Analytical solution of two model equations for shallow water waves and their extended model equations by Adomian's decomposition and He's variational iteration methods

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Abstract: - In this paper two model equations for shallow water waves and their extended models were considered. Adomian's decomposition method (ADM) and variational iteration method (VIM) have been employed to solve them. Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the ADM. The decomposition method provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems. This method efficiently works for initial- value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. The variational iteration method (VIM) established in (1999) by He is thoroughly used by many researchers to handle linear and nonlinear models. Finally the results of ADM and VIM methods have been compared and it is shown that the results of the VIM method are in excellent agreement with results of ADM method and the obtained solutions are shown graphically.

Key-Words: - Adomian's decomposition method, Variational iteration method, shallow water waves

1 Introduction

Clarkson et.al [1] investigated the generalized short water wave (GSWW) equation

$$u_t - u_{xxt} - \alpha u u_t - \beta u_x \int^x u_t dx + u_x = 0, \qquad (1)$$

where α and β are non-zero constants. Ablowitz et al. [2] studied the specific case $\alpha = 4$ and $\beta = 2$ where Eq. (1) is reduced to

$$u_t - u_{xxt} - 4uu_t - 2u_x \int^x u_t dx + u_x = 0, \qquad (2)$$

This equation was introduced as a model equation which reduces to the KdV equation in the long small amplitude limit [2, 3]. However, Hirota et al. [3] examined the model equation for shallow water waves

$$u_t - u_{xxt} - 3uu_t - 3u_x \int^x u_t dx + u_x = 0, \qquad (3)$$

obtained by substituting $\alpha = \beta = 3$ in (1).

Equation (2) can be transformed to the bilinear forms

$$\begin{bmatrix} D_x(D_t - D_t D_x^2 + D_x) + \\ \frac{1}{3} D_t (D_s + D_x^3) \end{bmatrix} f.f = 0, \qquad (4)$$

where s is an auxiliary variable, and f satisfies the bilinear equation

$$D_x(D_s + D_x^3) f.f = 0, \qquad (5)$$

However, Eq.(3) can be transformed to the bilinear form

$$D_x(D_t - D_t D_x^2 + D_x) f.f = 0, \qquad (6)$$

where the solution of the equation is

$$u(x,t) = 2(\ln f)_{xx}, \qquad (7)$$

where $f(x, t)$ is given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,t), \qquad (8)$$

where ε is a bookkeeping non-small parameter, and $f_n(x,t)$, n = 1, 2,are unknown functions that will be determined by substituting the last equation into the bilinear form and solving the resulting equations by equating different powers of ε to zero. The customary definition of the Hirota's bilinear operators are given by

$$D_t^n D_x^m a.b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m a(x,t) \, b(x',t') \mid x' = x, t' = t. \qquad (9)$$

Some of the properties of the *D*-operators are as follows

$$\begin{aligned}
&\frac{D_t^2 f \cdot f}{f^2} = \iint u_{tt} dx dx, \\
&\frac{D_t D_x^3 f \cdot f}{f^2} = u_{xt} + 3u \int x u_t dx', \\
&\frac{D_x^2 f \cdot f}{f^2} = u, \\
&\frac{D_x^4 f \cdot f}{f^2} = u_{2x} + 3u^2, \\
&\frac{D_t D_x f \cdot f}{f^2} = \ln(f^2)_{xt}, \\
&\frac{D_x^6 f \cdot f}{f^2} = u_{4x} + 15uu_{2x} + 15u^3,
\end{aligned} \qquad (10)$$

Where

$$u(x,t) = 2(\ln f(x,t))_{xx}, \qquad (11)$$

Also extended model of Eq.(2) is obtained by the operator D_x^4 to the bilinear forms (4) and (5)

$$\begin{bmatrix} D_x(D_t - D_t D_x^2 + D_x + D_x^3) + \\ \frac{1}{3} D_t (D_s + D_x^3) \end{bmatrix} f \cdot f = 0, \qquad (12)$$

where s is an auxiliary variable, and f satisfies the bilinear equation

$$D_x(D_s + D_x^3) f.f = 0, \qquad (13)$$

Using the properties of the D operators given above, and differentiating with respect to x we obtain the extended model for Eq. (2) given by

$$\begin{aligned} &u_t - u_{xxt} - 4uu_t - 2u_x \int^x u_t dx + u_x \\ &+ u_{xxx} + 6uu_x = 0, \end{aligned} \qquad (14)$$

In a like manner, we extend Eq.(3) by adding the operator D_x^4 to the bilinear forms (6) to obtain

$$D_x(D_t - D_t D_x^2 + D_x + D_x^3) f.f = 0, \qquad (15)$$

Using the properties of the D operators given above we obtain the extended model for Eq.(3) given by

$$\begin{aligned} &u_t - u_{xxt} - 3uu_t - 3u_x \int^x u_t dx + u_x + \\ &u_{xxx} + 6uu_x = 0, \end{aligned} \qquad (16)$$

In this paper, we use the Adomian's decomposition method (ADM) and He's variational iteration method (VIM) to obtain the solution of two considered equations above for shallow water waves. Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the ADM [4-9]. A reliable modification of ADM has been done by Wazwaz [10-12].The decomposition method provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems [13-15].This method efficiently works for initial- value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. Moreover, we have the advantage of a single global method for solving ordinary or partial differential equations as well as many types of other equations. The variational iteration method (VIM) [16-20] established in (1999) by He is thoroughly used by many researchers to handle linear and nonlinear models. The reliability of the method and the reduction in the size of computational domain gave this method a wider applicability. The method has been proved by many authors to be reliable and efficient for a wide variety of scientific applications, linear and nonlinear as well. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists. For concrete problems, a few numbers of approximations can be used for numerical purposes with high degree of accuracy. The VIM does not require specific transformations or nonlinear terms as required by some existing techniques.

