

# Robust Consensus Problem of Data-sampled Networked Multi-agent Systems with Delay and Noise

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**Abstract:** This paper proposes an observer-based control strategy for networked multi-agent systems with time-varying communication delays and random white noises under both fixed topology and Bernoulli switching topology. First, a queuing mechanism is introduced and thus a team of agents can be modeled as a system with constant delay. Then, using the system transformation method, the robust mean-square consensus problem of multi-agent systems can be converted into the robust mean-square stability problem of an equivalent system, and some equivalent conditions concerning the robust mean-square consensus of networked multi-agent systems are presented, whose related observer-based stabilizability criteria can be established in the form of linear matrix inequalities (LMIs). Furthermore, if the LMIs are feasible, the multi-agent systems achieve robust mean-square consensus if and only if the directed graph has a directed spanning tree (fixed topology) or the union of graphs has a directed spanning tree (Bernoulli switching topology). Finally, numerical simulations are given to illustrate the effectiveness of the obtained theoretical results.

**Key-Words:** Networked multi-agent systems, robust mean-square consensus, delay, sampled control, general linear dynamics. inequalities(LMIs).

## 1 Introduction

In recent years, distributed coordination of large numbers of autonomous individuals has attracted more and more attention in a wide range including system control theory, applied mathematics, biology, communication, computer science and so on. This is partly due to its challenging features and many applications, e.g., rescue mission, large object moving, troop hunting, formation control and satellite clustering. One critical and fundamental issue in the distributed coordination of multi-agent systems is the consensus problem, which generally means that as time goes, all agents can asymptotically reach an agreement on their states by designing a network protocol. Investigations of this problem are of interest in both theory and engineering applications [1]. Up to now, by using different analysis methods and tools including the graph theory [2], the Lyapunov function method [3], the frequency-domain analysis method [4], the matrix theory [5] and so on, many consensus criteria have been obtained for the systems under fixed topology [6-7], switching topology [8], and time delays [5-6,9,10-12]. See the survey [13-14] and the references therein for more details.

In the past decade, consensus problem of multi-

agent systems has developed very fast and several research topics have been addressed. But most of the results on consensus in the existing literature are developed under the assumption that exact model of the agent dynamics is known. However, there may exist disturbances and uncertainties in practical engineering. Recently, robust mean-square consensus problem with random measurement noises has attracted the attention of some researchers [15-20], which is more complicated than normal consensus problem when some noises are exerted on the interconnections among autonomous mobile agents. [19] studied the first-order consensus problem with least-mean-square error. Assuming that the system state can be obtained directly, [20] investigated the robust consensus of second-order integrator with variable delays and noises. It showed that the robust mean-square consensus problem is solvable if and only if the union of the topology set is connected. Compared with first-order dynamics and second-order dynamics, there are still lack of good results of general linear dynamics. For general linear dynamical systems, [21] proposed an observer-based control strategy for networked multi-agent systems and studied the mean-square consensus problem with constant communication delay, but ne-

glected the noise problem. This motivates us to write this paper.

In this paper, we focus on the robust mean-square consensus problem of networked multi-agent systems with time-varying communication delays and random white noises in a sampling setting. The main contributions of this paper are twofold: (i) compared with [20], we adopt an observer-based consensus control strategy as it is usually impossible to directly obtain all states of systems in practice due to economic costs or constraints on measurement, (ii) compared with [21], we investigate the robust mean-square consensus problem of general linear systems with time-varying communication delays and random white noises.

The rest of this paper is organized as follows. In Section 2, we introduce some graph knowledge, formulate our problems and give some useful lemmas as the preliminaries of our paper. Our main results are given in Section 3. Simulations are given in Section 4 to illustrate the effectiveness of the obtained theoretical results. In Section 5, we give our conclusions.

## 1.1 Notations

We use standard notations throughout this paper. Let  $\mathbb{R}^{n \times n}$  be the set of  $n \times n$  real matrix,  $M^T$  be the transpose of the matrix  $M$ .  $M > 0$  ( $M < 0$ ) means that matrix  $M$  is positive definite (negative definite).  $\mathbb{R}^n$  is the set of  $n$ -dimensional Euclidean space.  $I_n$  represents the identity matrix of dimension  $n$ , and  $I$  denotes the identity matrix of an appropriate dimension.  $\text{Diag}\{A_1, \dots, A_n\}$  represents a block-diagonal matrix with matrices  $A_i, i = 1, \dots, n$  on its diagonal. The symbol  $*$  will be used to denote a symmetric structure in a matrix, that is,

$$\begin{bmatrix} L & N \\ * & R \end{bmatrix} = \begin{bmatrix} L & N \\ N^T & R \end{bmatrix}.$$

$\mathbb{1}_n$  is a vector with all entries equal to 1.  $\rho(\cdot)$ ,  $\det(\cdot)$  represent the spectral radius, determinant of a matrix, respectively.  $E(\cdot)$  denotes the mathematical expectation.  $Pr\{\cdot\}$  denotes the occurrence probability of an event.  $\|x\|$ ,  $\|A\|$  denote the Euclidean norm of vector  $x$  and  $A$ , respectively.  $A \otimes B$  denotes the Kronecker product.  $A \sim B$  denotes that the matrix  $A$  is similar to the matrix  $B$ . Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2 Problem Formulations and Preliminaries

In this section, we first introduce some graph knowledge and the networked multi-agent systems model, then we formulate our problems and propose some lemmas as the preliminaries of our paper.

### 2.1 Graph Theory

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  denote a directed weighted graph, where  $\mathcal{V} = \{1, \dots, N\}$  is the node set,  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  denotes the edge set, and  $\mathcal{A} = [a_{ij}\omega_{ij}]$  is the weighted adjacency matrix with  $\omega_{ij} > 0$ . Here,  $\omega_{ij} > 0$  is said to be the weight between the agent  $i$  and the agent  $j$ , which reflects the dependence of the agent  $i$  on the agent  $j$ . A directed edge of  $\mathcal{G}$  is denoted by  $e_{ij} = (j, i)$ , where  $j$  is called the parent node of  $i$  and  $i$  is the child node of  $j$ . If the edge  $e_{ij} = (j, i) \in \mathcal{E}$ , then  $a_{ij} = 1$ , otherwise  $a_{ij} = 0$ . Suppose that each node has no self edge, i.e.,  $a_{ii} = 0$  for all  $i$ . The set of neighbors of node  $i$  is denoted by  $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ . The Laplacian matrix  $L = [l_{ij}]$  of digraph  $\mathcal{G}$  is defined by

$$l_{ij} = -a_{ij}\omega_{ij}, \quad \text{if } i \neq j$$

$$l_{ij} = \sum_{k=1, k \neq i}^N a_{ik}\omega_{ik}, \quad \text{if } i = j.$$

A path of  $\mathcal{G}$  from node  $i$  to node  $j$  is a sequence of finite ordered edges in the form of  $(i, k_1), (k_1, k_2), \dots, (k_l, j)$ . A directed graph is strongly connected if for any distinct nodes, there exists a path between them. A directed graph has or contains a directed spanning tree if there exists a node called root such that there exists a directed path from this node to every other node. A subgraph  $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1)$  of  $\mathcal{G}$  is a graph such that  $\mathcal{V}_1 \subset \mathcal{V}$  and  $\mathcal{E}_1 \subset \mathcal{E}$ .

