conditions are [22]

Table 1: The blowup times of the line movement of the heat source (36)

k = 0	t = 0.590721677
k = 1	t = 0.667461343
k = 2	t = 1.297749062
k = 3	not blowup

$\alpha = -(6R\sqrt{s_0}+1), \beta = -s_0(3+2R\sqrt{s_0}), \gamma = 2R$ and $\delta = 6Rs_0$, where we choose $s_0 = 4.0$.

The initial value of the temperature is

$$T_0(x,y) = \begin{cases} \cos^2(\frac{\pi(r-3)}{6}), & r \leq 3 \\ 0, & \text{other} \end{cases}$$

where $r = \sqrt{x^2+y^2}$.

4.1 linear motion of the heat source

First, we consider the case of the linear motion of the heat source. The location of the heat source gives

$$\{(x_0,y_0)|\, r_0(t) := \sqrt{x_0(t)^2+y_0(t)^2} = kt+3\}, \quad (35)$$

where $k \geq 0$ is a parameter which denotes the velocity of the heat source. In this case, the heat source is moving at a constant speed. The blowup times are listed in table 4.1. It shows that the blowup time increases as the heat source moves faster. And when the heat source moves fast enough, the blowup will not happen (see the case of $k = 3$ in the table). This is because the heat source is continually being exposed to the new cool surroundings as it moves and the heat is diffused. This phenomenon is consistent with the case of one-dimension [1, 3].

When $k = 0$, the heat source is fixed (figure 1). In this case, heat is supplied at the same location, and accumulated from time to time. So the blow-up happens at time $t = 0.590721677$. From the bottom-Left of figure 1, we can see the blow-up happens at the location of the heat source. The top-left and top-right subgraphs show the blow-up profiles in variable $\bar{r}$ and in computational variable ξ respectively. It reveals that the profiles in computational variable are much smoother. This demonstrates the effective of our moving collocation method. The blow-up profiles in variable x, y is shown in the bottom-left subgraph. It can be seen that the temperature is still very low at the locations far away from the heat source. The adaptive mesh is shown in the bottom-right subgraph. It is clear that a lot of mesh grids are gathered near the heat source.

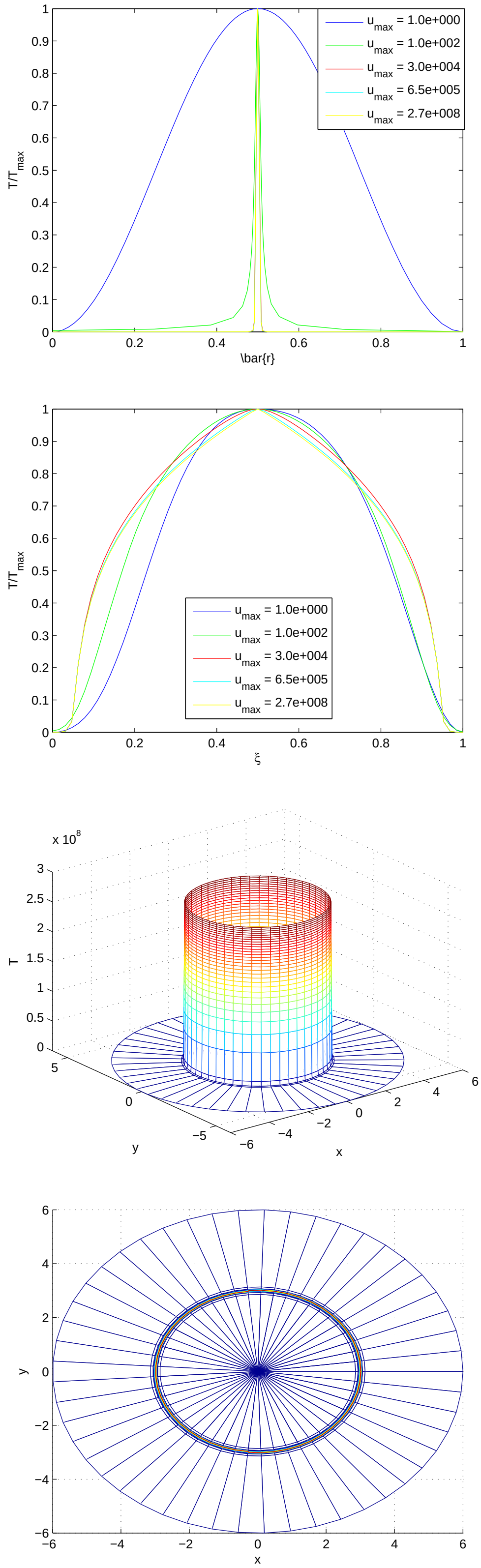

Figure 1: $r_0(t) = kt + 3$ with $k = 0$. Top-Left: The blow-up profiles T/T_{max} in the variable $\bar{r}$. Top-Right: The blow-up profiles T/T_{max} in the computational variable ξ. Bottom-Left: The blow-up profile in the variables x, y. Bottom-Right: The adaptive mesh.