2 Basic idea of Adomian's decomposition method

We begin with the equation

$$Lu + R(u) + F(u) = g(t), \tag{17}$$

where L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. The nonlinear term is represented by F (u). Thus we get

$$Lu = g(t) - R(u) - F(u), \tag{18}$$

The inverse L^{-1} is assumed an integral operator given by

$$L_t^{-1} = \int_0^t (\cdot)dt, \tag{19}$$

The operating with the operator L^{-1} on both sides of Eq. (18) we have

$$u = f_0 + L^{-1}(g(t) - R(u) - F(u)), \tag{20}$$

where f_0 is the solution of homogeneous equation

$$Lu = 0, \tag{21}$$

involving the constants of integration. The integration constants involved in the solution of homogeneous equation (21) are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem. The ADM assumes that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{22}$$

and the nonlinear operator $F(u)$ can be decomposed by an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{23}$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomials of $u_0, u_1, ..., u_n$ defined by

$$An = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F(\sum_{n=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}, \qquad n = 0,1,2... \tag{24}$$

3 Basic idea of He's variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \tag{25}$$

where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau, \tag{26}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variation theory. The subscript n indicates the $n\,th$ approximation and $\tilde{u}_n$ is considered as a restricted variation $\delta\,\tilde{u}_n = 0$.

4 ADM implement for first model of shallow water wave equation

We first consider the application of ADM to first model of shallow water wave equation. If Eq. (2) is dealt with this method, it is formed as

$$L_t u = L_{xxt}u + 4uL_t u + 2L_x u\int^x L_t u dx - L_x u, \qquad (27)$$

where

$$L_t = \frac{\partial}{\partial t}, \qquad L_x = \frac{\partial}{\partial x}, \qquad L_{xxt} = \frac{\partial^3}{\partial x^2 \partial t}, \qquad (28)$$

If the invertible operator $L_t^{-1} = \int_0^t (\cdot)dt$ is applied to Eq. (27), then

$$L_t^{-1}L_t u = L_t^{-1}(L_{xxt}u + 4uL_t u + 2L_x u\int^x L_t u dx - L_x u), \qquad (29)$$

is obtained. By this

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xxt}u + 4uL_t u + 2L_x u\int^x L_t u dx - L_x u), \qquad (30)$$

is found. Here the main point is that the solution of the decomposition method is in the form of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad (31)$$

Substituting from Eq. 31 in 30, we find

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}\begin{pmatrix} L_{xxt}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \\ 4\left(\sum_{n=0}^{\infty} u_n(x,t)\right)L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \\ 2L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\int^x L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right)dx \\ - L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \end{pmatrix}, \qquad (32)$$

is found.

According to Eq. (19) approximate solution can be obtained as follows

$$u_0(x,t) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}{2c} \qquad (33)$$

$$u_1(x,t) = \frac{(c-1)\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)\sqrt{\frac{c-1}{c}}t}{2c\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)} \qquad (34)$$

$$u_2(x,t) = \int_0^t (L_{xxt}u_1 + 4u_1L_t u_1 + 2L_x u_1\int^x L_t u_1 dx - L_x u_1)dt, \qquad (35)$$

Thus the approximate solution for first model of shallow water wave equation is obtained as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t), \qquad (36)$$

The terms $u_0(x,t), u_1(x,t), u_2(x,t)$ in Eq. (36), obtained from Esq. (33), (34), (35).

5 VIM implement for first model of shallow water wave equation

Now let us consider the application of VIM for first model of shallow water wave equation with the initial condition of:

$$u(x,0) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}{2c}, \qquad (37)$$

Its correction variational functional in x and t can be expressed, respectively, as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \begin{pmatrix} \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ -4u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} - \\ 2\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau}d\delta \\ + \frac{\partial u_n(x,\tau)}{\partial x} \end{pmatrix} d\tau, \qquad (38)$$

where λ is general Lagrangian multiplier. After some calculations, we obtain the following stationary conditions:

$$\lambda'(\tau) = 0, \tag{39a}$$

$$1 + \lambda(\tau)\big|_{\tau=t} = 0, \tag{39b}$$

Thus we have:

$$\lambda(t) = -1, \tag{40}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\begin{array}{l} \dfrac{\partial u_n(x,\tau)}{\partial \tau} - \dfrac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ -4u_n(x,\tau)\dfrac{\partial u_n(x,\tau)}{\partial \tau} - \\ 2\dfrac{\partial u_n(x,\tau)}{\partial x}\displaystyle\int_0^x \begin{array}{l} \dfrac{\partial u_n(\delta,\tau)}{\partial \tau} d\delta \\ + \dfrac{\partial u_n(x,\tau)}{\partial x} \end{array} \end{array} \right) d\tau \tag{41}$$

We start with the initial approximation of $u(x,0)$ given by Eq. (37). Using the above iteration formula (41), we can directly obtain the other components as follows:

$$u_0(x,t) = \frac{(c-1)\operatorname{sech}^2\left(\dfrac{1}{2}\sqrt{\dfrac{c-1}{c}}x\right)}{2c}, \tag{42}$$

$$u_1(x,t) = \frac{(c-1)\left(\begin{array}{l} \cosh\left(\dfrac{1}{2}\sqrt{\dfrac{c-1}{c}}x\right) \\ +\sinh\left(\dfrac{1}{2}\sqrt{\dfrac{c-1}{c}}x\right)\sqrt{\dfrac{c-1}{c}}t \end{array} \right)}{2c\cosh^3\left(\dfrac{1}{2}\sqrt{\dfrac{c-1}{c}}x\right)}, \tag{43}$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left(\begin{array}{l} \dfrac{\partial u_1(x,\tau)}{\partial \tau} - \dfrac{\partial^3 u_1(x,\tau)}{\partial x^2 \partial \tau} \\ -4u_1(x,\tau)\dfrac{\partial u_1(x,\tau)}{\partial \tau} - \\ 2\dfrac{\partial u_1(x,\tau)}{\partial x}\displaystyle\int_0^x \begin{array}{l} \dfrac{\partial u_1(\delta,\tau)}{\partial \tau} d\delta \\ + \dfrac{\partial u_1(x,\tau)}{\partial x} \end{array} \end{array} \right) d\tau, \tag{44}$$

If we assume c=2 then by drawing 3-D and 2-D figures of ADM and VIM solutions, we see those figures are similar to each other(Fig.1, Fig.2).

