Optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance (CEV) model

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Abstract: This paper studies the optimal investment problem of utility maximization with taxes, dividends and transaction costs under the constant elasticity of variance (CEV) model. The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is established via stochastic control approach. Applying power transform and variable change technique, we obtain explicit solutions for the logarithmic and exponential utility functions. For the quadratic utility function, we obtain the optimal strategy explicitly via Legendre transform and dual theory. Furthermore, we analyze the properties of the optimal strategies. Finally, a numerical simulation is presented to discuss the effects of market parameters on the strategies.

Key–Words: Optimal investment, Transaction cost, Constant elasticity of variance (CEV) model, Utility function, Hamilton-Jacobi-Bellman (HJB) equation, Legendre transform

1 Introduction

Optimal investment problem of utility maximization is an important issue in mathematical finance and has drawn great attention in recent years. Merton [1] proposed the stochastic control approach to study the investment problem for the first time. Pliska [2], Karatzas [3] adapted the martingale approach to investment problems of utility maximization. Zhang [4] investigated the utility maximization problem in an incomplete market using the martingale approach. Recently, more and more researches study the utility maximization problem via stochastic control theory, see e.g., Devolder et al.[5], Yang and Zhang [6] and Wang [7].

However, the price processes of risky assets in most of the above-mentioned literature are assumed to follow the geometric Brownian motion (GBM), which implies the volatility of risky asset is constant and deterministic. This is contradicted by empirical evidence. It is clear that a model with stochastic volatility is more practical.

The constant elasticity of variance (CEV) model is a stochastic volatility model and is a natural extension of the GBM model. The CEV model was proposed by Cox and Ross [8] and has the ability of capturing the implied volatility skew. Furthermore, this model is analytically tractable in comparison with other SV models. At first, the CEV model was usually applied to calculate the theoretical price, sensitivities and implied volatility of options, see e.g., Davydov and Linetsky [9], Jones [10]. Recently, Xiao [11] and Gao [12], [13] have begun to apply the CEV model to investigate the utility maximization problem for a participant in the defined contribution pension plan. Gu [14] used the CEV model for studying the optimal investment and reinsurance problem of utility maximization. Lin and Li [15] considered an optimal reinsurance investment problem for an insurer with jump-diffusion risk process under the CEV model. Zhao and Rong [16] studied the portfolio selection problem with multiple risky assets under the CEV model. Jung [17] gave an explicit optimal investment strategy which maximizes the expected HARA utility of the terminal wealth under the CEV model. In Gu and Guo [18], optimal strategies and optimal value functions are obtained under a CEV model on the condition that the insurer can purchase excess-of-loss reinsurance.

These researches of optimal investment problem under the CEV model generally suppose that there is no taxes, dividends and transaction costs, which is not practical. To make our model more realistic, we consider taxes, dividends and transaction costs for optimal investment problem under the CEV model. The investor aims to maximize the expected utility of his/her terminal wealth and is allowed to invest in a risk-free asset and a risky asset. In addition, we study

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this problem for logarithmic utility, exponential utility and quadratic utility, respectively. By applying the method of stochastic optimal control, the Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is established and transformed into a complicated non-linear partial differential equation (PDE). Due to the difficulty of solution characterization, we use power transform and variable change technique proposed by Cox [19] to simplify the PDE and obtain the explicit solutions for the logarithmic and exponential utility functions. For the quadratic utility function, we use the Legendre transform and dual theory to solve the HJB equation.

This paper proceeds as follows. In Section 2, we formulate the optimal investment problem. Section 3 derives the explicit optimal investment strategy for the logarithmic, exponential and quadratic utility functions respectively. In Section 4, we propose a numerical analysis to illustrate our results. Section 5 concludes the paper.

2 Problem formulation

In this section, we assume that there is a risk-free asset and a risky asset in the financial market. The price of risk-free asset is given by

$$\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \;\; S_0(0) = 1 \tag{1}$$

and the price of risky asset is described by the CEV model (cf. [11] and [12], [13])

$$\mathrm{d}S(t) = S(t)\left(\mu \mathrm{d}t + kS(t)^{\gamma}\mathrm{d}W(t)\right), \tag{2}$$

where μ is the appreciation rate of the risky asset and r_0 is the interest rate. $\{W(t), t > 0\}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. The $\mathcal{F} = (\mathcal{F}_t)$ is an augmented filtration generated by the Brownian motion. $kS(t)^{\gamma}$ is the instantaneous volatility and the elasticity γ is parameter which satisfies the general condition $\gamma \leq 0$.

Remark 1 *If the elasticity parameter $\gamma = 0$ in equation (2), then the CEV model reduces to a GBM.*

The investor is allowed to invest in the risky asset and the risk-free asset. Let $\pi(t)$ be the money amount invested in the risky asset at time t, then $V(t) - \pi(t)$ is the money amount invested in the risk-free asset, where $V(t)$ is the surplus process of the investor. Suppose that the rate of the taxes in financial market is α and b is the dividend income. The rate of transaction costs is θ, which consists of fees and stamp duty. In addition, we assume that $\mu - \theta + b > r_0$. Corresponding to a trading strategy $\pi(t)$ and an initial capital V_0, the wealth process $V(t)$ of the investor follows the dynamics

$$\begin{aligned}\mathrm{d}V(t) &= [\pi(t)(\mu + b - r_0 - \theta) + (r_0 - \alpha)V(t)]\mathrm{d}t \\ &\quad + \pi(t)kS(t)^{\gamma}\mathrm{d}W(t), \\ V(0) &= V_0.\end{aligned} \tag{3}$$

Suppose that the investor has a utility function $U(\cdot)$ which is strictly concave and continuously differentiable on $(-\infty, +\infty)$. The investor aims to maximize the expected utility of his/her terminal wealth, i.e.

$$\max_{(\pi(t))} \mathrm{E}[U(V(T))]. \tag{4}$$

In this paper, we consider the optimization problem (4) for three different utility functions, the logarithmic utility function, exponential utility function and quadratic utility function. They are given by: logarithmic utility

$$U(v) = \ln v, \tag{5}$$

exponential utility

$$U(v) = -\frac{1}{q}\mathrm{e}^{-qv}, \;\; q > 0, \tag{6}$$

and quadratic utility

$$U(v) = (v - c)^2, \tag{7}$$

where q and c are constant coefficients.

3 Optimal investment strategies for different utility functions

In this section, we try to find the explicit solutions for optimization problem (4) under the three utility functions via stochastic optimal control.

3.1 General framework

We define the value function as

$$H(t, s, v) = \max_{\pi} \mathrm{E}[U(V_t) | S = s, V = v], \tag{8}$$

where $0 < t < T$. The corresponding HJB equation is derived by maximum principle

$$\begin{aligned} &H_t + \mu s H_s + (r_0 - \alpha)vH_v + \frac{1}{2}k^2 s^{2\gamma+2}H_{ss} \\ &+ \sup_{\pi}\left\{\pi^2\left(\frac{1}{2}k^2 s^{2\gamma}H_{vv}\right) + \pi\left[k^2 s^{2\gamma+1}H_{sv}\right.\right. \\ &\qquad \left.\left. + (\mu + b - r_0 - \theta)H_v\right]\right\} = 0,\end{aligned} \tag{9}$$

where H_t, H_s, H_v, H_{ss}, H_{vv}, H_{sv} denote partial derivatives of first and second orders with respect to time t, risky asset's price s and wealth v. The boundary condition of this problem is $H(T, s, v) = U(v)$.