Figure 2 and figure 3 are corresponding to the cases of $k = 1$ and $k = 2$ respectively. In these cases, the heat source moves at a low speed. The heat is not able to diffuse in time. The blow-up happens. When $k = 3$, the heat source moves at a high speed. The heat diffuses in time, and the blow-up is prevented (figure 4). Interestingly, the highest temperature is not at the heat source but at the center point of the domain (see the Bottom-Left of figure 4). This can be explained that the heat is accumulated at the center.

4.2 Curvilinear motion of the heat source

Now, we consider the case of the curvilinear motion of the heat source. The location of the heat source is

$$\begin{aligned}\{(x_0, y_0)|r_0(t) \quad &:= \quad \sqrt{x_0(t)^2 + y_0(t)^2} \\ &= k\sin(\pi t) + 3\}, \quad (36)\end{aligned}$$

where k is a parameter. In this situation, the circle of the heat source amplifies or lessens.

Figure 5 is corresponding to the case $k = 1$. From the figure, we can see that the blow-up happens. The blow-up time for this case is 0.869099193. The top-left, top-right and bottom-right subgraphs show the profiles in the different variables. It shows that the profiles in the computational variable ξ are much smoother. And the bottom-right subgraph gives the adaptive mesh. Figure 6 shows the numerical results of the case $k = 2$. The blow-up happens at 3.234220953.

5 Conclusions

In this paper, a new mathematical model is introduced for the moving heat source problem in two-dimensions. Considering the symmetry, the model is reduced to the one-dimensional problem by using the artificial boundary condition method and coordinate transformations. Then a moving collocation method is proposed for solving the problem. Numerical results show that the blow-up time increases as the heat source moves faster. And the blow-up will be avoided when the speed is fast enough.

Our present work has mainly focused on the influence of the speed of the heat source on the blow-up in two-dimensions. In fact, the size and strength of the source may also influence the blow-up phenomenon [6]. The numerical investigation of blow-up in these factors are also very interesting. And this research is ongoing.