6 ADM implement for second model of shallow water wave equation

Now we consider the application of ADM to second model of shallow water wave equation. If Eq. (3) is dealt with this method, it is formed as

$$L_t u = L_{xxt}u + 3uL_t u + 3L_x u\int^x L_t u dx - L_x u, \tag{45}$$

where

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial}{\partial x}, \quad L_{xxt} = \frac{\partial^3}{\partial x^2 \partial t}, \tag{46}$$

If the invertible operator $L_t^{-1} = \int_0^t (\cdot)dt$ is applied to Eq. 45, then

$$\begin{aligned} L_t^{-1}L_t u &= L_t^{-1}(L_{xxt}u + 3uL_t u \\ &+ 3L_x u\int^x L_t u dx - L_x u), \end{aligned} \tag{47}$$

is obtained. By this

$$\begin{aligned} u(x,t) &= u(x,0) + L_t^{-1}(L_{xxt}u + 3uL_t u \\ &+ 3L_x u\int^x L_t u dx - L_x u), \end{aligned} \tag{48}$$

is found. Here the main point is that the solution of the decomposition method is in the form of

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{49}$$

Substituting from Eq. 49 in 48, we find

$$\sum_{n=0}^{\infty}u_n(x,t)=u(x,0)+ L_t^{-1}\begin{pmatrix} L_{xxt}\left(\sum_{n=0}^{\infty}u_n(x,t)\right)+ \\ 3\left(\sum_{n=0}^{\infty}u_n(x,t)\right)L_t\left(\sum_{n=0}^{\infty}u_n(x,t)\right)+ \\ 3L_x\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\int^x L_t\left(\sum_{n=0}^{\infty}u_n(x,t)\right)dx \\ -L_x\left(\sum_{n=0}^{\infty}u_n(x,t)\right)\end{pmatrix}, \tag{50}$$

is found.

According to Eq.19 approximate solution can be obtained as follows:

$$u_0(x,t)=\frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}{2c} \tag{51}$$

$$u_1(x,t)=\frac{(c-1)\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)\sqrt{\frac{c-1}{c}}t}{2c\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}, \tag{52}$$

$$u_2(x,t)=\int_0^t (L_{xxt}u_1+3u_1L_tu_1 + 3L_xu_1\int^x L_tu_1dx-L_xu_1)dt, \tag{53}$$

Thus the approximate solution for second model of shallow water wave equation is obtained as

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t), \tag{54}$$

The terms $u_0(x,t), u_1(x,t), u_2(x,t)$ in Eq. (54), obtained from Eqs. (51), (52), (53).

7 VIM implement for second model of shallow water wave equation

Here we consider the application of VIM for second model of shallow water wave equation with the initial condition of:

$$u(x,0)=\frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}{2c}, \tag{55}$$

Its correction variational functional in *x* and *t* can be expressed, respectively, as follows

$$u_{n+1}(x,t)=u_n(x,t) +\int_0^t\lambda\begin{pmatrix} \frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial^3u_n(x,\tau)}{\partial x^2\partial\tau} \\ -3u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial\tau}- \\ 3\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x\frac{\partial u_n(\delta,\tau)}{\partial\tau}d\delta \\ +\frac{\partial u_n(x,\tau)}{\partial x}\end{pmatrix}d\tau, \tag{56}$$

where λ is general Lagrangian multiplier. After some calculations, we obtain the following stationary conditions

$$\lambda'(\tau)=0, \tag{57a}$$

$$1+\lambda(\tau)\big|_{\tau=t}=0, \tag{57b}$$

Thus we have:

$$\lambda(t)=-1, \tag{58}$$

As a result, we obtain the following iteration formula

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\begin{pmatrix}\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial^3 u_n(x,\tau)}{\partial x^2\partial\tau}\\-3u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial\tau}-\\3\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x\frac{\partial u_n(\delta,\tau)}{\partial\tau}d\delta\\+\frac{\partial u_n(x,\tau)}{\partial x}\end{pmatrix}d\tau \tag{59}$$

We start with the initial approximation of $u(x,0)$ given by Eq. (55). Using the above iteration formula (59), we can directly obtain the other components as follows

$$u_0(x,t)=\frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}{2c} \tag{60}$$

$$u_1(x,t)=\frac{(c-1)\begin{pmatrix}\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)\\+\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)\sqrt{\frac{c-1}{c}}t\end{pmatrix}}{2c\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c}}x\right)}, \tag{61}$$

$$u_2(x,t)=u_1(x,t)-\int_0^t\begin{pmatrix}\frac{\partial u_1(x,\tau)}{\partial\tau}-\frac{\partial^3 u_1(x,\tau)}{\partial x^2\partial\tau}\\-3u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial\tau}-\\3\frac{\partial u_1(x,\tau)}{\partial x}\int_0^x\frac{\partial u_1(\delta,\tau)}{\partial\tau}d\delta\\+\frac{\partial u_1(x,\tau)}{\partial x}\end{pmatrix}d\tau \tag{62}$$

If we assume c=2 then by drawing 3-D and 2-D figures of ADM and VIM solutions, we see those figures are similar to each other (Fig.3, Fig.4).