Differentiating with respect to π in (9) gives the optimal policy

$$\pi^* = -\frac{(\mu + b - r_0 - \theta)H_v + k^2 s^{2\gamma+1} H_{sv}}{k^2 s^{2\gamma} H_{vv}}. \quad (10)$$

Putting (10) in (9), after simplification, we obtain

$$\begin{aligned} &H_t + \mu s H_s + (r_0 - \alpha) v H_v + \frac{1}{2} k^2 s^{2\gamma+2} H_{ss} \\ &-\frac{\left[(\mu + b - r_0 - \theta)H_v + k^2 s^{2\gamma+1} H_{sv}\right]^2}{2k^2 s^{2\gamma} H_{vv}} = 0 \end{aligned} \quad (11)$$

with $V(T, s, v) = U(v)$.

The problem now is to solve equation (11) for H and replace it in (10) to obtain the optimal strategy.

3.2 Optimal strategy for the logarithmic utility function

According to the logarithmic utility function described by (5), we construct the solution to (11) with the following form

$$H(t, s, v) = g(t, s) \ln v + a(t, s) \quad (12)$$

and the boundary conditions are given by $g(T, s) = 1, a(T, s) = 0$. Then

$$\begin{aligned} &H_t = g_t \ln v + a_t, \;\; H_s = g_s \ln v + a_s, \\ &H_v = \frac{g}{v}, \;\; H_{ss} = g_{ss} \ln v + a_{ss}, \\ &H_{vv} = -\frac{g}{v^2}, \;\; H_{sv} = \frac{g_s}{v}. \end{aligned}$$

Introducing these derivatives into (11), we derive

$$\begin{aligned} &\left(g_t + \mu s g_s + \frac{1}{2} k^2 s^{2\gamma+2} g_{ss}\right) \ln v + a_t \\ &+\mu s a_s + (r_0 - \alpha) g + \frac{1}{2} k^2 s^{2\gamma+2} a_{ss} \\ &+\frac{\left[(\mu + b - r_0 - \theta) g + k^2 s^{2\gamma+1} g_s\right]^2}{2g k^2 s^{2\gamma}} = 0. \end{aligned} \quad (13)$$

To solve this equation, we can split (13) into two equations

$$g_t + \mu s g_s + \frac{1}{2} k^2 s^{2\gamma+2} g_{ss} = 0, \quad (14)$$

$$\begin{aligned} &a_t + \mu s a_s + (r_0 - \alpha) g + \frac{1}{2} k^2 s^{2\gamma+2} a_{ss} \\ &+\frac{\left[(\mu + b - r_0 - \theta) g + k^2 s^{2\gamma+1} g_s\right]^2}{2g k^2 s^{2\gamma}} = 0. \end{aligned} \quad (15)$$

Taking the boundary condition $g(T, s) = 1$ into account, we obtain the solution to (14) by using power transformation and variable change technique.

Let

$$g(t, s) = h(t, y), \;\; y = s^{-2\gamma} \quad (16)$$

and the boundary condition is $h(T, y) = 1$. Then

$$\begin{aligned} &g_t = h_t, \;\; g_s = -2\gamma s^{-2\gamma-1} h_y, \\ &g_{ss} = 2\gamma(2\gamma + 1) s^{-2\gamma-2} h_y + 4\gamma^2 s^{-4\gamma-2} h_{yy}. \end{aligned}$$

Substituting these derivatives into (14), we derive

$$\begin{aligned} &h_t + \gamma \left[(2\gamma + 1) k^2 - 2\mu y\right] h_y \\ &+2\gamma^2 k^2 y h_{yy} = 0. \end{aligned} \quad (17)$$

We try to find a solution to (17) with the following form

$$h(t, y) = A(t) + B(t) y \quad (18)$$

with the boundary conditions $A(T) = 1$, $B(T) = 0$. Then

$$h_t = A_t + B_t y, \;\; h_y = B, \;\; h_{yy} = 0.$$

Introducing these derivatives into (17), we obtain

$$A_t + \gamma (2\gamma + 1) k^2 B + y \left[B_t - 2\gamma \mu B\right] = 0. \quad (19)$$

Equation (19) can be split into two equations

$$A_t + \gamma (2\gamma + 1) k^2 B = 0. \quad (20)$$

$$B_t - 2\gamma \mu B = 0. \quad (21)$$

Considering the boundary conditions $A(T) = 1$ and $B(T) = 0$, we get the solutions to (20) and (21)

$$A(t) = 1, \;\; B(t) = 0. \quad (22)$$

Then

$$h(t, y) = 1, \;\; g(t, s) = 1$$

and (15) is simplified into

$$\begin{aligned} &a_t + \mu s a_s + r_0 - \alpha + \frac{1}{2} k^2 s^{2\gamma+2} a_{ss} \\ &+\frac{(\mu + b - r_0 - \theta)^2}{2k^2 s^{2\gamma}} = 0. \end{aligned} \quad (23)$$

Let

$$a(t, s) = w(t, y), \;\; y = s^{-2\gamma}$$

and the boundary condition is $w(T, y) = 0$. Then

$$\begin{aligned} &a_t = w_t, \;\; a_s = -2\gamma s^{-2\gamma-1} w_y, \\ &a_{ss} = 2\gamma(2\gamma + 1) s^{-2\gamma-2} w_y + 4\gamma^2 s^{-4\gamma-2} w_{yy}. \end{aligned}$$

Putting these derivatives into (23), we derive

$$\begin{aligned} & w_t + \gamma[(2\gamma+1)k^2 - 2\mu y]w_y + 2k^2\gamma^2 y w_{yy} \\ & +r_0 - \alpha + \frac{(\mu+b-r_0-\theta)^2 y}{2k^2} = 0. \end{aligned} \tag{24}$$

Assume the solution to (24) is

$$w(t,y) = C(t) + D(t)y$$

with the boundary conditions $C(t) = 0$ and $D(t) = 0$. Then

$$w_t = C_t + D_t y, \;\; w_y = D, \;\; w_{yy} = 0.$$

Introducing these derivatives into (24), we obtain

$$\begin{aligned} & C_t + \gamma(2\gamma+1)k^2 D + r_0 - \alpha \\ & + \left\{ D_t - 2\mu\gamma D + \frac{(\mu+b-r_0-\theta)^2}{2k^2} \right\} y = 0. \end{aligned} \tag{25}$$

Splitting (25) into two equations, we have

$$C_t + \gamma(2\gamma+1)k^2 D + r_0 - \alpha = 0, \tag{26}$$

$$D_t - 2\mu\gamma D + \frac{(\mu+b-r_0-\theta)^2}{2k^2} = 0. \tag{27}$$

Considering the boundary conditions $C(T) = 0$ and $D(T) = 0$, we find the solutions to (26) and (27) are

$$\begin{aligned} & D(t) \\ & = \frac{(\mu+b-r_0-\theta)^2}{4k^2\gamma\mu}\left[1 - \exp(2\gamma\mu(t-T))\right], \end{aligned}$$

$$\begin{aligned} C(t) \;\; & = \frac{(2\gamma+1)(\mu+b-r_0-\theta)^2}{4\mu}\Bigg[T - t \\ & - \frac{1-\exp(2\gamma\mu(t-T))}{2\gamma\mu}\Bigg] + (r_0-\alpha)(T-t). \end{aligned}$$

Therefore,

$$a(t,s) = C(t) + D(t)s^{-2\gamma}.$$

The following theorem gives the optimal investment strategy for the logarithmic utility function.