### 2.2 System Model

Consider  $N$  agents with general linear dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t), t \in \mathbb{R}^+, \\ y_i(t) = Cx_i(t), i \in \{1, \dots, N\}, \end{cases} \quad (1)$$

where  $x_i(t) \in \mathbb{R}^p$  is the state,  $u_i(t) \in \mathbb{R}^q$  is the control input, and  $y_i(t) \in \mathbb{R}^m$  is the measured output.

The model of the networked multi-agent systems used in this paper is shown in Fig.1 below.

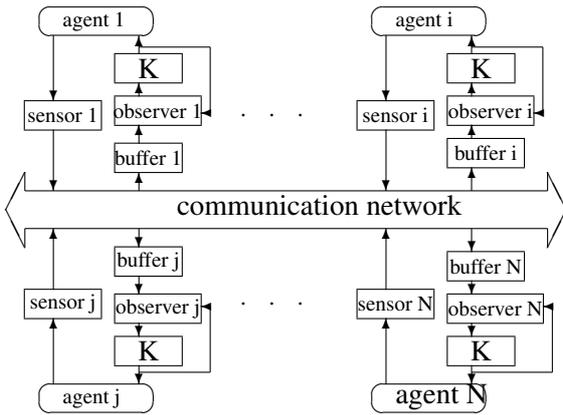


Fig. 1 The structure of observer-based multi-agent systems

### 2.3 Problem Formulations

#### 2.3.1 Fixed Topology Case

Throughout the paper, we need the following assumptions:

**Assumption 1.**  $(A, B)$  is controllable and observable. Matrix  $A$  described in (1) is not Hurwitz stable, i.e., the open-loop system is not stable.

**Assumption 2.** For simplicity, but without loss of generality, all the time delays exist in the communication channels between the sensors and the observers.

**Assumption 3.** Every agent is regarded as a plant. The plant output node (sensor) is assumed to be time-driven, and its sampling period is  $h$ , whereas the observer is event-driven.

**Assumption 4.** Here, we apply a queuing mechanism, set a buffer in the receiver of every agent. Let  $\tau_{ij}^k, 0 < \tau_{ij}^k < h, i = 1, \dots, N, j \in N_i$  be the communication delay from agent  $j$  to agent  $i$  during the  $k$ -th sampling period,  $\tau_k = \max_{i=1, \dots, N, j \in N_i} \{\tau_{ij}^k\}$  be the maximum delay during the  $k$ -th sampling period,  $\tau = \max_k \{\tau^k\}$  be the maximum delay of the multi-agent system. Let  $kh + \tau$  be the threshold time of all the buffers during every sampling period.

**Assumption 5.** There exists random white noises in the communication channels. The measured output of agent  $j$  at the time of  $kh$  is  $y_j(kh)$ . The information which agent  $i$  obtained from agent  $j$  at the time of  $kh + \tau$  is corrupted by channel noise  $\delta_{ij}(t)$ , and  $\delta_{ij}(t)$  is assumed to satisfy

$$E(\delta_{ij}(t)) = 0, \quad E(\delta_{ij}^T(t)\delta_{ij}(t)) \leq \Delta_{ij}$$

$0 < \tau < h, i = 1, \dots, N, j \in N_i$ , is the communication delay from agent  $j$  to agent  $i$  during the sampling period.

For agent  $i$ , suppose the obtained information at the time of  $kh + \tau$  is  $\eta_i(kh)$ , specifically,

$$\eta_i(kh) = \sum_{j=1}^N a_{ij}\omega_{ij}[y_j(kh) - y_i(kh) + C\delta_{ij}(kh)], \quad (2)$$

where  $a_{ij}, \omega_{ij}$  are the adjacency relationship, the connection weight from agent  $j$  and agent  $i$ , respectively.

**Remark 1.** The queuing mechanism works in the following way: during the  $k$ -th sampling period, when there arrives a packet in the first  $\tau$  time, i.e.,  $kh \leq t \leq kh + \tau$ , compare the time stamp of the packet with current time, if the arrived packet is new, put it in the queue; otherwise discard it. At the time  $kh + \tau$ , the queue unloads the packets inside to update the agent's control input. In this way, all agents update control inputs synchronously, and the outputs used for updating control inputs are all delayed by equal time  $\tau$ .

We design an observer-based agreement protocol as follows:

$$\begin{cases} \dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + G\eta_i(kh) \\ \quad - GC \sum_{j=1}^N a_{ij}\omega_{ij}[\hat{x}_j(kh) - \hat{x}_i(kh) + \hat{\delta}_{ij}(kh)], \\ u_i(t) = K \sum_{j=1}^N a_{ij}\omega_{ij}[\hat{x}_j(kh) - \hat{x}_i(kh) + \hat{\delta}_{ij}(kh)] \end{cases} \quad (3)$$

where  $t \in [kh + \tau, (k + 1)h + \tau)$ ,  $\hat{x}_i(t) \in \mathbb{R}^p$  is the protocol state,  $i \in \{1, 2, \dots, N\}$ ,  $G$  and  $K$  are the feedback gain matrices to be designed,  $a_{ij}$  and  $\omega_{ij}$  are defined as above.

**Remark 2.**  $\hat{\delta}_{ij}(kh)$  is the noise of the observers, in general,  $\hat{\delta}_{ij}(kh) \neq \delta_{ij}(kh)$ .

Then, by (2), (3), for  $\forall t \in [kh + \tau, (k + 1)h + \tau)$ , system (1) can be written as:

$$\begin{aligned} \dot{\bar{x}}_i(t) &= \bar{A}\bar{x}_i(t) + \bar{B} \sum_{j=1}^N a_{ij}\omega_{ij}[\bar{x}_j(kh) - \bar{x}_i(kh)] \\ &\quad + \bar{B} \sum_{j=1}^N a_{ij}\omega_{ij}\bar{\delta}_{ij}(kh) \end{aligned} \quad (4)$$

where

$$\bar{x}_i(t) = [\hat{x}_i^T(t), x_i^T(t)]^T, \bar{\delta}_{ij}(kh) = [\hat{\delta}_{ij}^T(kh), \delta_{ij}^T(kh)]$$

and

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} BK - GC & GC \\ BK & 0 \end{bmatrix}.$$

In this paper, we aim to design an observer-based control protocol to guarantee that system (1) can reach

robust mean-square consensus. Specifically, we focus on investigating the interdependency between the convergence properties of the observer-based agreement protocol and the structural attributes of the underlying network topology. Here, the concept of robust mean-square consensus is given as follows:

**Definition 3.** *Multi-agent system (1) with strategy (2), (3) reaches robust mean-square consensus if there exist gain matrices  $K, G$ , connection weights  $\omega_{ij}$  and monotonously increasing function  $c(\cdot)$  satisfying  $\lim_{\Delta \rightarrow 0} c(\Delta) = 0$ , such that the states of system (4) satisfy*

$$\lim_{t \rightarrow +\infty} E(\|\bar{x}_i(t) - \bar{x}_j(t)\|^2) \leq c(\Delta)$$

for arbitrary  $i, j \in \{1, 2, \dots, N\}$ .

Let  $z_i(t) = \bar{x}_i(t) - \bar{x}_1(t)$ ,  $i = 2, \dots, N$ . Define

$$z(t) = [z_2^T(t), \dots, z_N^T(t)]^T,$$

$$\tilde{\delta}(kh) = [\delta_2^T(kh) - \delta_1^T(kh), \dots, \delta_N^T(kh) - \delta_1^T(kh)]^T.$$

Then for  $\forall t \in [kh + \tau, (k+1)h + \tau)$ , we can equivalently obtain a reduced system:

$$\dot{z}(t) = Fz(t) + Hz(kh) + M\tilde{\delta}(kh), \quad (5)$$

where  $F = I_{N-1} \otimes \bar{A}$ ,  $H = -\tilde{L} \otimes \bar{B}$ ,  $M = I_{N-1} \otimes \bar{B}$ ,

$$\tilde{L} = \begin{bmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{bmatrix}$$

is defined as the reduced Laplacian matrix, where  $l_{ij}$  is the corresponding element in the Laplacian matrix  $L$ ,  $\bar{A}$ ,  $\bar{B}$  are defined as above.