8 ADM implement for first extended model of shallow water wave equation

We consider the application of ADM to first extended model of shallow water wave equation. If Eq. (14) is dealt with this method, it is formed as

$$\begin{aligned}L_t u&=L_{xxt}u+4uL_t u+2L_x u\int^x L_t u dx\\&-L_x u-L_{xxx}u-6uL_x u,\end{aligned} \tag{63}$$

where

$$L_t=\frac{\partial}{\partial t},\qquad L_x=\frac{\partial}{\partial x},\qquad L_{xxt}=\frac{\partial^3}{\partial x^2\partial t},\qquad L_{xxx}=\frac{\partial^3}{\partial x^3}, \tag{64}$$

If the invertible operator $L_t^{-1}=\int_0^t(\cdot)dt$ is applied to Eq. (63), then

$$\begin{aligned}L_t^{-1}L_t u&=L_t^{-1}(L_{xxt}u+4uL_t u+\\&2L_x u\int^x L_t u dx-L_x u-L_{xxx}u-6uL_x u),\end{aligned} \tag{65}$$

is obtained. By this

$$\begin{aligned}u(x,t)&=u(x,0)+L_t^{-1}(L_{xxt}u+4uL_t u\\&+2L_x u\int^x L_t u dx-L_x u-L_{xxx}u-6uL_x u)\end{aligned} \tag{66}$$

is found. Here the main point is that the solution of the decomposition method is in the form of

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{67}$$

Substituting from Eq. (67) in (66), we find

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}\left(\begin{array}{l} L_{xxt}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \\ 4\left(\sum_{n=0}^{\infty} u_n(x,t)\right) L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \\ 2L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \\ \int_0^x L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right) dx \\ -L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \\ -L_{xxx}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \\ -6\left(\sum_{n=0}^{\infty} u_n(x,t)\right) L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \end{array} \right) \quad (68)$$

is found.

According to Eq.(19) approximate solution can be obtained as follows:

$$u_0(x,t) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \quad (69)$$

$$u_1(x,t) = \frac{\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\sqrt{\frac{c-1}{c+1}}tc(c-1)}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)(c+1)^2}, \quad (70)$$

$$\begin{aligned} u_2(x,t) &= \int_0^t \ (L_{xxt}u_1 + 4u_1 L_t u_1 \\ &+ 2L_x u_1 \int^x L_t u_1 dx - L_x u_1 \\ &- L_{xxx}u_1 - 6u_1 L_x u_1) dt, \end{aligned} \quad (71)$$

Thus the approximate solution for first extended model of shallow water wave equation is obtained as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t)\,, \quad (72)$$

The terms $u_0(x,t), u_1(x,t), u_2(x,t)$ in Eq.(72), obtained from Eqs.(69), (70), (71).

9 VIM implement for first extended model of shallow water wave equation

Now let us consider the application of VIM for first extended model of shallow water wave equation with the initial condition of

$$u(x,0) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2}, \quad (73)$$

Its correction variational functional in x and t can be expressed, respectively, as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left(\begin{array}{l} \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ -4u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} - \\ 2\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau} d\delta \\ +\frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \\ +6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x} \end{array} \right) d\tau, \quad (74)$$

where λ is general Lagrangian multiplier. After some calculations, we obtain the following stationary conditions:

$$\lambda'(\tau) = 0\,, \quad (75a)$$

$$1 + \lambda\ (\tau)\big|_{\tau=t} = 0\,, \quad (75b)$$

Thus we have

$$\lambda(t) = -1, \quad (76)$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \begin{pmatrix} \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ -4u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} - \\ 2\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau}d\delta \\ +\frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \\ +6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x} \end{pmatrix} d\tau \quad (77)$$

We start with the initial approximation of $u(x,0)$ given by Eq. (73). Using the above iteration formula (77), we can directly obtain the other components as follows:

$$u_0(x,t) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \quad (78)$$

$$u_1(x,t) = \frac{1}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)(c+1)^2} \left(\frac{1}{2}(c-1)\begin{pmatrix} c\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right) \\ +\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right) \\ +2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\sqrt{\frac{c-1}{c+1}}tc \end{pmatrix}\right), \quad (79)$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \begin{pmatrix} \frac{\partial u_1(x,\tau)}{\partial \tau} - \frac{\partial^3 u_1(x,\tau)}{\partial x^2 \partial \tau} \\ -4u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial \tau} - \\ 2\frac{\partial u_1(x,\tau)}{\partial x}\int_0^x \frac{\partial u_1(\delta,\tau)}{\partial \tau}d\delta \\ +\frac{\partial u_1(x,\tau)}{\partial x} + \frac{\partial^3 u_1(x,\tau)}{\partial x^3} \\ +6u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} \end{pmatrix} d\tau \quad (80)$$

If we assume c=2 then by drawing 3-D and 2-D figures of ADM and VIM solutions, we see those figures are similar to each other(Fig.5, Fig.6).