Theorem 2 *The optimal strategy invested in the risky asset under the logarithmic utility function is given by*

$$\pi^*(t) = \frac{\mu+b-r_0-\theta}{k^2 s^{2\gamma}} v. \tag{28}$$

Proof: From (10), (12), (16), (18) and (22), we have

$$\begin{aligned} \pi^* \;\; & = -\frac{(\mu+b-r_0-\theta)H_v + k^2 s^{2\gamma+1} H_{sv}}{k^2 s^{2\gamma} H_{vv}} \\ & = -\frac{(\mu+b-r_0-\theta)\dfrac{g}{v} + k^2 s^{2\gamma+1}\dfrac{g_s}{v}}{-k^2 s^{2\gamma}\dfrac{g}{v^2}} \\ & = \frac{\mu+b-r_0-\theta}{k^2 s^{2\gamma}} v. \end{aligned}$$

□

Remark 3 *From Theorem 2, we find that taxes have no effect on the optimal strategy. The optimal investment increases with dividends and decreases with transaction costs. In addition, the optimal strategy is an increasing function of appreciation rate μ and decreasing function of the elasticity parameter γ.*

Remark 4 *If taxes, dividends and transaction costs are not considered in this paper, i.e. $\alpha = b = \theta = 0$, (28) reduces to the optimal strategy derived by Xiao [11].*

3.3 Optimal strategy for the exponential utility function

According to the exponential utility function described by (6), we conjecture a solution to (11) in the following form

$$\begin{aligned} & H(t,s,v) \\ & = -\frac{1}{q}\exp\{-q[a(t)(v-d(t)) + g(t,s)]\} \end{aligned} \tag{29}$$

with $g(T,s) = 0$, $a(T) = 1$ and $d(T) = 0$. Then

$$\begin{aligned} & H_t = -q[a_t(v-d) - d_t a + g_t]H, \\ & H_s = -q g_s H, \;\; H_v = -qaH, \\ & H_{ss} = (q^2 g_s^2 - q g_{ss})H, \\ & H_{vv} = q^2 a^2 H, \;\; H_{sv} = q^2 a g_s H. \end{aligned}$$

Introducing these derivatives into (11), we derive

$$\begin{aligned} & [a_t + (r_0-\alpha)a]\, v - (d_t + a_t d a^{-1})a \\ & + \Bigg\{ g_t - (b-r_0-\theta)s g_s \\ & + \frac{1}{2}k^2 s^{2\gamma+2} g_{ss} + \frac{(\mu+b-r_0-\theta)^2}{2qk^2 s^{2\gamma}} \Bigg\} = 0. \end{aligned} \tag{30}$$

We can decompose (30) into three equations

$$a_t + (r_0-\alpha)a(t) = 0, \tag{31}$$

$$d_t + a_t d(t) a^{-1} = 0, \tag{32}$$

$$\begin{aligned} & g_t - (b - r_0 - \theta) s g_s + \frac{1}{2} k^2 s^{2\gamma+2} g_{ss} \\ & + \frac{(\mu + b - r_0 - \theta)^2}{2qk^2 s^{2\gamma}} = 0. \end{aligned} \tag{33}$$

Taking the boundary condition $a(T) = 1$ into account, we find the solution to (31) is

$$a(t) = \exp\{(r_0 - \alpha)(T - t)\}. \tag{34}$$

Similar to the logarithmic utility, we assume that

$$g(t, s) = h(t, y), \;\; y = s^{-2\gamma} \tag{35}$$

and $h(T, y) = 0$. Then

$$\begin{aligned} & g_t = h_t, \;\; g_s = -2\gamma s^{-2\gamma-1} h_y, \\ & g_{ss} = 2\gamma(2\gamma + 1) s^{-2\gamma-2} h_y + 4\gamma^2 s^{-4\gamma-2} h_{yy}. \end{aligned}$$

Substituting these derivatives into (33), we obtain

$$\begin{aligned} & h_t + \gamma[(2\gamma + 1)k^2 + 2(b - r_0 - \theta)y] h_y \\ & + 2\gamma^2 k^2 y h_{yy} + \frac{(\mu + b - r_0 - \theta)^2 y}{2qk^2} = 0. \end{aligned} \tag{36}$$

We try ro find a solution to (36) with the following form

$$h(t, y) = P(t) + Q(t) y \tag{37}$$

and $P(T) = 0$, $Q(T) = 0$. Then

$$h_t = P_t + Q_t y, \;\; h_y = Q, \;\; h_{yy} = 0.$$

Introducing these derivatives into (36), we derive

$$\begin{aligned} P_t + \gamma(2\gamma + 1)k^2 Q + y \Big\{ & Q_t + 2\gamma(b - r_0 - \theta) Q \\ & + \frac{(\mu + b - r_0 - \theta)^2}{2qk^2} \Big\} = 0. \end{aligned} \tag{38}$$

Again we can split (38) into two equations

$$P_t + \gamma(2\gamma + 1) k^2 Q = 0. \tag{39}$$

$$Q_t + 2\gamma(b - r_0 - \theta) Q + \frac{(\mu + b - r_0 - \theta)^2}{2qk^2} = 0. \tag{40}$$

According to $P(T) = 0$, $Q(T) = 0$, we obtain the solutions to (39) and (40)

$$\begin{aligned} Q(t) \;\; & = -\frac{(\mu + b - r_0 - \theta)^2}{4qk^2 \gamma (b - r_0 - \theta)} \\ & \cdot [1 - \exp(-2\gamma(b - r_0 - \theta)(t - T))], \end{aligned} \tag{41}$$

$$\begin{aligned} P(t) \;\; & = -\tfrac{(2\gamma+1)(\mu+b-r_0-\theta)^2}{4q(b-r_0-\theta)} \\ & \Big[T - t + \tfrac{1-\exp(-2\gamma(b-r_0-\theta)(t-T))}{2\gamma(b-r_0-\theta)} \Big]. \end{aligned} \tag{42}$$

Therefore,

$$g(t, s) = P(t) + Q(t) s^{-2\gamma}.$$

Equation (32) is transformed into

$$d_t - (r_0 - \alpha) d(t) = 0.$$

With $d(T) = 0$, we obtain $d(t) = 0$.

From the above analysis, we obtain the optimal investment strategy under the exponential utility function.