### 2.3.2 Bernoulli Switching Topology Case

Throughout the paper, we need the following assumptions. Assumptions 1-5 are the same as the fixed topology case.

**Assumption 6.** Let  $a_{ij}(kh + \tau)$  denote the connection relationship from agent  $j$  to  $i$  at  $kh + \tau$ , which can be discussed in two cases: (i) if the measured output information  $y_j(kh)$  can be achieved by agent  $i$  at  $kh + \tau$ , then  $a_{ij}(kh + \tau) = 1$ , (ii) if the measured output information  $y_j(kh)$  can't be achieved by agent  $i$  at  $kh + \tau$ , then  $a_{ij}(kh + \tau) = 0$ .

**Assumption 7.** Let  $r_{ij} = \Pr\{a_{ij}(kh + \tau) = 0\}$  be a constant satisfying  $0 < r_{ij} < 1$ ,  $i, j \in \{1, 2, \dots, N\}$ . Obviously, the connection relationship from agent  $j$  to  $i$  at  $kh + \tau$  is subject to a Bernoulli distribution.

Suppose all the communication channels are independent of each other, thus all the probabilities are mutually independent and the multi-agent system can be described by the Bernoulli network.

**Assumption 8.** For simplicity, but without loss of generality, suppose there are  $M$  possible stochastic switching graphs in the topology set  $\{\mathcal{G}_1, \dots, \mathcal{G}_M\}$ , the occurrence probability of each graph  $\mathcal{G}_l$  is  $\pi_l$ , and satisfying  $0 < \pi_l < 1$  and  $\sum_{l=1}^M \pi_l = 1$ . Denote the varying topology process as  $\{\sigma(kh + \tau), k \geq 0\}$ ,  $\sigma(\cdot) : \mathbb{R}^+ \rightarrow \{1, 2, \dots, M\}$  is a piecewise-constant stochastic switching signal.

For agent  $i$ , suppose the obtained information at the time of  $kh + \tau$  is  $\eta_i(kh + \tau)$ , then

$$\eta_i(kh + \tau) = \sum_{j=1}^N a_{ij}(kh + \tau)\omega_{ij} \times [y_j(kh) - y_i(kh) + \delta_{ij}(kh)]. \quad (6)$$

Similar to the fixed topology case, for  $\forall t \in [kh + \tau, (k+1)h + \tau)$ , we can get an observer-based consensus protocol as follows:

$$\left\{ \begin{array}{l} \dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + G\eta_i(kh + \tau) \\ \quad - GC \sum_{j=1}^N a_{ij}(kh + \tau)\omega_{ij} \\ \quad \times [\hat{x}_j(kh) - \hat{x}_i(kh) + \hat{\delta}_{ij}(kh)], \\ u_i(t) = K \sum_{j=1}^N a_{ij}(kh + \tau)\omega_{ij} [\hat{x}_j(kh) - \hat{x}_i(kh)] \\ \quad + K \sum_{j=1}^N a_{ij}(kh + \tau)\omega_{ij} \hat{\delta}_{ij}(kh) \end{array} \right. \quad (7)$$

where  $\hat{x}_i(t)$ ,  $u_i(t)$ ,  $a_{ij}$ ,  $\omega_{ij}$ ,  $G$ ,  $K$ , are defined as above.

Then, for  $\forall t \in [kh + \tau, (k+1)h + \tau)$ , system (1) can be rewritten as:

$$\begin{aligned} & \dot{\hat{x}}_i(t) \\ &= \bar{A}\hat{x}_i(t) + \bar{B} \sum_{j=1}^N a_{ij}(kh + \tau)\omega_{ij}(\bar{x}_j(kh) - \bar{x}_i(kh)) \\ & \quad + \bar{B} \sum_{j=1}^N a_{ij}(kh + \tau)\omega_{ij}\bar{\delta}_{ij}(kh), \end{aligned} \quad (8)$$

where  $\bar{A}$ ,  $\bar{B}$ ,  $a_{ij}$ ,  $\omega_{ij}$ ,  $\bar{x}_i(t)$ ,  $\bar{\delta}_{ij}(kh)$ , are defined as above. Similarly, the concept of robust mean-square consensus is given as follows:

**Definition 4.** *Multi-agent system (1) with strategy (6), (7) reaches robust mean-square consensus if there exist gain matrices  $K, G$ , connection weights  $\omega_{ij}$  and monotonously increasing function  $c(\cdot)$  satisfying*

$\lim_{\Delta \rightarrow 0} c(\Delta) = 0$ , such that the states of system (4) satisfy

$$\lim_{t \rightarrow +\infty} E(\|\bar{x}_i(t) - \bar{x}_j(t)\|^2) \leq c(\Delta)$$

for arbitrary  $i, j \in \{1, 2, \dots, N\}$ .

Let  $z_i(t) = \bar{x}_i(t) - \bar{x}_1(t)$ ,  $i = 2, \dots, N$ . Define

$$z(t) = [z_2^T(t), \dots, z_N^T(t)]^T,$$

$$\tilde{\delta}(kh) = [\bar{\delta}_2^T(kh) - \bar{\delta}_1^T(kh), \dots, \bar{\delta}_N^T(kh) - \bar{\delta}_1^T(kh)],$$

then for  $\forall t \in [kh + \tau, (k + 1)h + \tau)$ , we can equivalently obtain a reduced system:

$$\dot{z}(t) = F_\sigma z(t) + H_\sigma z(kh) + M_\sigma \tilde{\delta}(kh), \quad (9)$$

where  $F_\sigma = I_{N-1} \otimes \bar{A}$ ,  $H_\sigma = -\tilde{L}_\sigma \otimes \bar{B}$ ,

$$\tilde{L}_\sigma = \begin{bmatrix} l_{22}^\sigma - l_{12}^\sigma & \cdots & l_{2N}^\sigma - l_{1N}^\sigma \\ \vdots & \ddots & \vdots \\ l_{N2}^\sigma - l_{12}^\sigma & \cdots & l_{NN}^\sigma - l_{1N}^\sigma \end{bmatrix}$$
 is de-

defined as the reduced Laplacian matrix, where  $l_{ij}^\sigma$  is the corresponding element in the Laplacian matrix  $L_\sigma$ .

**Remark 5.** Obviously,  $\lim_{t \rightarrow +\infty} E(\|\bar{x}_i(t) - \bar{x}_j(t)\|) \leq c(\Delta)$  is equivalent to  $\lim_{t \rightarrow +\infty} E(\|z_i(t)\|) \leq c(\Delta)$ ,  $\forall i, j \in \{2, \dots, N\}$ , i.e., the robust mean-square consensus problem of system (1) can be transformed into the stability problem of a reduced system (5) (fixed topology) or (9) (Bernoulli switching topology). Hence, in the following discussions, we will focus on seeking the necessary and sufficient conditions to guarantee the stability of system (5) or (9).

Next we will analyze the robust mean-square consensus of general linear systems under both fixed topology and Bernoulli switching topology. First, we propose some lemmas, which will play an important role in the proof of our main theorems in Section 3.