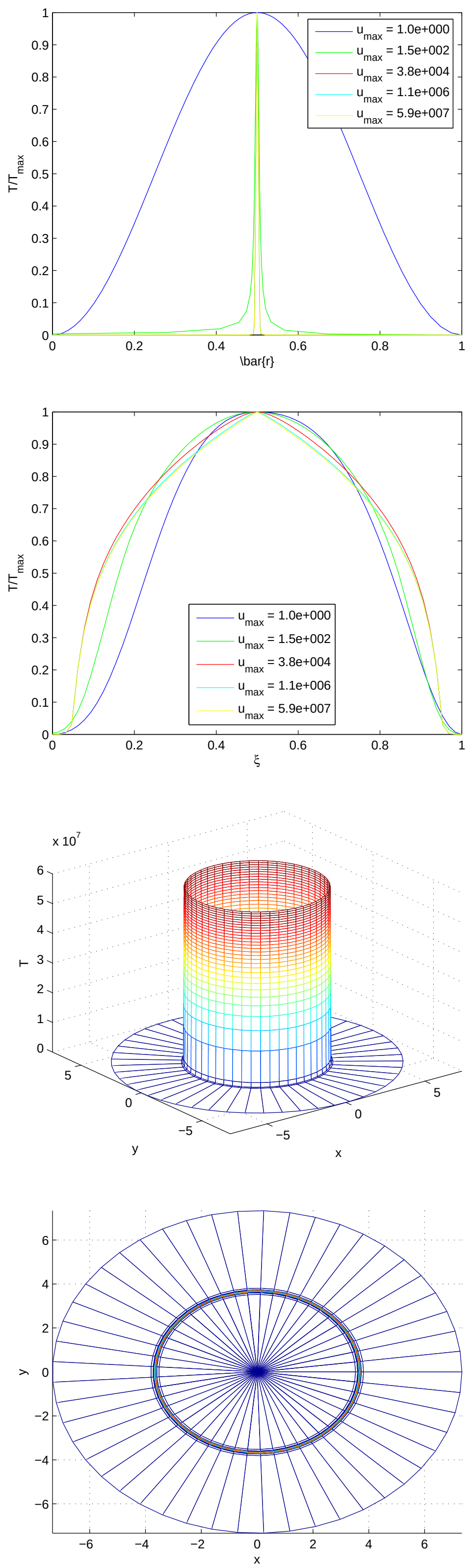

Figure 2: $r_0(t) = kt + 3$ with $k = 1$. Top-Left: The blow-up profiles T/T_{max} in the variable $\bar{r}$. Top-Right: The blow-up profiles T/T_{max} in the computational variable ξ. Bottom-Left: The blow-up profile in the variables x, y. Bottom-Right: The adaptive mesh.

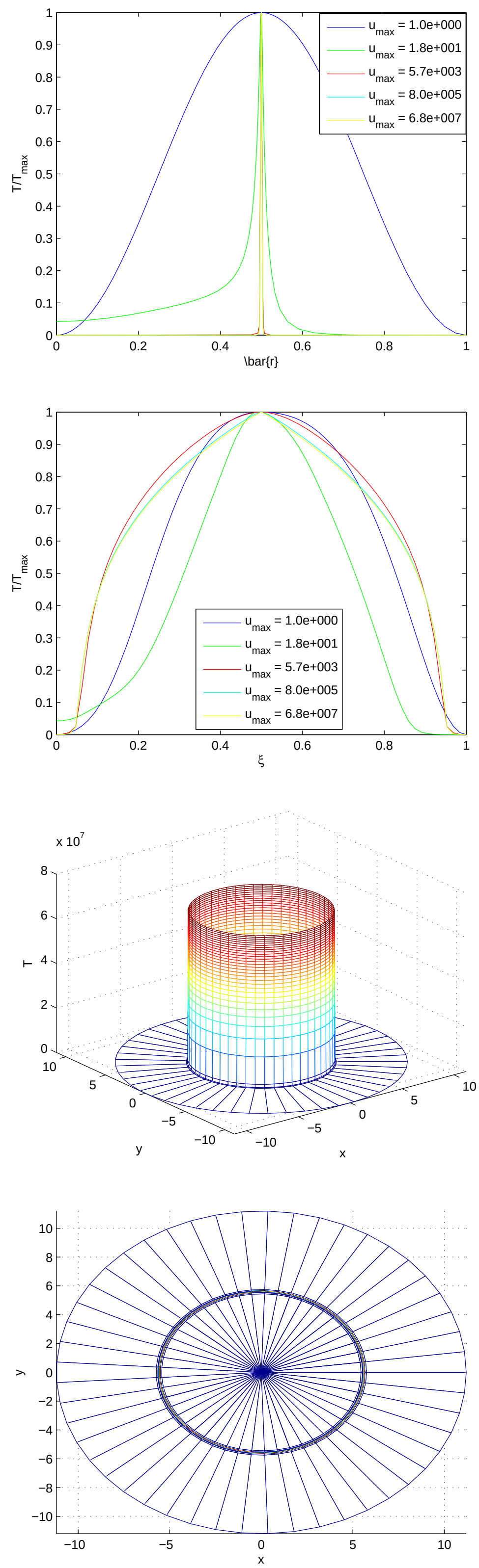

Figure 3: $r_0(t) = kt + 3$ with $k = 2$. Top-Left: The blow-up profiles T/T_{max} in the radius variable $\bar{r}$. Top-Right: The blow-up profiles T/T_{max} in the computational variable ξ. Bottom-Left: The blow-up profile in the variable x, y. Bottom-Right: The adaptive mesh.