Theorem 5 *The optimal strategy invested in the risky asset for the exponential utility function is given by*

$$\begin{aligned} \pi^*(t) \;\; & = \frac{\mu + b - r_0 - \theta}{qk^2 s^{2\gamma}} \exp\{(r_0 - \alpha)(t - T)\} \\ & \cdot \Big\{ 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)} \\ & \cdot [1 - \exp(-2\gamma(b - r_0 - \theta)(t - T))] \Big\}. \end{aligned} \tag{43}$$

Proof: From (10), (29), (34), (35), (37) and (41), we derive

$$\begin{aligned} & \pi^* \\ = & -\frac{(\mu + b - r_0 - \theta) H_v + k^2 s^{2\gamma+1} H_{sv}}{k^2 s^{2\gamma} H_{vv}} \\ = & -\frac{-(\mu + b - r_0 - \theta) q a H + k^2 s^{2\gamma+1} q^2 a g_s H}{k^2 s^{2\gamma} q^2 a^2 H} \\ = & \frac{(\mu + b - r_0 - \theta) + 2k^2 \gamma q h_y}{q a(t) k^2 s^{2\gamma}} \\ = & \frac{\mu + b - r_0 - \theta}{q a(t) k^2 s^{2\gamma}} [1 + \frac{2q\gamma k^2 Q(t)}{\mu + b - r_0 - \theta}] \\ = & \frac{\mu + b - r_0 - \theta}{qk^2 s^{2\gamma}} \exp\{(r_0 - \alpha)(t - T)\} \\ & \cdot \Big\{ 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)} \\ & \cdot [1 - \exp(-2\gamma(b - r_0 - \theta)(t - T))] \Big\}. \end{aligned}$$

□

Remark 6 *For exponential utility function, we conclude that the wealth has no influence on the optimal strategy from Theorem 5. This can be explained by the risk tolerance of the exponential utility function. According to (6), the risk tolerance is* $-U'(v)/U''(v) = 1/q$*, which is independent of the wealth.*

In addition, the optimal amount invested in the risky asset increases with dividends and decreases with transaction costs.

If $\alpha = b = \theta = 0$, the optimal strategy (43) degenerates to that of [12],[13].

Remark 7 *In the case that the risky asset's price follows the GBM model, the optimal strategy is*

$$\pi^*(t) = \frac{\mu + b - r_0 - \theta}{qk^2} \exp\{(r_0 - \alpha)(t - T)\}. \quad (44)$$

Compared to (44), we see that optimal strategy under the CEV model can be decomposed into two parts. One is

$$M(t) = \frac{\mu + b - r_0 - \theta}{qk^2 s^{2\gamma}} \exp\{(r_0 - \alpha)(t - T)\},$$

which is similar to the optimal strategy under the GBM model, but the volatility is stochastic. Thus we call $M(t)$ as the moving GBM strategy. The other one is

$$\begin{aligned} N(t) \quad &= 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)} \\ &\cdot [1 - \exp(-2\gamma(b - r_0 - \theta)(t - T))], \end{aligned}$$

which reflects an investor's decision to hedge the volatility risk and we regard it as a correction factor.

In the following, we discuss the properties of the correction factor.

Corollary 8 *The correction factor $N(t)$ is a monotone increasing function with respect to time t and satisfies*

$$\begin{aligned} 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)} &[1 - \exp(2\gamma(b - r_0 - \theta)T)] \\ &\leq N(t) \leq 1, \end{aligned}$$

where $t \in [0, T]$.

Proof: Note that

$$\begin{aligned} N(t) \quad &= 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)} \\ &\cdot [1 - \exp(-2\gamma(b - r_0 - \theta)(t - T))] \end{aligned}$$

and $\mu > r_0 + \theta - b$, $\gamma < 0$. Then

$$N_t = -\gamma(\mu + b - r_0 - \theta) \exp(-2\gamma(b - r_0 - \theta)(t - T)).$$

We find that $N_t > 0$, which implies that the correction factor is a monotone increasing function with respect to time t. Since

$$N(0) = 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)}[1 - \exp(2\gamma(b - r_0 - \theta)T)]$$

and $N(T) = 1$, we have

$$\begin{aligned} 1 - \frac{\mu + b - r_0 - \theta}{2(b - r_0 - \theta)} &[1 - \exp(2\gamma(b - r_0 - \theta)T)] \\ &\leq N(t) \leq 1, \end{aligned}$$

where $t \in [0, T]$. □

Corollary 8 shows that the correction factor advises the investor to invest a lower proportion of wealth in risky asset at the beginning of the investment horizon and steadily increase the amount as time goes on.

3.4 Optimal strategy for the quadratic utility function

According to the quadratic utility function described by (7), we cannot conjecture a solution to (11) directly. Therefore we solve problem (4) for the quadratic utility function by applying Legendre transform and dual theory.

Definition 9 *Let $f : R^n \to R$ be a convex function, for $z > 0$, define a Legendre transform*

$$L(z) = \max_x \{f(x) - zx\}. \quad (45)$$

The function $L(z)$ is called the Legendre dual of function $f(x)$ (cf. [20]).

If $f(x)$ is strictly convex, the maximum in (45) will be attained at just one point, which we denote by x_0 and then

$$L(z) = f(x_0) - zx_0. \quad (46)$$

Following [11] and [13], we define a Legendre transform

$$\hat{H}(t, s, z) = \sup_{v>0} \{H(t, s, v) - zv | 0 < v < \infty\}, \quad (47)$$

where $0 < t < T$, and $z > 0$ denotes the dual variable to v. The value of v where this optimum is attained is denoted by $g(t, s, z)$. Therefore,

$$g(t, s, z) = \inf_{v>0} \left\{ v | H(t, s, v) \geq zv + \hat{H}(t, s, z) \right\},$$

where $0 < t < T$. The function $\hat{H}$ is related to g by

$$g = -\hat{H}_z. \quad (48)$$

At the terminal time, we denote

$$\begin{aligned} \hat{U}(z) &= \sup_{v>0} \{U(v) - zv | 0 < v < \infty\}, \\ G(z) &= \inf_{v>0} \left\{ v | U(v) \geq zv + \hat{U}(z) \right\}. \end{aligned}$$

As a result,

$$G(z) = (U')^{-1}(z).$$

Since $H(T, s, v) = U(v)$, we have

$$g(T, s, v) = \inf_{v>0} \left\{ v | U(v) \geq zv + \hat{H}(T, s, z) \right\},$$
$$\hat{H}(T, s, z) = \sup_{v>0} \left\{ U(v) - zv \right\}.$$

Therefore,

$$g(T, s, z) = (U')^{-1}(z). \tag{49}$$

According to (46) and (47), we derive

$$H_v = z \tag{50}$$

and thus

$$\hat{H}(t, s, z) = H(t, s, g) - zg, \quad g(t, s, z) = v. \tag{51}$$

Differentiating(50) and (51) with respect to t, s and z, we obtain the following derivatives of H and $\hat{H}$

$$\begin{aligned} H_t = \hat{H}_t, \quad H_s = \hat{H}_s, \quad H_{ss} = \hat{H}_{ss} - \frac{\hat{H}_{sz}^2}{\hat{H}_{sz}}, \\ \hat{H}_{sv} = -\frac{\hat{H}_{sz}}{\hat{H}_{zz}}, \quad \hat{H}_{vv} = -\frac{1}{\hat{H}_{zz}}. \end{aligned} \tag{52}$$