### 2.4 Lemmas

**Lemma 6.** [22] For identical matrix  $I$  and arbitrary matrices  $A, B, C, D$ ,

$$e^{I \otimes A} = I \otimes e^A, \quad (A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

**Lemma 7.** [21] Suppose  $G_k \in \mathbb{R}^{n \times n}$ ,  $A_k, B_k, C_k, D_k \in \mathbb{R}^{m \times m}$ , then there exists a common inverse matrix  $P$  such that for  $k = 1, 2, \dots$ ,

$$G_k \otimes \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = P^{-1} \begin{bmatrix} G_k \otimes A_k & G_k \otimes B_k \\ G_k \otimes C_k & G_k \otimes D_k \end{bmatrix} P,$$

**Lemma 8.** [8]  $\tilde{L}$  has no zero eigenvalue, if and only if the Laplacian matrix  $L$  has only one zero eigenvalue, if and only if the graph  $\mathcal{G}$  has a directed spanning tree.

$$\tilde{L} = \begin{bmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{bmatrix}$$
 is defined as

the reduced Laplacian matrix, where  $l_{ij}$  is the corresponding element in the Laplacian matrix  $L$ .

**Lemma 9.** [20] For stochastic square matrices  $E(\Phi_\sigma), E(\Phi_\sigma \otimes \Phi_\sigma)$ , where  $\sigma$  is a piecewise-constant stochastic switching signal,  $\rho(E(\Phi_\sigma)) \leq \rho(E(\Phi_\sigma \otimes \Phi_\sigma))$ .

**Lemma 10.** [21] For stochastic matrices  $\Phi_\sigma, \tilde{\Phi}_\sigma$ , if there exists a common inverse matrix  $P$  such that  $\Phi_\sigma = P^{-1} \tilde{\Phi}_\sigma P$ , then  $E(\Phi_\sigma \otimes \Phi_\sigma) \sim E(\tilde{\Phi}_\sigma \otimes \tilde{\Phi}_\sigma)$ .

## 3 Robust Consensus Analysis

In this section, we aim to establish the necessary and sufficient conditions to guarantee that system (1) reaches robust mean-square consensus under both fixed topology and Bernoulli switching topology.

### 3.1 Fixed Topology Case

First, the decartelization models of system (5) are given as follows:

$$\begin{aligned} & z((k+1)h) \\ &= (e^{Fh} + \int_0^{h-\tau} e^{Fs} ds H) z(kh) \\ & \quad + \int_{h-\tau}^h e^{Fs} ds H z((k-1)h) + \int_0^{h-\tau} e^{Fs} ds M \tilde{\delta}(kh) \\ & \quad + \int_{h-\tau}^h e^{Fs} ds M \tilde{\delta}((k-1)h). \end{aligned} \quad (10)$$

By Lemma 6,

$$\begin{aligned} & z((k+1)h) \\ &= (I_{N-1} \otimes e^{Ah} - \Gamma) z(kh) - \tilde{L} \otimes \Theta z((k-1)h) \\ & \quad + I_{N-1} \otimes \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} \tilde{\delta}(kh) \\ & \quad + I_{N-1} \otimes \Theta \tilde{\delta}((k-1)h), \end{aligned} \quad (11)$$

where

$$\Gamma = \tilde{L} \otimes \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B}, \quad \Theta = \int_{h-\tau}^h e^{\bar{A}s} ds \bar{B}.$$

Define  $\bar{z}(k) = [z^T(kh), z^T((k+1)h)]^T$ ,  $\check{\delta}(k) = [\check{\delta}^T(kh), \check{\delta}^T((k+1)h)]$ , from (10) and (11), we get

$$\bar{z}(k) = \Phi(\tau)\bar{z}(k-1) + D\check{\delta}(k-1), \quad (12)$$

where

$$\Phi(\tau) = \begin{bmatrix} 0 & I_{2p \times (N-1)} \\ -\tilde{L} \otimes \Theta & I_{N-1} \otimes e^{Ah} - \Gamma \end{bmatrix}$$

and

$$D(\tau) = \begin{bmatrix} 0 & 0 \\ I_{N-1} \otimes \Theta & I_{N-1} \otimes e^{\tilde{A}h} - \Gamma \end{bmatrix}.$$

**Theorem 11.** For a fixed topology, system (1) under (2), (3) reaches robust mean-square consensus, if and only if  $\rho(\Phi(\tau) \otimes \Phi(\tau)) < 1$ .

**Proof:** Defining  $\zeta(k) = E[\bar{z}(k) \otimes \bar{z}(k)]$ , by system (12), we get that

$$\zeta(k+1) = (\Phi(\tau) \otimes \Phi(\tau))\zeta(k) + (D \otimes D)E[\check{\delta}(k) \otimes \check{\delta}(k)]. \quad (13)$$

*Necessity* Now we prove the necessity by contradiction. Suppose  $\rho(\Phi(\tau) \otimes \Phi(\tau)) \geq 1$ , then system (13) can't be robust asymptotically stable, i.e., system (12) can't be robust mean-square stable. By  $E(\|\bar{z}(k)\|^2) \leq \|\zeta(k)\|_1 \leq 4(\alpha - 1)E(\|\bar{z}(k)\|^2)$ , we get that (5) can't be robust mean-square stable. Thus system (1) can't be robust mean-square consensus.

*Sufficiency* If  $\rho(\Phi(\tau) \otimes \Phi(\tau)) < 1$ , there exists one matrix norm  $\|\cdot\|_\alpha$ , satisfying  $\|\Phi(\tau) \otimes \Phi(\tau)\|_\alpha = \lambda < 1$ . By system (13), it is easy to get

$$\begin{aligned} \|\zeta(k+1)\|_\alpha &\leq \|\Phi(\tau) \otimes \Phi(\tau)\|_\alpha \|\zeta(k)\|_\alpha \\ &\quad + \|(D \otimes D)\|_\alpha E(\|\check{\delta}(k) \otimes \check{\delta}(k)\|_\alpha), \\ &\leq \lambda^{k+1} \|\zeta(0)\|_\alpha + \sum_{i=1}^k \lambda^i \bar{\Delta}, \end{aligned} \quad (14)$$

where

$$\bar{\Delta} = 2(n-1)^2 \|D \otimes D\|_\alpha (\rho(\alpha))^{1/2} \Delta.$$

Since  $\lambda < 1$ , then  $\lim_{k \rightarrow +\infty} \|\zeta(k)\|^2 \leq \bar{\Delta}/(1-\lambda)$ . Hence, there exists positive constant  $M$ , such that

$$\lim_{k \rightarrow +\infty} E(\|\bar{z}(k)\|^2) \leq M\Delta,$$

which means

$$\lim_{k \rightarrow +\infty} E(\|\bar{x}_i(kh) - \bar{x}_j(kh)\|^2) \leq M\Delta$$

$$\lim_{k \rightarrow +\infty} E(\|\bar{x}_i((k+1)h) - \bar{x}_j((k+1)h)\|^2) \leq M\Delta.$$

Moreover,  $\forall i \neq j$ , by system (3), we can get

$$\begin{aligned} \bar{x}_i(t) &= e^{\tilde{A}(t-kh)} \bar{x}_i(kh) + \int_0^{t-kh} e^{\tilde{A}s} ds \\ &\quad \times \bar{B} \sum_{j=1}^N a_{ij} \omega_{ij} (\bar{x}_j(kh) - \bar{x}_i(kh) + \bar{\delta}_{ij}(kh)). \end{aligned}$$