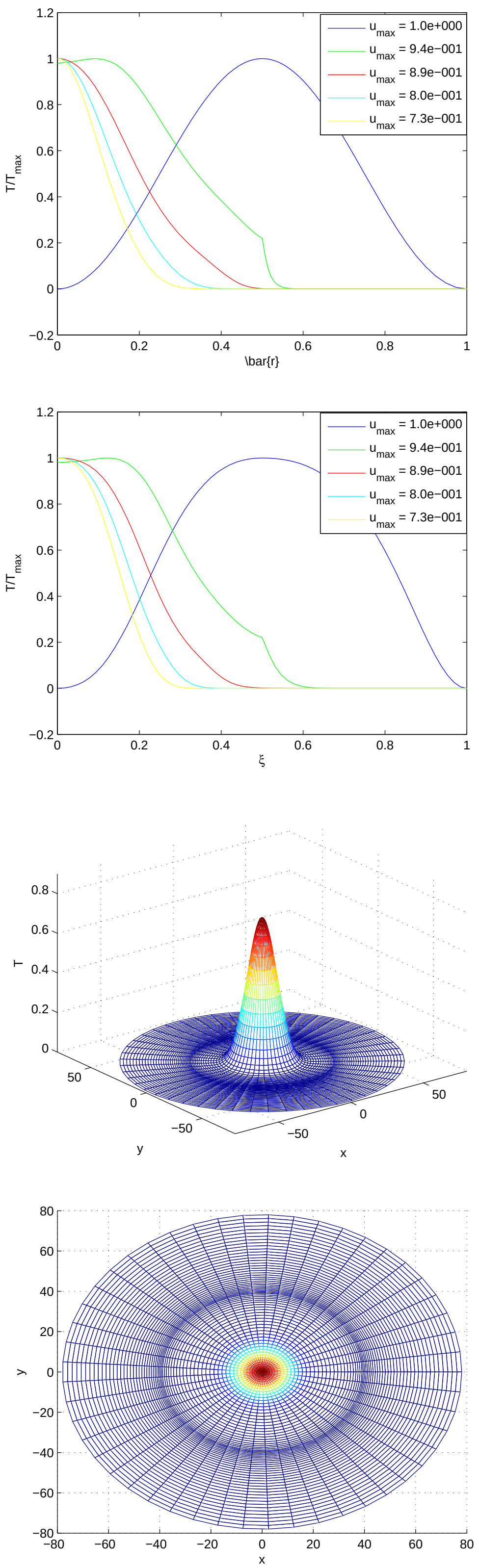

Figure 4: $r_0(t) = kt + 3$ with $k = 3$. Top-Left: The blow-up profiles T/T_{max} in the radius variable $\bar{r}$. Top-Right: The blow-up profiles T/T_{max} in the computational variable ξ. Bottom-Left: The blow-up profile in the variable x, y. Bottom-Right: The adaptive mesh.