Introducing (50), (52) into (11) results

$$\begin{aligned} &\hat{H}_t + \mu s \hat{H}_s + (r_0 - \alpha) vz + \frac{1}{2} k^2 s^{2\gamma+2} \hat{H}_{ss} \\ &+ \frac{(\mu + b - r_0 - \theta)^2}{2k^2 s^{2\gamma}} z^2 \hat{H}_{zz} \\ &- z(\mu + b - r_0 - \theta) s \hat{H}_{sz} = 0. \end{aligned}$$

From (50) and (51), the above equation can be transformed into

$$\begin{aligned} &g_t + \mu s g_s - (r_0 - \alpha)(g + z g_z) \\ &+ \frac{1}{2} k^2 s^{2\gamma+2} g_{ss} + \frac{(\mu + b - r_0 - \theta)^2}{k^2 s^{2\gamma}} z g_z \\ &+ \frac{(\mu + b - r_0 - \theta)^2}{2k^2 s^{2\gamma}} z^2 g_{zz} \\ &- s(\mu + b - r_0 - \theta)(g_s + z g_{sz}) = 0. \end{aligned} \tag{53}$$

According to (10), (48), (50), (51) and (52), the optimal policy is rewritten as

$$\begin{aligned} &\pi^* \\ &= -\frac{(\mu + b - r_0 - \theta) H_v + k^2 s^{2\gamma+1} H_{sv}}{k^2 s^{2\gamma} H_{vv}} \\ &= \frac{z(\mu + b - r_0 - \theta) + k^2 s^{2\gamma+1} (-\frac{\hat{H}_{sz}}{\hat{H}_{zz}})}{-\frac{1}{\hat{H}_{zz}} k^2 s^{2\gamma}} \\ &= \frac{k^2 s^{2\gamma+1} g_s - z(\mu + b - r_0 - \theta) g_z}{k^2 s^{2\gamma}}. \end{aligned} \tag{54}$$

From (7) and (49), the boundary condition is

$$g(T, s, z) = \frac{1}{2} z + c.$$

So we conjecture a solution to (53) with the following structure

$$g(t, s, z) = z h(t, s) + a(t) \tag{55}$$

and the boundary conditions are given by $a(T) = c$, $h(T, s) = \frac{1}{2}$. Then

$$g_t = z h_t + a_t, \quad g_s = z h_s, \quad g_{ss} = z h_{ss},$$
$$g_z = h, \quad g_{zz} = 0, \quad g_{sz} = h_s.$$

Introducing these derivatives into (53), we derive

$$\begin{aligned} &\Big\{ h_t + (2r_0 + 2\theta - \mu - 2b) s h_s \\ &+ \frac{1}{2} k^2 s^{2\gamma+2} h_{ss} + \frac{(\mu + b - r_0 - \theta)^2}{k^2 s^{2\gamma}} h \\ &- 2(r_0 - \alpha) h \Big\} z + a_t - (r_0 - \alpha) a(t) = 0. \end{aligned} \tag{56}$$

We split (56) into two equations

$$\begin{aligned} &h_t + (2r_0 + 2\theta - \mu - 2b) s h_s + \frac{1}{2} k^2 s^{2\gamma+2} h_{ss} \\ &+ \frac{(\mu + b - r_0 - \theta)^2}{k^2 s^{2\gamma}} h - 2(r_0 - \alpha) h = 0, \end{aligned} \tag{57}$$

$$a_t - (r_0 - \alpha) a(t) = 0. \tag{58}$$

Taking the boundary condition $a(T) = c$ into account, we find that the solution to (58) is

$$a(t) = c \exp \left\{ (r_0 - \alpha)(t - T) \right\}. \tag{59}$$

It is difficult to solve equation (57), so we use a power transformation and a variable change technique to transform it into a linear one.

Let

$$h(t, s) = f(t, y), \quad y = s^{-2\gamma} \tag{60}$$

and the boundary condition is $f(T, y) = \frac{1}{2}$. Then

$$h_t = f_t, \quad h_s = -2\gamma s^{-2\gamma-1} f_y,$$
$$h_{ss} = 2\gamma(2\gamma + 1) s^{-2\gamma-2} f_y + 4\gamma^2 s^{-4\gamma-2} f_{yy}.$$

Substituting these derivatives into (57), we have

$$\begin{aligned} &f_t + \gamma \left[-2(2r_0 + 2\theta - \mu - 2b) y \right. \\ &\left. + k^2 (2\gamma + 1) \right] f_y + 2k^2 \gamma^2 y f_{yy} \\ &+ \frac{(\mu + b - r_0 - \theta)^2}{k^2} y f - 2(r_0 - \alpha) f = 0. \end{aligned} \tag{61}$$

Let

$$f(t,y) = Y(t)\exp\{Z(t)y\} \tag{62}$$

with the boundary conditions given by $Y(T) = \frac{1}{2}$, $Z(T) = 0$. Then

$$f_t = Y_t \exp(Zy) + YZ_t \exp(Zy),$$
$$f_y = YZ\exp(Zy), \quad f_{yy} = YZ^2 \exp(Zy).$$

Substituting these derivatives in (61) leads to

$$\begin{aligned} &\frac{Y_t}{Y} + \gamma(2\gamma+1)k^2 Z - 2(r_0 - \alpha) \\ &+y\Big\{ Z_t - 2\gamma(2r_0 + 2\theta - \mu - 2b)Z \\ &+2k^2\gamma^2 Z^2 + \frac{(\mu + b - r_0 - \theta)^2}{k^2}\Big\} = 0. \end{aligned} \tag{63}$$

Similarly, equation (63) is decomposed into two equations

$$\begin{aligned} Z_t - 2\gamma(2r_0 + 2\theta - \mu - 2b)Z + 2k^2\gamma^2 Z^2 \\ + \frac{(\mu + b - r_0 - \theta)^2}{k^2} = 0, \end{aligned} \tag{64}$$

$$\frac{Y_t}{Y} + \gamma(2\gamma+1)k^2 Z - 2(r_0 - \alpha) = 0. \tag{65}$$

Let $l = -2\gamma^2$, $m = 2\gamma(2r_0 + 2\theta - \mu - 2b)$ and $n = -(\mu + b - r_0 - \theta)^2$. Thus, (64) is written as

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = lk^2 Z^2 + mZ + \frac{n}{k^2}, \quad Z(T) = 0,$$

or

$$\frac{\mathrm{d}Z}{lk^2 Z^2 + mZ + \frac{n}{k^2}} = \mathrm{d}t, \quad Z(T) = 0. \tag{66}$$

Next we solve (66) under two cases.

Case 1. $m^2 - 4nl > 0$.