So

$$\begin{aligned} &E(\|\bar{x}_i(t) - \bar{x}_j(t)\|^2) \\ &\leq e^{2\|A\|(t-kh)} E(\|\bar{x}_i(kh) - \bar{x}_j(kh)\|^2) \\ &\quad + 4N \|\bar{A}\|^{-2} \|\bar{B}\|^2 e^{2\|\tilde{A}\|(t-kh)} \\ &\quad \times \left[ \sum_{j=1}^N a_{ij} \omega_{ij} E(\|\bar{x}_j(kh) - \bar{x}_i(kh) + \bar{\delta}_{ij}(kh)\|^2) \right. \\ &\quad \left. + \sum_{j=1}^N a_{1j} \omega_{1j} E(\|\bar{x}_j(kh) - \bar{x}_1(kh) + \bar{\delta}_{1j}(kh)\|^2) \right]. \end{aligned}$$

Hence, there exists a positive constant  $\bar{M}$ , such that

$$\lim_{k \rightarrow +\infty} E(\|\bar{x}_i(t) - \bar{x}_j(t)\|^2) \leq \bar{M}\Delta, \forall t \in [kh, (k+1)h).$$

Thus, by Definition 3, multi-agent system (1) reaches robust mean-square consensus.

Based on Theorem 11, now we focus on seeking the necessary and sufficient conditions to guarantee that  $\rho(\Phi(\tau) \otimes \Phi(\tau)) < 1$ .

Since

$$\begin{aligned} \Phi(\tau) &= \begin{bmatrix} 0 & 0 \\ -\tilde{L} \otimes \int_{h-\tau}^h e^{\tilde{A}s} ds \bar{B} & -\tilde{L} \otimes \int_0^{h-\tau} e^{\tilde{A}s} ds \bar{B} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & I_{2p \times (N-1)} \\ 0 & I_{N-1} \otimes e^{\tilde{A}h} \end{bmatrix} \\ &= P^{-1} \Delta(\tau) P, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \Delta(\tau) &= I_{N-1} \otimes \begin{bmatrix} 0 & I_{2p} \\ 0 & e^{\tilde{A}h} \end{bmatrix} - \tilde{L} \\ &\quad \otimes \begin{bmatrix} 0 & 0 \\ \int_{h-\tau}^h e^{\tilde{A}s} ds \bar{B} & \int_0^{h-\tau} e^{\tilde{A}s} ds \bar{B} \end{bmatrix} \end{aligned} \quad (16)$$

Then, by Lemma 10

$$\rho(\Phi(\tau) \otimes \Phi(\tau)) < 1 \iff \rho(\Delta(\tau) \otimes \Delta(\tau)) < 1.$$

Based on the above discussions, we get the following proposition.

**Proposition 12.** If the multi-agent system (1) can reach robust mean-square consensus, then the graph  $\mathcal{G}$  contains a directed spanning tree.

**Proof:** We prove the proposition by contradiction. Let  $\tilde{L} = T^{-1}JT$ , where  $J$  is the Jordan canonical form of  $\tilde{L}$  with diagonal elements  $\lambda_1, \lambda_2, \dots, \lambda_{N-1}$ . We obtain

$$\Delta(\tau) = (T^{-1} \otimes I_{4p})\Omega(\tau)(T \otimes I_{4p}),$$

where

$$\Omega(\tau) = I_{N-1} \otimes \begin{bmatrix} 0 & I_{2p} \\ 0 & e^{\tilde{A}(h)} \end{bmatrix} - J \otimes \begin{bmatrix} 0 & 0 \\ \int_{h-\tau}^h e^{\tilde{A}s} ds \tilde{B} & \int_0^{h-\tau} e^{\tilde{A}s} ds \tilde{B} \end{bmatrix}.$$

Suppose that there is no directed spanning tree in the graph, by Lemma 8,  $\tilde{L}$  has at least one zero eigenvalue under arbitrary connection weights. Without loss of generality, suppose  $\lambda_1 = 0$ . Let

$$\Gamma_0 = \begin{bmatrix} 0 & I_{2p} \\ 0 & e^{\tilde{A}h} \end{bmatrix},$$

$$\Gamma_1 = \Gamma_0 - \begin{bmatrix} 0 & 0 \\ \int_{h-\tau}^h e^{\tilde{A}s} ds \tilde{B} & \int_0^{h-\tau} e^{\tilde{A}s} ds \tilde{B} \end{bmatrix},$$

then

$$\Gamma_0 \otimes \Gamma_0 = \begin{bmatrix} 0 & 0 & 0 & I_{2p} \otimes I_{2p} \\ 0 & 0 & 0 & I_{2p} \otimes e^{\tilde{A}h} \\ 0 & 0 & 0 & e^{\tilde{A}h} \otimes I_{2p} \\ 0 & 0 & 0 & e^{\tilde{A}h} \otimes e^{\tilde{A}h} \end{bmatrix}.$$

Moreover, it is easy to prove

$$\begin{aligned} & \det[sI_{4p \times 4p} - E(\Gamma_0 \otimes \Gamma_0)] \\ = & \det[sI_{4p \times 4p} - \Gamma_0 \otimes \Gamma_0], \\ = & \begin{vmatrix} sI_{4p^2} & 0 & 0 & -I_{2p} \otimes I_{2p} \\ 0 & sI_{4p^2} & 0 & -I_{2p} \otimes e^{\tilde{A}h} \\ 0 & 0 & sI_{4p^2} & -e^{\tilde{A}h} \otimes I_{2p} \\ 0 & 0 & 0 & sI_{4p^2} - e^{\tilde{A}h} \otimes e^{\tilde{A}h} \end{vmatrix}, \\ = & \begin{vmatrix} \Upsilon & 0 & 0 & 0 \\ 0 & \Upsilon & 0 & 0 \\ 0 & 0 & \Upsilon & 0 \\ 0 & 0 & 0 & \Upsilon \end{vmatrix} \times |sI_{4p^2}|^3, \end{aligned} \quad (17)$$

where  $\tilde{A}$  is defined above,  $\Upsilon = sI_{2p} - e^{\tilde{A}h} \otimes e^{\tilde{A}h}$ . Let  $A = P_A^{-1}J_AP_A$ , where  $J_A$  is the Jordan canonical form of  $A$  with diagonal elements  $\mu_1, \dots, \mu_p$ . We conclude

$$\det(sI_p - e^{Ah}) = \prod_{i=1}^p (s - e^{\mu_i h}).$$

Because  $A$  is not Hurwitz stable, thus there exists at least one eigenvalue  $\mu_i$  satisfying  $Re(\mu_i) \geq 0, i \in \{1, \dots, p\}$ . As a result,  $|e^{\mu_i h}| \geq 1$  and  $\rho(e^{Ah}) \geq 1$ . By Lemma 9,  $\rho(e^{Ah} \otimes e^{Ah}) \geq 1$ . Then  $\rho(E(\Gamma_0 \otimes \Gamma_0)) \geq 1, \rho(E(\Delta_\tau \otimes \Delta_\tau)) \geq 1$ . Thus,  $\rho(\Phi(\tau) \otimes \Phi(\tau)) \geq 1$ . Furthermore, by Theorem 11, the multi-agent system (1) can't reach robust mean-square consensus. Therefore, the graph  $\mathcal{G}$  contains a directed spanning tree.

