Randomly $M_t$—decomposable Multigraphs and $M_2$—equipackable Multigraphs

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Abstract: A graph $G$ is called randomly $H$—decomposable if every maximal $H$ — packing in $G$ uses all edges in $G$. $G$ is called $H$ — equipackable if every maximal $H$ — packing in $G$ is also a maximum $H$ — packing in $G$. $M_2$—decomposable graphs, randomly $M_2$—decomposable graphs and $M_2$—equipackable graphs have been characterized. The definitions could be generalized to multigraphs. And $M_2$—decomposable multigraphs has been characterized. In this paper, all randomly $M_2$—decomposable multigraphs and $M_2$—equipackable multigraphs are characterized, and some notes about randomly $M_t$—decomposable multigraphs are given.

Key Words: Multigraph, packing, decomposable, randomly decomposable, equipackable, matching.

1 Introduction

The common notes and definitions of graphs can be found in [1]. The path and cycle on $k$ vertices are denoted by $P_k$ and $C_k$, respectively. The star with $l$ edges is denoted by $K_{1,l}$. A matching in a graph is a set of independent edges. By $M_t$ $(t \geq 1)$, we denote a matching having $t$ edges. Let $H$ be a subgraph of $G$. By $G - H$, here we denote the graph left after deleting the edges of $H$ from $G$ and any resulting isolates.

Definition 1. A collection of disjoint copies of $H$, say $H_1, H_2, \cdots, H_k$, where each $H_i$ is a subgraph of $G$, is called an $H$ — packing in $G$.

An $H$ — packing in $G$ with $k$ copies $H_1, H_2, \cdots, H_k$ of $H$ is called maximal if $G - \bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to $H$. An $H$ — packing in $G$ with $k$ copies $H_1, H_2, \cdots, H_k$ of $H$ is called maximum if no more than $k$ disjoint copies of $H$ can be packed into $G$. Let $p(G : H)$ denote the number of copies of $H$ in the maximum $H$—packing of $G$.

Definition 2. A graph $G$ is called $H$ — decomposable if there exists an $H$ — packing of $G$ which uses all edges in $G$.

Definition 3. A graph $G$ is called randomly $H$ — decomposable if every maximal $H$ — packing in $G$ uses all edges in $G$.

Definition 4. A graph $G$ is called $H$ — equipackable if every maximal $H$ — packing in $G$ is also a maximum $H$ — packing in $G$.

Definition 5. A graph $G$ is $H$ — decomposable, if $F$ is a subgraph of $G$, and $F$ is $H$ — decomposable but not randomly $H$ — decomposable, then $F$ is called $H$ — forbidden.

There have been many results on $H$ — decomposable graphs, randomly $H$ — decomposable graphs and $H$ — equipackable graphs. Here give some results.

Theorem 6 ([2]) Let $G$ be a graph of size $2m > 0$ and without isolates. Then $G$ is $M_2$—decomposable if and only if $\Delta(G) \leq m$ and $G$ is not isomorphic to $K_3 \cup K_2$.

Theorem 7 ([2]) Let $G$ be a graph of size $2m > 0$ without isolates and $G$ isn’t isomorphic to $M_2$, then $G$ is randomly $M_2$—decomposable if and only if $G \in F$, where $F = \{K_4, C_4, 2K_3, K_3 \cup K_{1,3}\} \cup \{2mK_2, 2K_{1,m}|m \geq 2\}$.

Theorem 8 ([11]) A graph $G$ is randomly $M_3$—decomposable if and only if it is isomorphic to one of the following: $3nK_2, 3K_{1,n}$, $C_6$, $C_4 \cup K_{1,2}$, $3K_3$, $2K_3 \cup K_{1,3}$, $2K_{1,3} \cup K_3$, $K_4 \cup K_3$, $K_4 \cup K_{1,3}$ or $K_3$.

Theorem 9 ([2]) The only connected randomly $K_{1,2}$—decomposable graphs are the cycle $C_4$ and the stars $K_{1,2t}$.
Theorem 10 ([11]) For \( r \geq 2 \), a connected graph \( G \) is randomly \( K_{1,r} \)-decomposable if and only if it is \( K_{r, r} \) or it is bipartite with all degrees in one partite set being multiples of \( r \) and all degrees in the other set being less than \( r \).

Theorem 11 ([11]) Let \( G \) be a graph with \( q \) edges and maximum degree \( d \). For \( q > \frac{8d^2}{3} - 2t \), \( G \) is \( M_t \)-packable if and only if \( t \mid q \) and \( q \geq td \).

Theorem 12 ([11]) For a given integer \( t \geq 2 \), a graph with at least \( 2t^3 - t^2 \) edges is randomly \( M_t \)-decomposable if and only if it is isomorphic to \( tH \), where \( H \) is either \( nK_2 \) or \( K_{1,n} \), for some \( n \geq 1 \).

All \( M_2 \)-equipackable graphs have been characterized in [5].

Theorem 13 ([5]) If \( G \) is a graph with size \( 2m \), then \( G \) is \( M_2 \)-equipackable if and only if \( G \) satisfies one of the following:

1. \( G \cong K_3 \cup K_2 \);
2. \( G \in \mathcal{F} \), where \( \mathcal{F} = \{K_4, C_4, 2K_2, 2K_3, K_3 \cup K_{1,3} \} \cup \{2mK_2, 2K_{1,m} | m \geq 2 \} \);
3. \( \Delta(G) = d > m \), and for any vertex \( v \) whose degree is \( d \), the induced subgraph by \( E(G - v) \) must be \( K_{1,2m-1} \) or \( K_3 \).