10 ADM implement for second extended model of shallow water wave equation

Here we consider the application of ADM to second extended model of shallow water wave equation. If Eq. (16) is dealt with this method, it is formed as

$$L_t u = L_{xxt}u + 3uL_t u + 3L_x u\int^x L_t u dx \\ - L_x u - L_{xxx}u - 6uL_x u \quad , \quad (81)$$

where

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial}{\partial x}, \quad L_{xxt} = \frac{\partial^3}{\partial x^2 \partial t}, \\ L_{xxx} = \frac{\partial^3}{\partial x^3}, \quad (82)$$

If the invertible operator $L_t^{-1} = \int_0^t (\cdot) dt$ is applied to Eq. (81), then

$$L_t^{-1}L_t u = L_t^{-1}(L_{xxt}u + 3uL_t u + \\ 3L_x u\int^x L_t u dx - L_x u - L_{xxx}u - 6uL_x u), \quad (83)$$

is obtained. By this

$$u(x,t) = u(x,0) + L_t^{-1}(L_{xxt}u + 3uL_t u \\ + 3L_x u\int^x L_t u dx - L_x u - L_{xxx}u - 6uL_x u), \quad (84)$$

is found. Here the main point is that the solution of the decomposition method is in the form of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (85)$$

Substituting from Eq. (85) in (84), we find

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}\left(\begin{array}{l} L_{xxt}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \\ 3\left(\sum_{n=0}^{\infty} u_n(x,t)\right) L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \\ 3L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \\ \int_0^x L_t\left(\sum_{n=0}^{\infty} u_n(x,t)\right) dx \\ -L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - L_{xxx}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \\ -6\left(\sum_{n=0}^{\infty} u_n(x,t)\right) L_x\left(\sum_{n=0}^{\infty} u_n(x,t)\right) \end{array} \right) \qquad (86)$$

is found. According to Eq.(19) approximate solution can be obtained as follows

$$u_0(x,t) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \qquad (87)$$

$$u_1(x,t) = \frac{\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\sqrt{\frac{c-1}{c+1}}tc(c-1)}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)(c+1)^2}, \qquad (88)$$

$$u_2(x,t) = \int_0^t \; (L_{xxt}u_1 + 3u_1L_tu_1 + 3L_xu_1\int^x L_tu_1dx - L_xu_1 - L_{xxx}u_1 - 6u_1L_xu_1)dt, \qquad (89)$$

Thus the approximate solution for second extended model of shallow water wave equation is obtained as

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t)\,, \qquad (90)$$

The terms $u_0(x,t), u_1(x,t), u_2(x,t)$ in Eq.(90), obtained from Eqs. (87), (88), (89).

11 VIM implement for second extended model of shallow water wave equation

At last we consider the application of VIM for second extended model of shallow water wave equation with the initial condition given by Eq.(73). Its correction variational functional in x and t can be expressed, respectively, as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left(\begin{array}{l} \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ -3u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} \\ -3\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau} d\delta + \\ \frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \\ +6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x} \end{array} \right) d\tau \qquad (91)$$

where λ is general Lagrangian multiplier. After some calculations, we obtain the following stationary conditions

$$\lambda'(\tau) = 0\,, \qquad (92a)$$

$$1 + \lambda\,(\tau)\big|_{\tau=t} = 0\,, \qquad (92b)$$

Thus we have

$$\lambda(t) = -1, \qquad (93)$$

As a result, we obtain the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\begin{array}{l} \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^3 u_n(x,\tau)}{\partial x^2 \partial \tau} \\ -3u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial \tau} - \\ 3\frac{\partial u_n(x,\tau)}{\partial x}\int_0^x \frac{\partial u_n(\delta,\tau)}{\partial \tau} d\delta \\ +\frac{\partial u_n(x,\tau)}{\partial x} + \frac{\partial^3 u_n(x,\tau)}{\partial x^3} \\ +6u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x} \end{array} \right) d\tau \tag{94}$$

We start with the initial approximation of $u(x,0)$ given by Eq. (73). Using the above iteration formula (94), we can directly obtain the other components as follows

$$u_0(x,t) = \frac{(c-1)\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)}{2c+2} \tag{95}$$

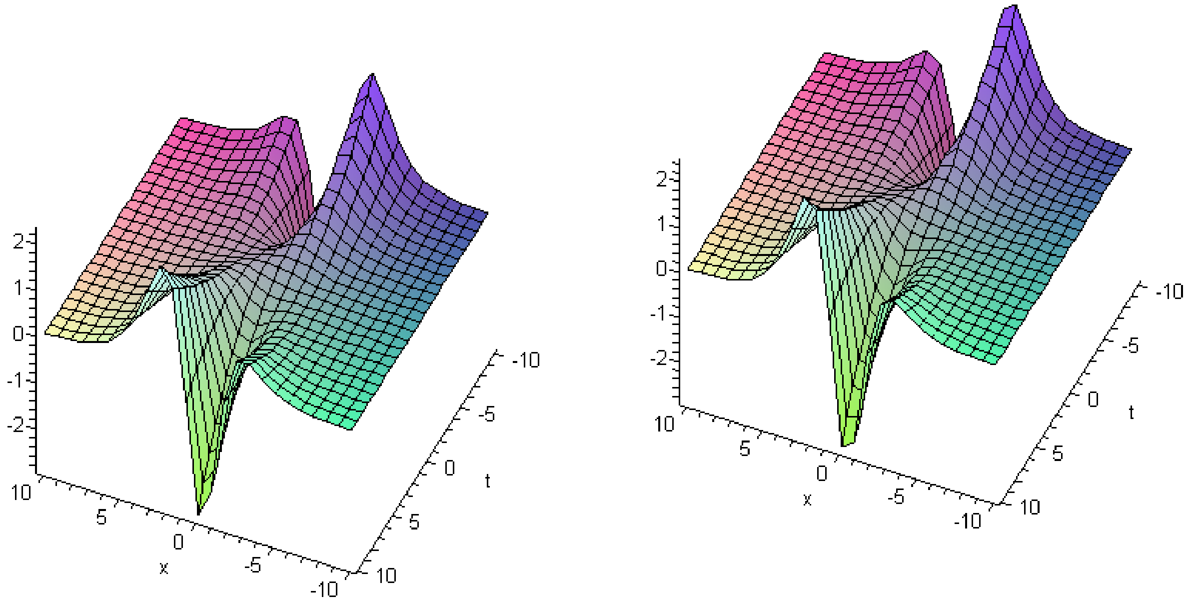
$$u_1(x,t) = \frac{1}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)(c+1)^2} \left(\frac{1}{2}(c-1) \left(\begin{array}{l} c\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right) \\ +\cosh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right) \\ +2\sinh\left(\frac{1}{2}\sqrt{\frac{c-1}{c+1}}x\right)\sqrt{\frac{c-1}{c+1}}tc \end{array} \right) \right), \tag{96}$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left(\begin{array}{l} \frac{\partial u_1(x,\tau)}{\partial \tau} - \frac{\partial^3 u_1(x,\tau)}{\partial x^2 \partial \tau} \\ -3u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial \tau} - \\ 3\frac{\partial u_1(x,\tau)}{\partial x}\int_0^x \frac{\partial u_1(\delta,\tau)}{\partial \tau} d\delta \\ +\frac{\partial u_1(x,\tau)}{\partial x} + \frac{\partial^3 u_1(x,\tau)}{\partial x^3} \\ +6u_1(x,\tau)\frac{\partial u_1(x,\tau)}{\partial x} \end{array} \right) d\tau \tag{97}$$