**Theorem 13.** *suppose there exist gain matrices  $K, G$  such that system*

$$\dot{\xi}_i(t) = \bar{A}\xi_i(t) - \lambda_i \bar{B}\xi_i(kh), \quad (18)$$

*is robust mean-square stable, where  $t \in [kh + \tau, (k + 1)h + \tau), \lambda_i, i \in \{1, \dots, N - 1\}$  is the eigenvalue of  $\tilde{L}$ . Then, there exist connection weights  $\omega_{ij}$  such that system (1) reaches robust mean-square consensus if and only if the graph  $\mathcal{G}$  contains a directed spanning tree.*

**Proof:** Necessity follows from Proposition 12.

(Sufficiency) If the graph  $\mathcal{G}$  contains a directed spanning tree, we can introduce a method to choose the connection weights such that all the eigenvalues of  $\tilde{L}$  are equal and not zero. Suppose  $\mathcal{G}_0 = (\mathcal{V}, \mathcal{E}_0, \mathcal{A}_0)$  is a subgraph which is composed of a directed spanning tree. Obviously,  $\mathcal{E}_0 \in \mathcal{E}$ . First, we renumber the agents in the following way: the number of the agent which corresponds to the root in the  $\mathcal{G}_0$  is 1, whereas for the nodes corresponding to the remaining agents, the number of the child node is larger than the number of its parent node.

Then let

$$\omega_{ij} = \begin{cases} 1, & \text{if } \omega_{ij} \in \mathcal{E}_0; \\ 0, & \text{if } \omega_{ij} \in \mathcal{E} \setminus \mathcal{E}_0; \\ \text{arbitrary,} & \text{other case.} \end{cases}$$

For the given connection weights above,

$$\tilde{L} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & 1 \end{bmatrix}.$$

Obviously,  $\lambda_1 = \lambda_2 = \dots = \lambda_{N-1} = 1$ .

$$\Delta(\tau) = \begin{bmatrix} \Delta_1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \dots & \Delta_i & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \dots & * & \dots & \Delta_{N-1} \end{bmatrix},$$

where

$$\Delta_i = \begin{bmatrix} 0 & I_{2p} \\ 0 & e^{\bar{A}h} \end{bmatrix} - \begin{bmatrix} 0 & o \\ \int_{h-\tau}^h e^{\bar{A}s} ds \bar{B} & \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} \end{bmatrix}.$$

Furthermore, by Lemma 7, we can get

$$\begin{aligned} & \Delta(\tau) \otimes \Delta(\tau) \\ = & \begin{bmatrix} \Delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & \Delta_{N-1} \end{bmatrix} \otimes \begin{bmatrix} \Delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & \Delta_{N-1} \end{bmatrix}, \\ = & \begin{bmatrix} \Delta_1 \otimes \Delta(\tau) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & \Delta_{N-1} \otimes \Delta(\tau) \end{bmatrix}, \\ = & \begin{bmatrix} Q^{-1}\Theta_1Q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & Q^{-1}\Theta_{N-1}Q \end{bmatrix}, \\ = & \text{Diag}\{Q^{-1}, \dots, Q^{-1}\} \Psi \text{Diag}\{Q, \dots, Q\}, \end{aligned} \quad (19)$$

where

$$\Theta_i = \begin{bmatrix} \Delta_i \otimes \Delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & \Delta_i \otimes \Delta_{N-1} \end{bmatrix},$$

$$\Psi = \begin{bmatrix} \Delta_1 \otimes \Delta_1 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ * & \cdots & \Delta_1 \otimes \Delta_{N-1} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ * & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ * & \cdots & * & \cdots \\ \\ 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \Delta_{N-1} \otimes \Delta_1 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ * & \cdots & \Delta_{N-1} \otimes \Delta_{N-1} & \cdots \end{bmatrix},$$

$$Q = \begin{bmatrix} I & O & \cdots & O & O & O & \cdots \\ O & O & \cdots & O & I & O & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & O & O & O & \cdots \\ O & I & \cdots & O & O & O & \cdots \\ O & O & \cdots & O & O & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & O & O & O & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & I & O & O & \cdots \\ O & O & \cdots & O & O & O & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & O & O & O & \cdots \\ \\ O & \cdots & O & O & \cdots & O \\ O & \cdots & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & \cdots & I & O & \cdots & O \\ O & \cdots & O & O & \cdots & O \\ O & \cdots & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & \cdots & O & I & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & \cdots & O & O & \cdots & O \\ I & \cdots & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O & \cdots & O & O & \cdots & I \end{bmatrix},$$

$\Theta_i, \Psi, Q \in R^{16(N-1)p^2 \times 16(N-1)p^2}, I, O \in R^{4p \times 4p}$ . Obviously, if  $i \neq j$ ,  $\lambda_i$  and  $\lambda_j$  are mutually independent and  $E(\Delta_i) = E(\Delta_j)$ .

By Lemma 10,  $E(\Delta(\tau) \otimes \Delta(\tau)) \sim E(\Psi)$ . Hence, by choosing the appropriate connection weights, we get  $E(\Phi(\tau) \otimes \Phi(\tau)) \sim E(\Psi)$ . Obviously

$$\begin{aligned} & \rho(\Phi(\tau) \otimes \Phi(\tau)) < 1, \\ \Leftrightarrow & \rho(E(\Delta_i \otimes \Delta_j)) < 1, i, j \in \{1, 2, \dots, N-1\}, \\ \Leftrightarrow & \rho(E(\Delta_i \otimes \Delta_i)) < 1, i \in \{1, 2, \dots, N-1\}, \\ \Leftrightarrow & \text{System (18) is mean-square stable.} \end{aligned}$$

By the given conditions, sufficiency can be directly proved. By Theorem 11, system (1) reach robust mean-square consensus.

**Proposition 14.** *If there exist gain matrices  $G, K$  and matrices  $Z_i > 0$  satisfying the following matrix in-*

equalities:

$$\begin{bmatrix} Z_i & * & * \\ 0 & Z_i & * \\ \Delta_i Z_i & 0 & Z_i \end{bmatrix} > 0, \quad (20)$$

where  $i \in \{1, 2, \dots, N-1\}$ ,  $\Delta_i$  are defined as above, then, system (18) is mean-square stable.

### 3.2 Bernoulli Switching Topology Case

First, the decartelization models of system (9) are given as follows:

$$\begin{aligned} & z((k+1)h + \tau) \\ = & e^{F_\sigma h} z(kh + \tau) + \int_0^h e^{F_\sigma s} ds H_\sigma z(kh) \\ & + \int_0^h e^{F_\sigma s} ds M_\sigma \tilde{\delta}(kh). \end{aligned} \quad (21)$$

$$\begin{aligned} & z((k+1)h) \\ = & e^{F_\sigma(h-\tau)} z(kh + \tau) + \int_0^{h-\tau} e^{F_\sigma s} ds H_\sigma z(kh) \\ & + \int_0^{h-\tau} e^{F_\sigma s} ds M_\sigma \tilde{\delta}(kh). \end{aligned} \quad (22)$$

By Lemma 6

$$\begin{aligned} & z((k+1)h + \tau) \\ = & I_{N-1} \otimes e^{\bar{A}h} z(kh + \tau) \\ & - \tilde{L}_\sigma \otimes \int_0^h e^{\bar{A}s} ds \bar{B} z(kh) \\ & + I_{N-1} \otimes \int_0^h e^{\bar{A}s} ds \bar{B} \tilde{\delta}(kh). \end{aligned} \quad (23)$$

$$\begin{aligned} & z((k+1)h) \\ = & I_{N-1} \otimes e^{\bar{A}(h-\tau)} z(kh + \tau) \\ & - \tilde{L}_\sigma \otimes \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} z(kh) \\ & + I_{N-1} \otimes \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} \tilde{\delta}(kh). \end{aligned} \quad (24)$$