Integrating (66) on both sides with respect to the time t, we obtain

$$\frac{\int \left(\frac{1}{Z(t) - x_1} - \frac{1}{Z(t) - x_2}\right)\mathrm{d}Z(t)}{lk^2(x_1 - x_2)} = t + C_1,$$

where C_1 is a constant and $x_{1,2}$ are the solutions of the quadratic equation

$$lk^2 Z^2 + mZ + \frac{n}{k^2} = 0.$$

Namely,

$$x_{1,2} = \frac{-m \pm \sqrt{m^2 - 4nl}}{2lk^2}.$$

Considering the boundary condition $Z(T) = 0$, we have

$$Z(t) = \frac{x_1 - x_1 \exp\{lk^2(x_1 - x_2)(t-T)\}}{1 - \frac{x_1}{x_2}\exp\{lk^2(x_1 - x_2)(t-T)\}}. \tag{67}$$

Define

$$\lambda_{1,2} = \frac{-m \pm \sqrt{m^2 - 4nl}}{2l},$$

$$I(t) = \frac{\lambda_1 - \lambda_1 \exp\{2\gamma^2(\lambda_1 - \lambda_2)(t-T)\}}{1 - \frac{\lambda_1}{\lambda_2}\exp\{2\gamma^2(\lambda_1 - \lambda_2)(t-T)\}} \tag{68}$$

and then (67) is rewritten as

$$Z(t) = k^{-2} I(t). \tag{69}$$

Putting (69) into (65), we get

$$\frac{\mathrm{d}Y}{Y} = [-\gamma(2\gamma+1)I(t) + 2(r_0 - \alpha)]\mathrm{d}t.$$

Note that

$$\begin{aligned} &\int I(t)\mathrm{d}t = \lambda_1 t \\ &+\frac{1}{2\gamma^2}\ln\{\lambda_2 - \lambda_1 \exp(2\gamma^2(\lambda_1 - \lambda_2)(T-t))\} + C_2, \end{aligned}$$

where C_2 is a constant, then

$$\begin{aligned} Y(t) &= \frac{1}{2}\exp\{[\lambda_1\gamma(2\gamma+1) - 2(r_0 - \alpha)](T-t)\} \\ &\cdot \left\{\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1 \exp(2\gamma^2(\lambda_1 - \lambda_2)(T-t))}\right\}^{\frac{2\gamma+1}{2\gamma}}. \end{aligned}$$

Case 2. $m^2 - 4nl \le 0$.

Equation (66) is written as

$$\frac{\mathrm{d}Z}{(Z + \frac{m}{2lk^2})^2 + \frac{4nl - m^2}{4l^2k^4}} = lk^2 \mathrm{d}t. \tag{70}$$

Let

$$p^2 = \frac{4nl - m^2}{4l^2k^4},$$

integrating (70) on both sides with respect to t, we derive

$$\arctan\left(\frac{Z + \frac{m}{2lk^2}}{p}\right) = plk^2 t + C_3,$$

where C_3 is a constant. Considering the boundary condition, we have

$$\begin{aligned} Z(t) &= -\frac{m}{2lk^2} \\ &+p\tan\left[plk^2(t-T) + \arctan\left(\frac{m}{2plk^2}\right)\right]. \end{aligned} \tag{71}$$

Rewriting (65) as

$$\frac{Y_t}{Y} = -\gamma(2\gamma+1)k^2 Z + 2(r_0 - \alpha). \qquad (72)$$

and integrating (72) on both sides with respect to t, we obtain

$$\ln Y(t) = 2(r_0-\alpha)t - \gamma(2\gamma+1)k^2 \int_0^t Z(s)\mathrm{d}s + C_4,$$

where C_4 is a constant. From (71), we obtain

$$\int_0^t Z(s)\mathrm{d}s = -\frac{1}{lk^2}\ln\left\{\cos\left[plk^2(t-T) + \arctan\left(\frac{m}{2plk^2}\right)\right]\right\} - \frac{m}{2lk^2}t + C_5,$$

where C_5 is a constant. Let

$$G(t) = -\frac{1}{l}\ln\left\{\cos\left[plk^2(t-T) + \arctan\left(\frac{m}{2plk^2}\right)\right]\right\} - \frac{m}{2l}t.$$

Combining with $Y(T) = \frac{1}{2}$, we obtain

$$Y(t) = \frac{1}{2}\exp\left\{2(r_0-\alpha)(t-T) + \gamma(2\gamma+1)(G(T)-G(t))\right\}.$$

Now we have the following conclusion.

Theorem 10 *The optimal strategy for the quadratic utility function is given by*

Case 1. $m^2 - 4nl > 0$,

$$\pi^*(t) = \frac{\mu+b-r_0-\theta}{k^2 s^{2\gamma}}(v-a(t)) \cdot \left(-\frac{2\gamma I(t)}{\mu+b-r_0-\theta} - 1\right),$$

Case 2. $m^2 - 4nl \le 0$,

$$\pi^*(t) = \frac{\mu+b-r_0-\theta}{k^2 s^{2\gamma}}(v-a(t)) \cdot \left(-\frac{2\gamma J(t)}{\mu+b-r_0-\theta} - 1\right),$$

where $l = -2\gamma^2$, $m = 2\gamma(2r_0 + 2\theta - \mu - 2b)$, $n = -(\mu+b-r_0-\theta)^2$, $J(t) = k^2 Z(t)$ *and* $a(t)$, $I(t)$ *are shown by (59), (68).*

Proof: From (54), (55), (59), (60) and (62), we obtain

$$\begin{aligned}
\pi^* &= \frac{k^2 s^{2\gamma+1} g_s - z(\mu+b-r_0-\theta)g_z}{k^2 s^{2\gamma}} \\
&= \frac{k^2 s^{2\gamma+1} z h_s - z(\mu+b-r_0-\theta)h}{k^2 s^{2\gamma}} \\
&= \frac{k^2 s^{2\gamma+1}\frac{g-a(t)}{h}(-2\gamma s^{-2\gamma-1} f_y)}{k^2 s^{2\gamma}} - \frac{\frac{g-a(t)}{h}(\mu+b-r_0-\theta)h}{k^2 s^{2\gamma}} \\
&= \frac{(g-a(t))[-2\gamma k^2 \frac{f_y}{f} - (\mu+b-r_0-\theta)]}{k^2 s^{2\gamma}} \\
&= \frac{(v-a(t))[-2\gamma k^2 Z - (\mu+b-r_0-\theta)]}{k^2 s^{2\gamma}},
\end{aligned}$$

then according to (68), (69) and (71), we have that when $m^2 - 4nl > 0$,

$$\begin{aligned}
\pi^* &= \frac{(v-a(t))[-2\gamma k^2 Z - (\mu+b-r_0-\theta)]}{k^2 s^{2\gamma}} \\
&= \frac{(v-a(t))[-2\gamma I(t) - (\mu+b-r_0-\theta)]}{k^2 s^{2\gamma}} \\
&= \frac{\mu+b-r_0-\theta}{k^2 s^{2\gamma}}(v-a(t)) \cdot \left(-\frac{2\gamma I(t)}{\mu+b-r_0-\theta} - 1\right),
\end{aligned}$$

and when $m^2 - 4nl \le 0$,

$$\begin{aligned}
\pi^* &= \frac{(v-a(t))[-2\gamma k^2 Z - (\mu+b-r_0-\theta)]}{k^2 s^{2\gamma}} \\
&= \frac{(v-a(t))[-2\gamma J(t) - (\mu+b-r_0-\theta)]}{k^2 s^{2\gamma}} \\
&= \frac{\mu+b-r_0-\theta}{k^2 s^{2\gamma}}(v-a(t)) \cdot \left(-\frac{2\gamma J(t)}{\mu+b-r_0-\theta} - 1\right),
\end{aligned}$$

where $l = -2\gamma^2$, $m = 2\gamma(2r_0 + 2\theta - \mu - 2b)$, $n = -(\mu+b-r_0-\theta)^2$, $J(t) = k^2 Z(t)$ and $a(t)$, $I(t)$ are shown by (59), (68). □