If we assume c=2 then by drawing 3-D and 2-D figures of ADM and VIM solutions, we see those figures are similar to each other(Fig.7, Fig.8).

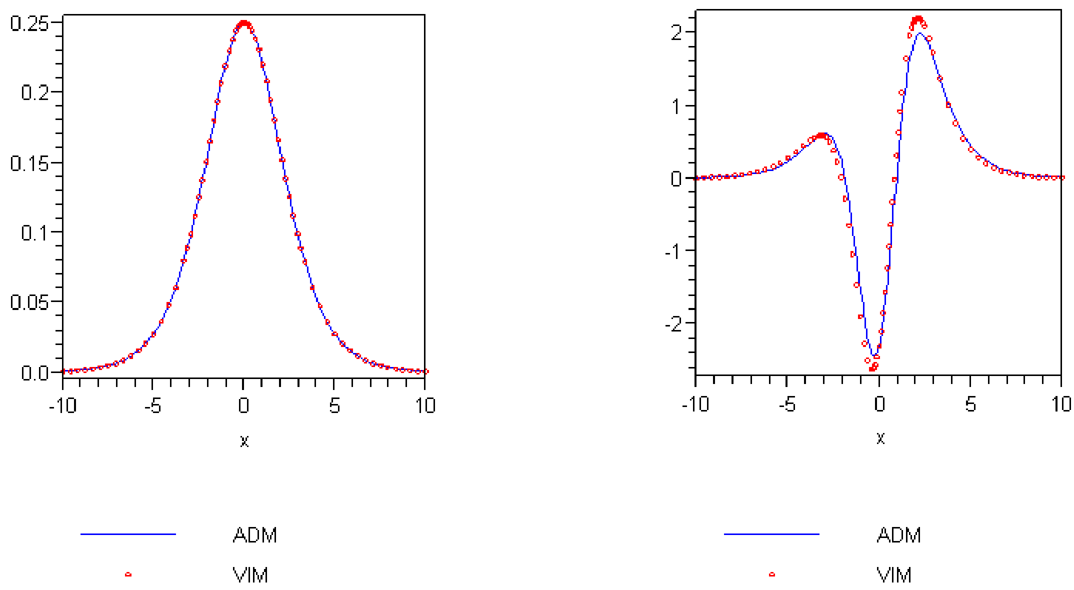
12 Conclusion

In this paper, Adomian's decomposition and He's variational iteration methods have been successfully applied to find the solution of two model equations for shallow water waves and their extended models. In addition, we compared these two methods to each other and showed that the results of the VIM method are in excellent agreement with results of ADM method. Also the obtained results were showed graphically. In our work; we used the Maple Package to calculate the functions obtained from the Adomian's decomposition method and He's variational iteration method.



1a 1b

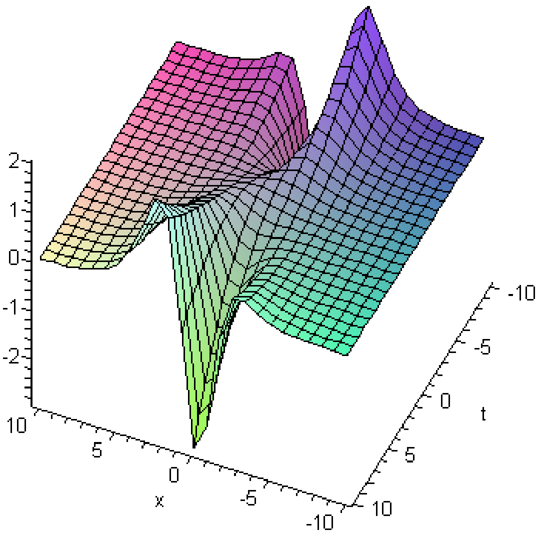
Fig.1. For the first model of shallow water wave equation with the first initial condition (33) of Eq. (2), ADM result for $u(x,t)$ is, respectively (1a) and VIM result (1b), when c=2.