Defining  $\bar{z}(k) = [z^T(kh + \tau), z^T(kh)]^T$ , by (23) and (24), we get

$$\bar{z}(k) = \Phi_\sigma(\tau) \bar{z}(k) + D_\sigma \tilde{\delta}(kh), \quad (25)$$

where

$$\Phi_\sigma(\tau) = \begin{bmatrix} I_{N-1} \otimes e^{\bar{A}h} & -\tilde{L}_\sigma \otimes \int_0^h e^{\bar{A}s} ds \bar{B} \\ I_{N-1} \otimes e^{\bar{A}(h-\tau)} & -\tilde{L}_\sigma \otimes \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} \end{bmatrix},$$

$$D_\sigma = \begin{bmatrix} I_{N-1} \otimes \int_0^h e^{\bar{A}s} ds \bar{B} \\ I_{N-1} \otimes \int_0^{h-\tau} e^{\bar{A}s} ds \bar{B} \end{bmatrix}.$$

**Theorem 15.** For a Bernoulli switching network, system (1) reaches robust mean-square consensus, if and only if  $\rho(\Xi) < 1$ , where  $\Xi = E(\Phi_\sigma \otimes \Phi_\sigma)$ .

**Proof:** Similar to the proof of Theorem 11, it is easy to establish this theorem.

Based on Theorem 15, now we focus on seeking the necessary and sufficient conditions to guarantee that  $\rho(\Xi) < 1$ . From [21], we can get the following results.

**Corollary 16.** Suppose there exist gain matrices  $K, G$  such that system

$$\dot{\xi}_i(t) = \bar{A}\xi_i(t) - \lambda_{i,\sigma} \bar{B}\xi_i(kh), \quad (26)$$

is robust mean-square stable, where  $t \in [kh + \tau, (k+1)h + \tau)$ ,  $i \in \{1, \dots, N-1\}$ ,  $\lambda_{i,\sigma} \in \{0, 1\}$ ,  $Pr\{\lambda_{i,\sigma} = 0\} = r_i$ ,  $Pr\{\lambda_{i,\sigma} = 1\} = 1 - r_i$ . Then, there exist connection weights  $\omega_{ij}$  such that system (1) reaches robust mean-square consensus if and only if the union of graphs in the switching topology set contains a directed spanning tree.

**Corollary 17.** If there exist gain matrices  $G, K$  and matrices  $Z_i > 0$  satisfying the following matrix inequalities

$$\begin{bmatrix} Z_i & * & * \\ \sqrt{r_i} \Gamma_0 Z_i & Z_i & * \\ \sqrt{1-r_i} \Gamma_1 Z_i & 0 & Z_i \end{bmatrix} > 0, \quad (27)$$

where  $i \in \{1, 2, \dots, N-1\}$ ,  $r_i, \Gamma_0, \Gamma_1$  are defined as above, then system (26) is mean-square stable.

## 4 Simulations

In this section, numerical simulations will be given to illustrate the theoretical results obtained in this paper. In consideration of the result of fixed topology case is a special case of Bernoulli switching topology case with  $r_i = 0$ , then we only give simulation of Bernoulli switching topology case below.

**Example.** Consider a multi-agent system with 4 agents, satisfying

$$\begin{cases} \dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.2 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_i(t), \\ y_i(t) = [1 \ 0] x_i(t), \quad i = 1, 2, 3, 4. \end{cases}$$

Suppose there are eight possible stochastic switching graphs in the topology set, the union of graphs is given by Fig. 2 below. It has a directed spanning tree.



Fig. 2 The union of graphs

Select the connection weights as follows:  $\omega_{21} = \omega_{32} = \omega_{43} = 1$ , others are 0. Suppose the connection loss probability of each communication channel is 0.5 and the eight graphs in the switching topology set with equal occurrence probability 0.125. Let  $G = [-0.2456; 0.167]$ ,  $K = [0.1186 - 0.1826]$ . The initial values are given as

$$\begin{aligned} \bar{x}_1(0) &= [2, -5, 8, -1], & \bar{x}_2(0) &= [9, -4, 13, -6], \\ \bar{x}_3(0) &= [7, -2, 9, -3], & \bar{x}_4(0) &= [8, -4, 16, -7]. \end{aligned}$$

The noises are chosen as

$$\begin{aligned} \bar{\delta}_1(0) &= [1, -1, 0.8, -0.8], \\ \bar{\delta}_2(0) &= [0.7, -0.7, 0.6, -0.6], \\ \bar{\delta}_3(0) &= [0.4, -0.4, 0.3, -0.3], \\ \bar{\delta}_4(0) &= [0.2, -0.2, 0.1, -0.1]. \end{aligned}$$

When  $h = 0.05, \tau = 0$ , Fig. 3 and Fig. 4 show the simulation results.

When  $h = 0.05, \tau = 0.01$ , Fig. 5 and Fig. 6 show the simulation results.

When  $h = 0.05, \tau = 0.1$ , Fig. 7 and Fig. 8 show the simulation results.

**Remark 18.** From Fig. 7 and Fig. 8, it is easy to find that even though  $\tau > h$ , the multi-agent system can reach robust mean-square consensus.

## 5 Conclusions

In this paper, the robust mean-square consensus problem of data-sampled networked multi-agent systems with time-varying communication delays and random white noises has been investigated, respectively. A queuing mechanism is introduced and thus a team of agents can be modeled as a system with

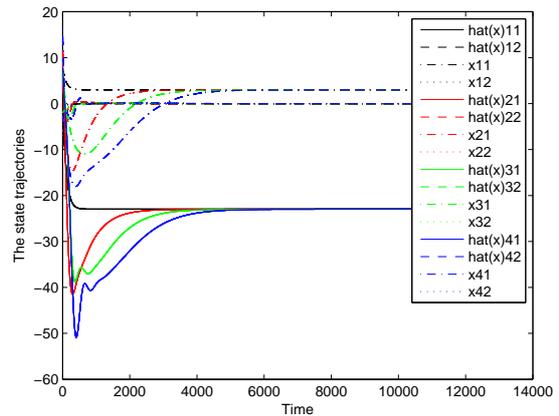


Fig. 3 Simulation result of the states  $x_i, \hat{x}_i, i = 1, 2, 3, 4$  with noise.

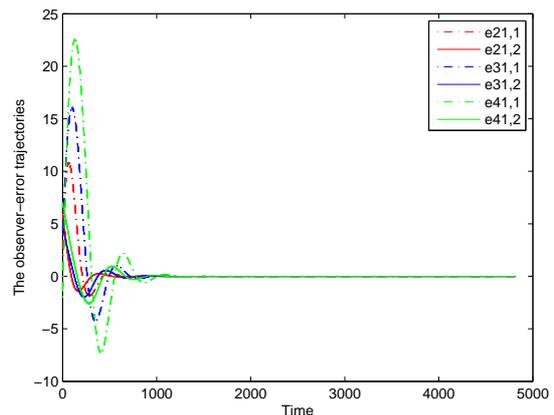


Fig. 4 Simulation result of the observer-error  $e_{21}, e_{31}, e_{41}$  with noise.

constant delay. Some necessary and sufficient conditions for the robust mean-square consensus problem have been obtained. To conclude this paper, the authors would like to note that how to select gain matrix and connection weight is important. Our future work will focus on the robust consensus problem of discrete-time networked multi-agent systems with Markovian packet losses, time-varying communication delays and random white noises under both the fixed topology and the stochastic switching topology.