Remark 11 *For the quadratic utility function, there are two expressions of the optimal investment strategy for different parameters.*

4 Numerical analysis

This section provides some numerical analysis to illustrate our results. Gao [12], Zhao and Rong [16] have analyzed the sensitivities of the correction factor and optimal strategies with respect to the parameter of CEV model. Thus, we mainly consider the effect of taxes, dividends and transaction costs on the optimal strategies. Throughout numerical analysis, unless otherwise stated, the basic parameters are given by: $r_0 = 0.03$, $\mu = 0.12$, $k = 16.16$, $b = 0.05$, $\theta = 0.004$, $\alpha = 0.01$, $S(0) = 67$, $v = 1000$, $\gamma = 0.12$, $t = 5$, $T = 20$, $q = 0.05$, $c = 1$.

4.1 Numerical analysis under the logarithmic utility case

Figure 1 shows the effect of dividend rate b on the optimal strategy for the logarithmic utility function. From Figure 1, we find that the amount invested in risky asset increases with dividend rate. This is because that as b increases, investors will get more profits from risky assets. Therefore, investors would like to put more money in the risky asset to gain more profits.

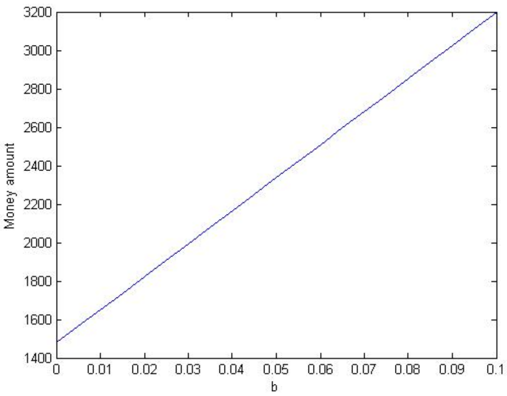

Figure 1: Sensitivity of the optimal strategy w.r.t b

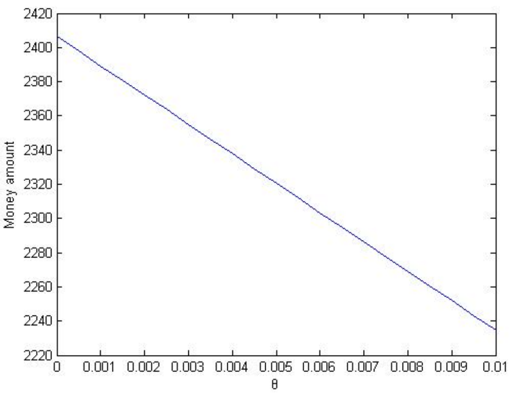

Figure 2: Sensitivity of the optimal strategy w.r.t θ

From Figure 2, we see that the rate of transaction cost θ exerts a negative effect on the optimal strategy.

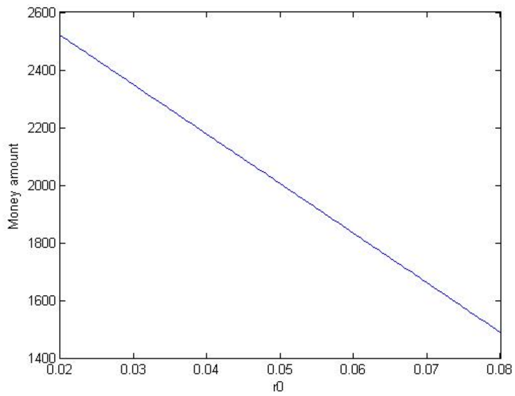

Figure 3: Sensitivity of the optimal strategy w.r.t r_0

Transaction cost arises from investing in the risky asset. Thus, the optimal strategy decreases with θ.

In Figure 3, we find that the optimal investment strategy is a decreasing function of the interest rate r_0. When the interest rate r_0 increases, the risk-free asset is more attractive. Therefore, investors will invest more in risk-free asset.

4.2 Numerical analysis under the exponential utility case

Figure 4, Figure 5 and Figure 6 show the effect of dividend rate b, the rate of transaction cost θ and interest rate r_0 on the optimal strategy for the exponential utility function respectively. As shown in these figures, we find that the effects are similar to those under the logarithmic utility cases.

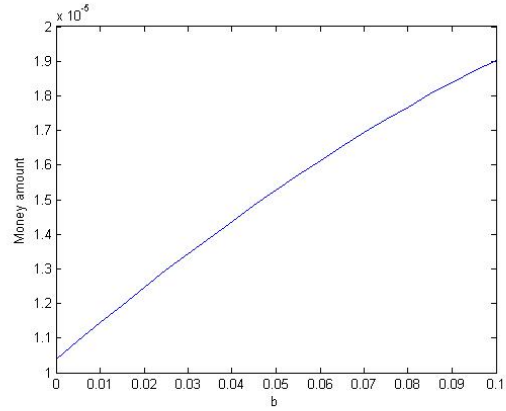

Figure 4: Sensitivity of the optimal strategy w.r.t b

Figure 7 illustrates the effect of the taxes rate α on the optimal strategy under the exponential utility function. we see that α has a positive effect on the optimal strategy. Since investors should pay taxes as long as he/she gets revenue, taxes lead to the decrease of