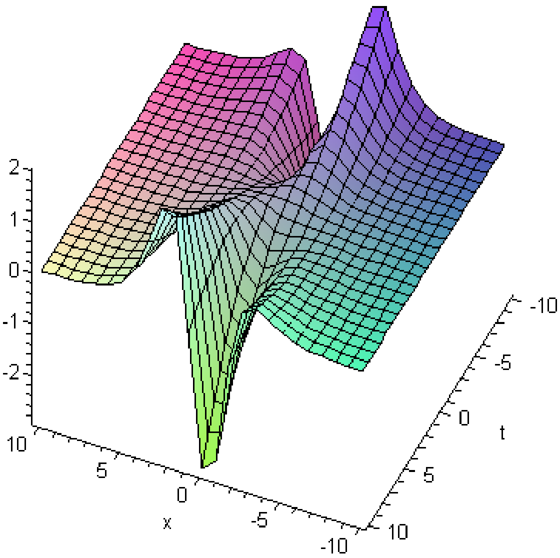
2a 2b

Fig.2.The comparison of ADM and VIM results for the first model of shallow water wave equation at

t=0 (2a) and **t=9** (2b)

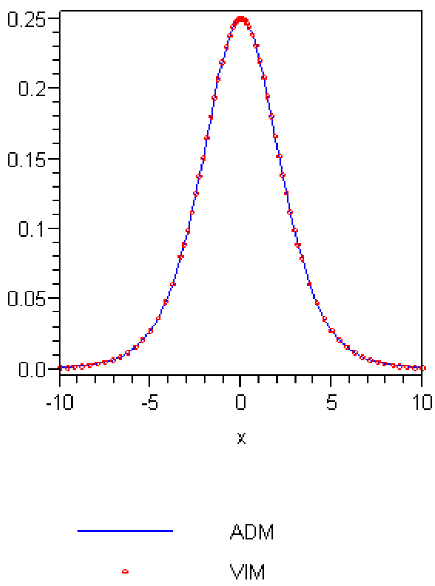


3a

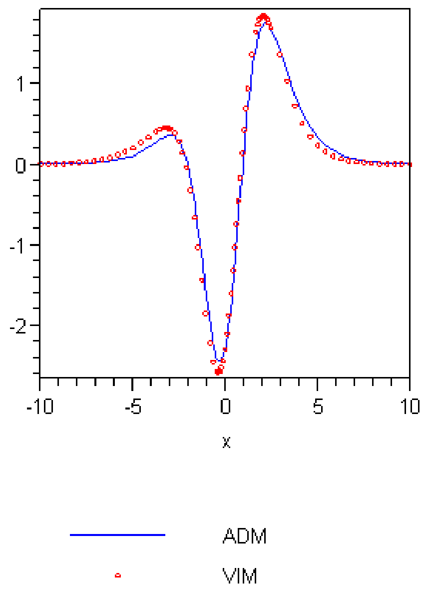


3b

Fig.3. For the second model of shallow water wave equation with the first initial condition (33) of Eq. (3), ADM result for $u(x,t)$ is, respectively (3a) and VIM result (3b), when c=2.



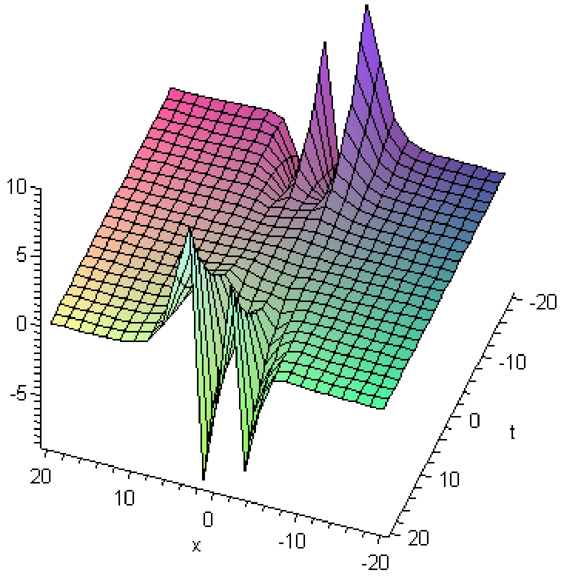
4a

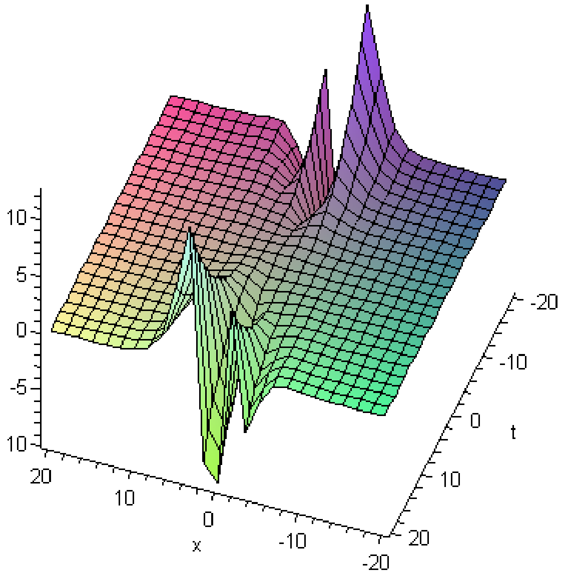


4b

Fig.4.The comparison of ADM and VIM results for the second model of shallow water wave equation, at

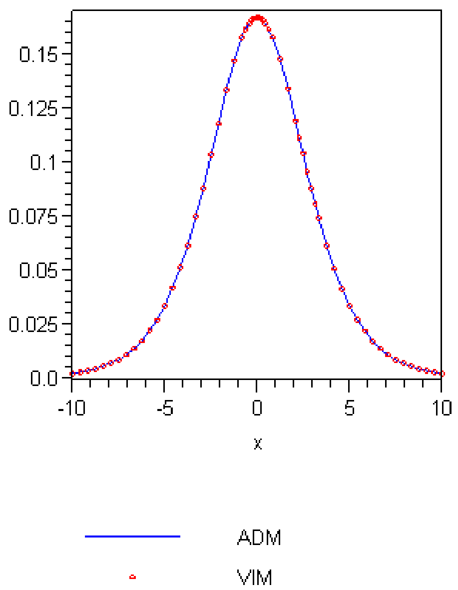
t=0 (4a) and at **t=9** (4b)