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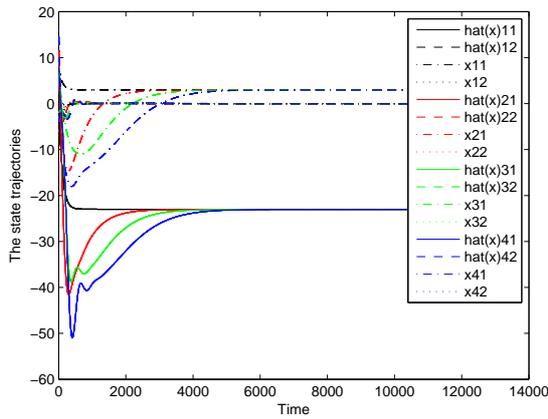


Fig. 5 Simulation result of the states  $x_i, \hat{x}_i, i = 1, 2, 3, 4$  with small delay and noise.

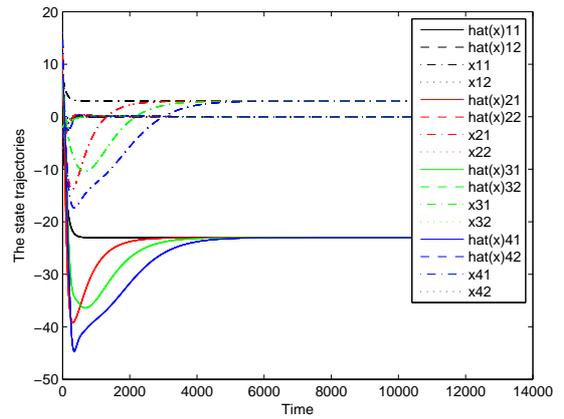


Fig. 7 Simulation result of the states  $x_i, \hat{x}_i, i = 1, 2, 3, 4$  with long delay and noise.

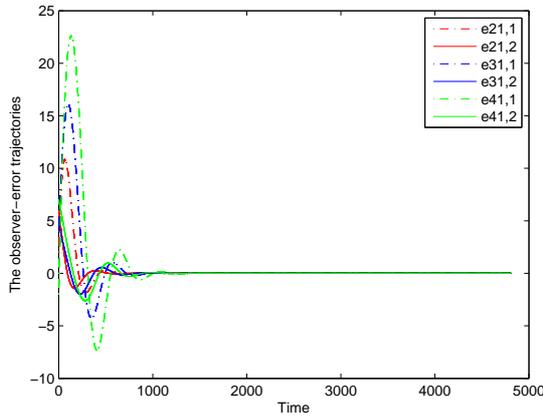


Fig. 6 Simulation result of the observer-error  $e_{21}, e_{31}, e_{41}$  with small delay and noise.

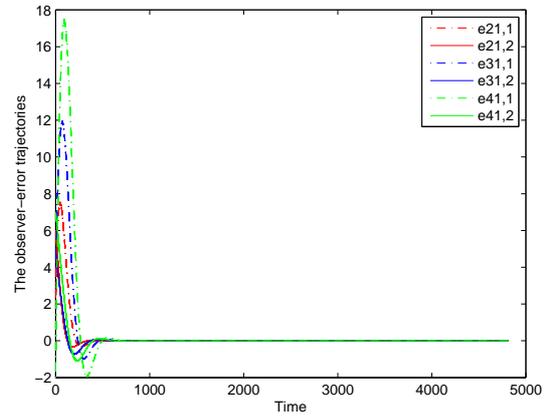


Fig. 8 Simulation result of the observer-error  $e_{21}, e_{31}, e_{41}$  with long delay and noise.

References:

- [1] W. Ren, R. W. Beard, Distributed consensus in multi-vehicle cooperative control: theory and application. 2008.
- [2] F. Xiao, L. Wang, Asynchronous consensus in continuous-time multi-agents with switching topology and time-varying delays. *IEEE.Trans.Automatic.Control.* 2008, pp. 1804-1816.
- [3] J. P. Hu and Y. G. Hong, Leader-following coordination of multi-agent systems with coupling time delays. *Physica.A.* 2007, pp. 853-863.
- [4] Y. P. Tian and C. L. Liu, Robust Consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica.* vol.45, no.5, 2009, pp.1374-1353.
- [5] R. Olfati-Saber and R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays. *IEEE.Trans.Automatic.Control.* 2004, pp. 1520-1532.
- [6] P. Lin, Y. M. Jia, and S.Y. Yuan, Distributed consensus control for second-order agents with fixed topology and time-delay. *26th chinese control conf.* 2007, pp.577-581.
- [7] D. M. Xie, S. K. Wang, Consensus of second-order discrete-time multi-agent systems with fixed topology. *Journal of Mathematical Analysis.* 2012, pp.8-16.
- [8] Y. Zhang, Y. P. Tian, Consentability and protocol design of multi-agent systems with stochastic switching topology, *Automatica.* vol.45, no.5, 2009, pp.1195-1201.

- [9] P. Lin, Y. M. Jia, Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatic.* vol.45, 2009, pp.2154–2158.
- [10] P. Lin and Y. M. Jia, Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topology. *IEEE.Trans.Atomic.Control.* vol.55, no.3, 2010, pp. 778–784.
- [11] X. W. Liu, W. L. Lu, Consensus of a class of multi-agent systems with unbounded time-varying delays. *IEEE.Trans.Atomic.Control.* vol.55, no.10, 2010, pp. 2396–2401.
- [12] Y. P. Tian, C. L. Liu, Consensus of multi-agent systems with diverse input and communication delays, *IEEE.Trans.Autom.Control.* vol.53, no.9, 2008, pp.2122–2128.
- [13] W. Ren, R. W. Beard, E. M. Atkins, Information consensus in multivehicle cooperative control. *IEEE Control Systems Magazine.* 2007, pp. 71–82.
- [14] R. Olfati-Saber, Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE* 2007, pp. 215–233.
- [15] S. Y. Zheng and R. Jiong, Consensus problems of multi-agent systems with noise perturbation. *Chinese Phys.B.* 2008, pp. 4137–4141.
- [16] S. Liu, L. H. Xie and H. S. Zhang, Distributed consensus for multi-agent systems with delays and noise in transmission channels. *Automatic.* vol.10, 2011, pp. 920–934.
- [17] C. Q. Ma and Tao. Li and J. F. zhang, Leader-following consensus control for multi-agent systems with measurement noises. *Journal of systems science complex.* vol.23, 2010, pp. 33–49.
- [18] Wang. Lin, Z. X. Lin, Guo. Lei, Robust consensus of multi-agent systems with noise. *Control theory and application.* vol.5, 2011, pp. 1881–1888.
- [19] L. Xiao, S. Boyd, and S. J. Kim, Distributed average consensus with least-mean-square deviation. *J.Parallel and Distrib.Compute.* vol.67, 2007, pp.33–46.
- [20] Y. Zhang and Y. P. Tian, Consensus of data-sampled multi-agent systems with random communication delay packet loss. *IEEE. Trans. Atomic. Control.* vol.55, no.3, 2010, pp. 939–943.
- [21] J. H. Chen, D. M. Xie and Mei. Yu, Consensus Problem of Networked Multi-agent Systems with Constant Communication Delay: Stochastic Switching Topology Case. *Internationnal Journal of Control.* 2012.
- [22] J. W. Brewer, Kronecker products and matrix calculus in system theory. *IEEE Transaction on Circuits and Systems.* vol.25, no.9, 1978, pp.772–781.