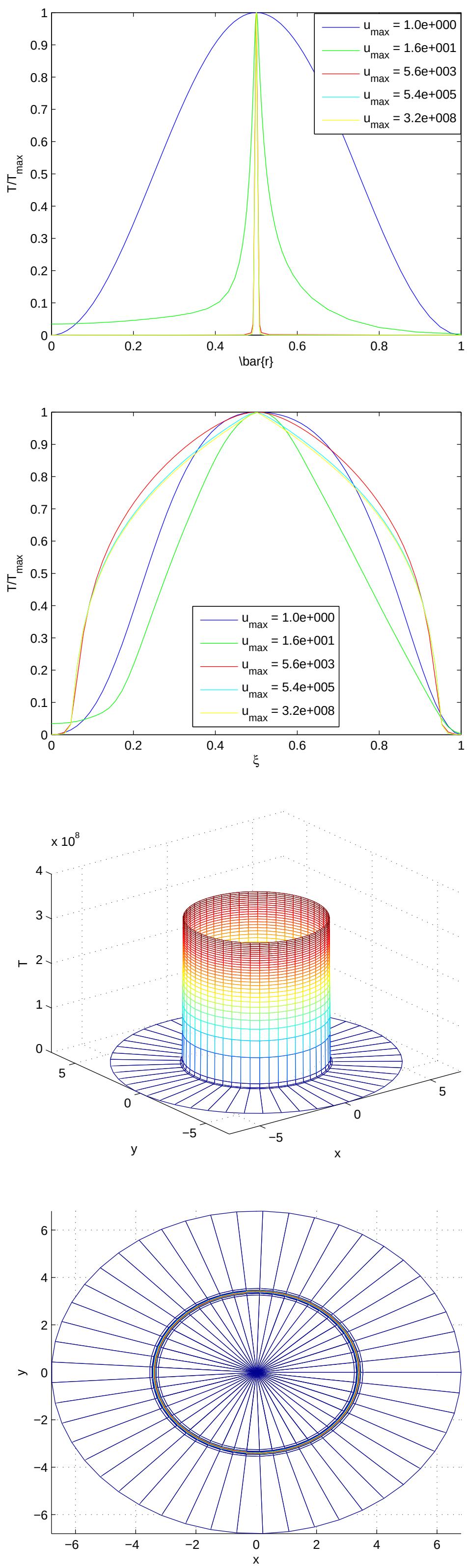

Figure 5: $r_0(t) = k\sin(\pi t) + 3$ with $k = 1$. Top-Left: The blow-up profiles T/T_{max} in the radius variable $\bar{r}$. Top-Right: The blow-up profiles T/T_{max} in the computational variable ξ. Bottom-Left: The blow-up profile in the variable x, y. Bottom-Right: The adaptive mesh.

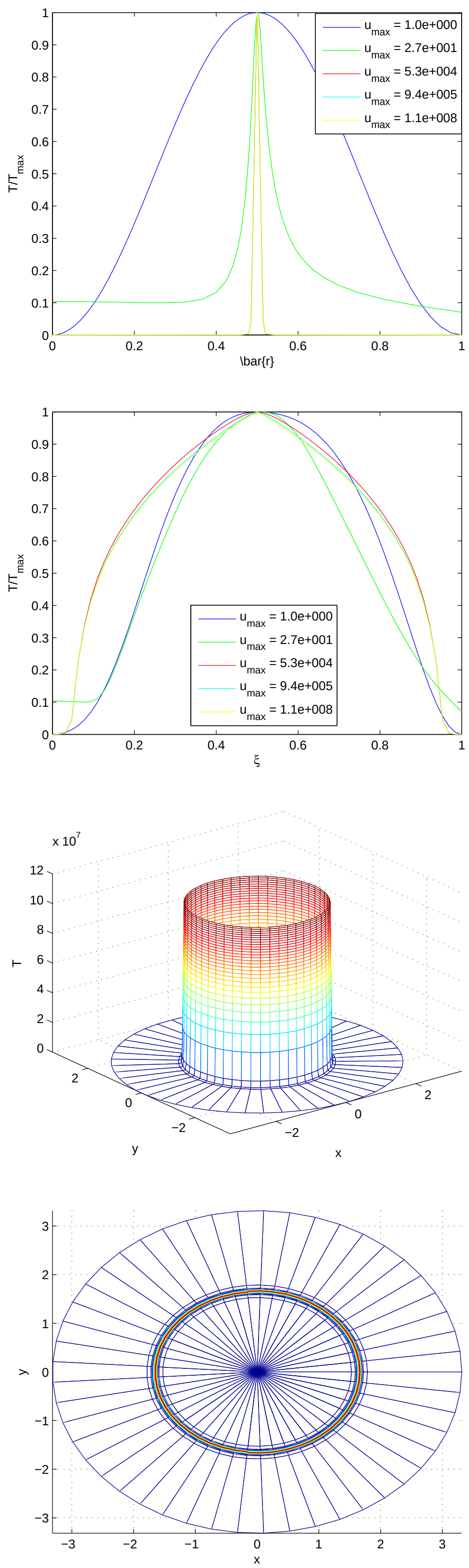

Figure 6: $r_0(t) = k\sin(\pi t) + 3$ with $k = 2$. Top-Left: The blow-up profiles T/T_{max} in the radius variable $\bar{r}$. Top-Right: The blow-up profiles T/T_{max} in the computational variable ξ. Bottom-Left: The blow-up profile in the variable x, y. Bottom-Right: The adaptive mesh.

Acknowledgements: This work was supported by the TianYuan Special Funds of the National Natural Science Foundation of China (Grant No. 11226311), the National Natural Science Foundation of China (Grant No. 11171305), Zhejiang Provincial Natural Science Foundation of China (Grant No. Y6110252) and Scientific research funds of Shaoxing university (Grant No. 20115013).

References:

[1] C. M. Kirk and W. E. Olmstead, Blow-up in a reactive-diffusive medium with a moving heat source, *Z. angew. Math. Phys.* 53, 2002, pp. 147–159.

[2] C. M. Kirk, Numerical and asymptotic analysis of a localized heat source undergoing periodic motion, *Nonlinear Anal.* 71, 2009, pp. e2168–e2172.