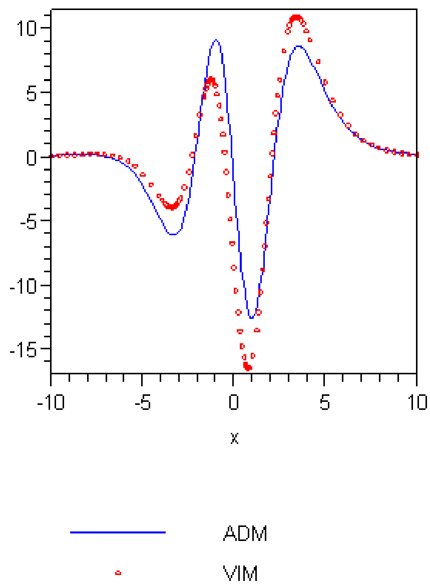




5a 5b

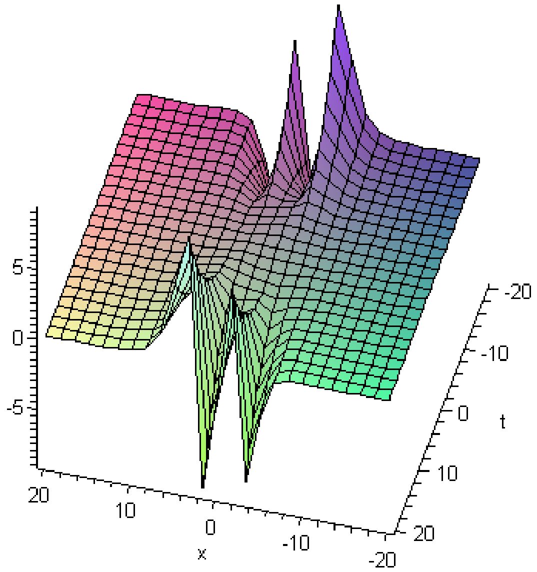
Fig.5. For the first extended model of shallow water wave equation with the first initial condition (69) of Eq. (14), ADM result for $u(x,t)$ is, respectively (5a) and VIM result (5b), when c=2.



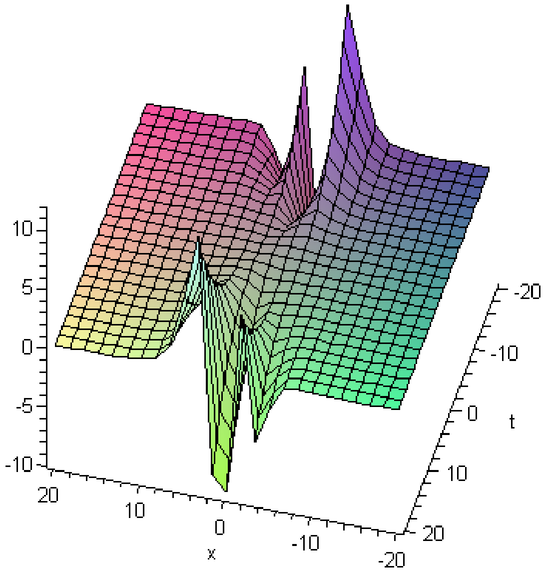


6a 6b

Fig.6.The comparison of ADM and VIM results for the first extended model of shallow water wave equation, at **t=0** (6a) and at **t=19** (6b)

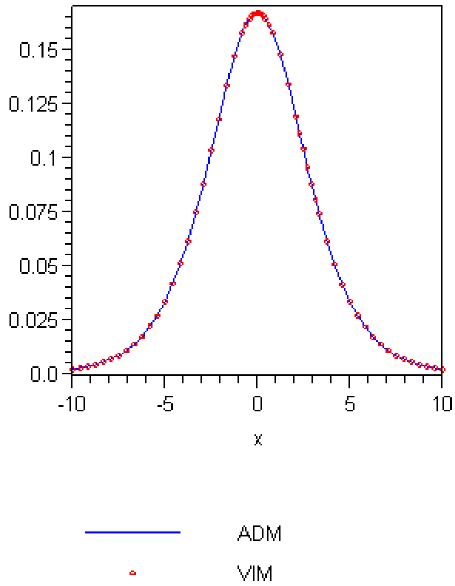


7a

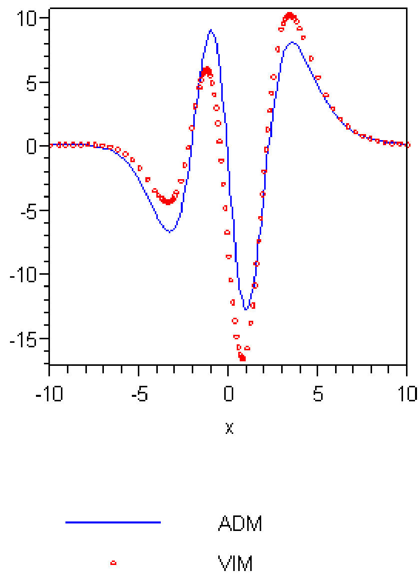


7b

Fig.7. For the second extended model of shallow water wave equation with the first initial condition (69) of Eq. (16), ADM result for $u(x,t)$ is, respectively (7a) and VIM result (7b), when c=2.



8a



8b

Fig.8.The comparison of ADM and VIM results for the second extended model of shallow water wave equation, at **t=0** (8a) and at **t=19** (8b)

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