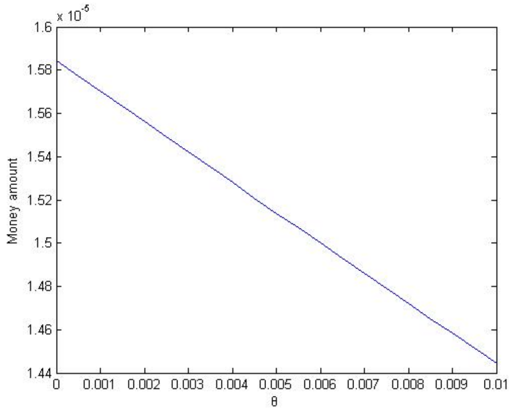

Figure 5: Sensitivity of the optimal strategy w.r.t θ

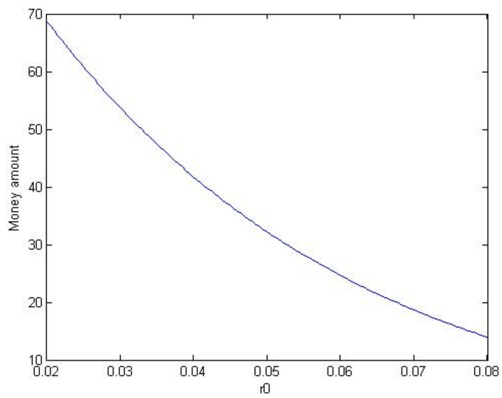

Figure 6: Sensitivity of the optimal strategy w.r.t r_0

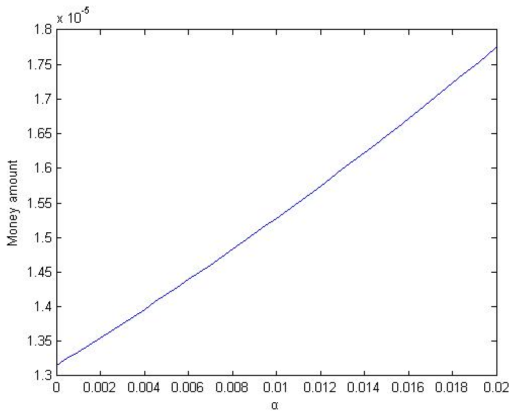

Figure 7: Sensitivity of the optimal strategy w.r.t α

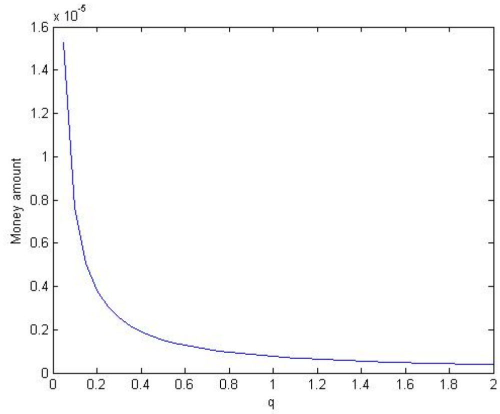

Figure 8: Sensitivity of the optimal strategy w.r.t q

wealth. Therefore, investors will invest more money in risky assets to increase wealth.

In Figure 8, we plot the effect of the risk aversion coefficient q on the optimal strategy for the exponential utility function. We see that q exerts a negative effect on the optimal strategy. An investor who has higher q will invest less in risky asset. This is consistent with intuition.

4.3 Numerical analysis under the quadratic utility function case

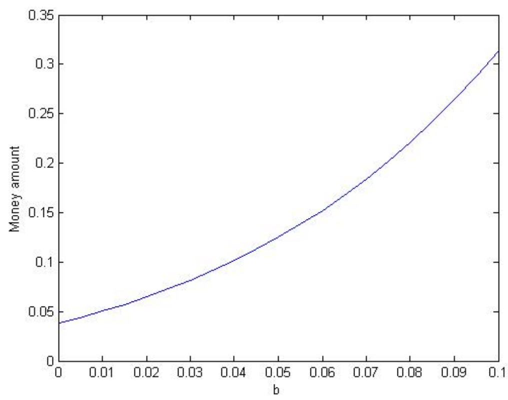

Figure 9: Sensitivity of the optimal strategy w.r.t b

As shown in Figure 9, Figure 10, Figure 11 and Figure 12, we plot the effect of market parameters on the optimal strategy under the quadratic utility function. The effects of dividend rate b, the rate of transaction cost θ, interest rate r_0 and taxes rate α on the optimal strategy are similar to that under the first two cases.

Figure 13 plots the effect of the coefficient c of quadratic utility function on the optimal strategy and

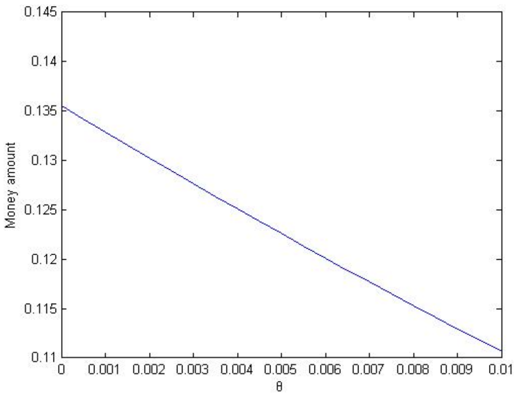

Figure 10: Sensitivity of the optimal strategy w.r.t θ

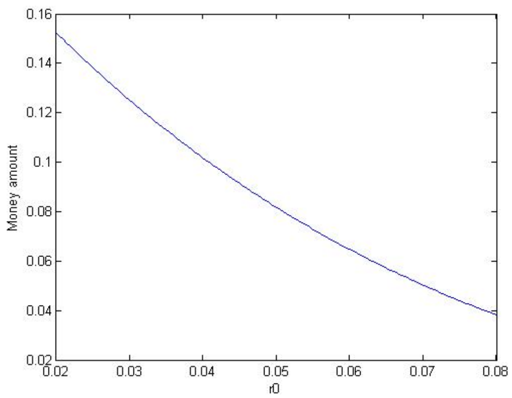

Figure 11: Sensitivity of the optimal strategy w.r.t r_0

we see that the optimal investment strategy increases with coefficient c.

5 Conclusion

In this paper, we study the optimal investment problem for utility maximization with taxes, dividends and transaction costs. We adopted the constant elasticity of variance (CEV) model to describe the dynamic movements of the risky asset's price. By applying stochastic optimal control, we establish the corresponding Hamilton-Jacobi-Bellman (HJB) equation. By using the Legendre transform, dual theorem, power transform and variable change technique, we obtain explicit solutions for the logarithmic, exponential and quadratic utility functions. At last, a numerical simulation is presented to analyze the properties of the optimal strategy. From the numerical simulation, we find that taxes and dividends have positive effects on the optimal strategies, while transaction costs exert a negative effect on the optimal strategies.

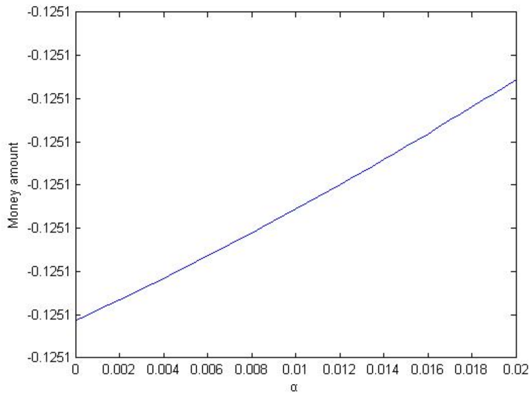

Figure 12: Sensitivity of the optimal strategy w.r.t α

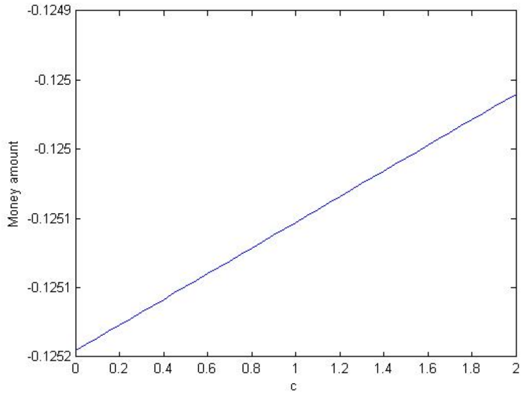

Figure 13: Sensitivity of the optimal strategy w.r.t c

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