[3] J. Ma, Y. Jiang, Moving mesh methods for blowup in reaction-diffusion equations with traveling heat source, *J. Comput. Phys.* 228, 2009, pp. 6977–6990.

[4] H. Zhu, K. Liang, X. Cheng, A numerical investigation of blow-up in reactionCdiffusion problems with traveling heat sources, *J. Comput. Appl. Math.* 234, 2010, pp. 3332–3343.

[5] J. M. Chadam and H. M. Yin, A diffusion equation with localized chemical reactions, *Proc. Edinburegh Math. Soc.* 37, 1993, pp. 101–118.

[6] C. M. Kirk and W. E. Olmstead, Blow-up solutions of the two-dimensional heat equation due to a localized moving source, *Anal. Appl.* 3, 2005, pp. 1–16.

[7] W. Huang, Y. Ren, R. D. Russell, Moving mesh partial differential equations (MMPDEs) based upon the equidistribution principle, *SIAM J. Number. Anal.* 31, 1994, pp. 709–730.

[8] W. Huang, R.D. Russell, Adaptive Moving Mesh Methods, New York: Springer, 2011.

[9] C. J. Budd, W. Huang, R. D. Russell, Moving mesh methods for problems with blow-up, *SIAM J. Sci. Comput.* 17, 1996, pp. 305–327.

[10] W. Huang, J. Ma, R. D. Russell, A study of moving mesh PDE methods for numerical simulation of blowup in reaction diffusion equations, *J. Comput. Phys.* 227, 2008, pp. 6532–6552.

[11] J. Ma and Y. Jiang, K. Xiang, Numerical simulation of blowup in nonlocal reaction-diffusion equations using a moving mesh method, *J. Comput. Appl. Math.* 230, 2009, pp. 8–21.

[12] A. R. Soheili, J. M. Stockie, A moving mesh method with variable mesh relaxation time, *Appl. Numer. Math.* 58, 2008, pp. 249–263.

[13] J. T. Ma, W. Z. Huang and R. D. Russell, Analysis of a moving collocation method for one-dimensional partial differential equations, *Sci China Math.* 55, 2012, pp. 827–840.

[14] J. Ma, Y. Jiang, Moving collocation methods for time fractional differential equations and simulation of blowup, *Sci. China Math.* 54, 2011, pp. 611–622.

[15] R. D. Russell , J. F. Williams, X. Xu, Movcol4: A moving mesh code for fourth-order time-dependent partial differential equations, *SIAM J. Sci. Comput.* 29, 2007, pp. 197–220.

[16] R. J. Leveque, Z. Li, The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, *SIAM J. Number. Anal.* 31, 1994, pp. 1019–1044.

[17] H. Qiang, K. Liang, Moving mesh method for problems with blow-up on unbounded domains, *Numer. Algor.* 59, 2012, pp. 63–77.

[18] H. Brunner, X. Wu, J. Zhang, Computational solution of blow-up problems for semilinear parabolic PDEs on unbounded domains, *SIAM J. Sci. Comput.* 31, 2010, pp. 4478–4496.

[19] W. Ren, X. Wang, An iterative grid redistribution method for singular problems in multiple dimensions, *J. Comput. Phys.* 159, 2000, pp. 246–273.

[20] H. D. Cenicerosa, T.Y. Hou, An efficient dynamically adaptive mesh for potentially singular solutions, *J. Comput. Phys.* 172, 2001, pp. 609–639.

[21] K. Liang, P. Lin, A splitting moving mesh method for 3-D quenching and blow-up problems, Recent advances in adaptive computation, 311–324, Contemp. Math., 383, Amer. Math. Soc., Providence, RI, 2005.

[22] J. Zhang, H. Han, and H. Brunner, Numerical Blow-up of Semilinear Parabolic PDEs on Unbounded Domains in $\mathbb{R}^2$, *J. Sci. Comput.* 49, 2011, pp. 367–382.

[23] W. Huang, R. D. Russell, A moving collocation method for solving time dependent partial differential equations, *Appl. Numer. Math.* 20, 1996, pp. 101–116.

[24] J. A. Mackenzie, Analysis of stability and convergence of finite-difference methods for a reaction-diffusion problem on a one-dimensional growing domain, *IMA J. Numer. Anal.* 31, 2011, pp. 212–232.

[25] W. Huang, Y. Ren and R. D. Russell, Moving mesh methods based on moving mesh partial differential equations, *J. Comput. Phys.* 113, 1994, pp. 279–290.