Theorem 14 ([5]) If \( G \) is a graph with size \( 2m + 1 \) and \( \Delta(G) = d \geq m + 2 \), then \( G \) is \( M_2 \)-equipackable if and only if for any vertex \( v \) whose degree is \( d \), the induced subgraph by \( E(G - v) \) must be \( K_{1,2m+1-d} \) or \( K_3 \).

Theorem 15 ([5]) If \( G \) is a graph with size \( 2m + 1 > 1 \) and \( \Delta(G) = d \leq m + 2 \), then \( G \) is \( M_2 \)-equipackable if and only if \( G \) satisfies one of the following:

1. Neither \( K_3 \) nor \( K_{1,3} \) is contained in \( G \) as a subgraph.
2. At least one copy of \( K_3 \) or \( K_{1,3} \) is contained in \( G \) as a subgraph and for any subgraph \( H \) of \( G \) which is isomorphic to \( K_3 \) or \( K_{1,3} \), \( \Delta(G - H) > m - 1 \) or \( G - H \cong K_3 \cup K_2 \).

A graph is called a multigraph if it contains loops or has two edges joining two common vertices. All multigraphs considered in the following are loopless and without isolates. For any pair of adjacent vertices of \( M \), say \( x \) and \( y \), let \( n(x, y) \) denote the number of multiple edges joining \( x \) and \( y \), called the multiplicity of \( x \) and \( y \). The underlying simple graph of \( M \) is a simple spanning subgraphs of \( M \), obtained by deleting all loops and all but one multiple edges, such that \( n(x, y) = 1 \), for any adjacent vertices \( x \) and \( y \) of \( M \).

Definition 16 The multistar \( S^{w_1, \ldots, w_4} \) is the multigraph, whose underlying graph is \( K_{1,t} \), and the multiplicities of its edges are \( w_1, \ldots, w_t \).

Definition 17 The multitriangle \( T^{x_1, \ldots, x_3} \) is the multigraph, whose underlying graph is a triangle \( C_3 \), and the multiplicities of its three edges are \( x_1, x_2, x_3 \).

Given a submultigraph (or a subset of vertices) \( S \), let \( M[S] = M[x_1, y_1, \ldots, z_1] \) denote the submultigraph of \( M \) induced by vertices of \( S \).

Definition 18 An edge set in which all edges are mutually adjacent is called a cluster or edge clique.

Therefore a cluster is a subset of edges of a submultigraph induced by vertices of either a star or a triangle.

Definition 19 The maximum size among clusters in \( M \) is called the cluster number (or edge clique number) of \( M \) and is denoted by \( \omega_1 = \omega_1(M) \).

Thus \( \omega_1 = \omega(L(M)) \), the clique number of the line graph \( L(M) \) of \( M \). Hence

\[
\omega_1(M) = \max\{\Delta(M), \max_{K_3 \subseteq M} e(M[K_3])\}.
\]

A cluster of size \( e(M)/2 \) is called a critical cluster in \( M \). By a critical triangle and a critical star we mean a critical cluster induced by vertices of a triangle and a star, respectively. The center of a critical star is called a critical vertex of the multigraph.

The concept of decomposable has been first extended to multigraphs by Skupień.

Theorem 20 ([4]) A multigraph \( M \) is \( M_2 \)-decomposable if and only if the number of edges \( e(M) \) is even and every edge-clique includes no more than half of the edges, i.e., \( \omega_1(M) \leq e(M)/2 \).

Theorem 21 ([9]) A multigraph \( M \) is \( M_3 \)-decomposable if and only if \( 3|e(M) \), \( \omega_1(M) \leq e(M)/3 \) and \( \omega_2(M) \leq 2e(M)/3 \), where \( \omega_2(M) = \max\{\max_{V_5 \subseteq V(M), |V_5| = 5} e(M[V_5])\} \). In this paper, some of the results will be generalized to multigraphs. All randomly \( M_2 \)-decomposable multigraphs and \( M_2 \)-equipackable multigraphs will be characterized in section 2 and section 4, and in section 3, there are some notes about randomly \( M_1 \)-decomposable multigraphs. When \( M \cong K_2, M \cong C_2 \) or \( M \cong P_3 \), there exists no copy of \( M_2 \) in \( M \), it has no meaning. When \( M \cong M_2 \), it is obviously \( M_2 \)-decomposable and \( M_2 \)-equipackable. So in the following, we consider the multigraphs with size larger than 2.
2 Randomly $M_2$ -decomposable multigraphs

In order to describe the randomly $M_2$-decomposable multigraphs, give a family of simple graphs $\mathcal{R}$ defined by

$$\mathcal{R} = \{C_4, K_4, 2K_3, K_3 \cup K_{1,r}, K_{1,r} \cup K_{1,s}, 2nK_2 | r = 1, 2, \cdots, n \geq 2\}.$$ 

Let $\mathcal{G}$ be a family of multigraphs with $\mathcal{R}$ as the underlying simple graphs.

See Figure 1. Here $e_i$ has the same multiplicity with $e_i'$ ($i = 1, 2, 3$) in (a) and (b). In (c), (d) and (e),

$$t_c = \sum_{i,j=1, i \neq j} 3 n(v_i, v_j), t_c' = \sum_{i,j=1, i \neq j} 3 n(v_i', v_j');$$

$$t_d = \sum_{i,j=1, i \neq j} 3 n(v_i, v_j), s_d = \sum_{1 \leq i \leq s} n(v', v_i');$$

$$s_e = \sum_{1 \leq i \leq r} n(v, v_i), s_e' = \sum_{1 \leq i \leq s} n(v', v_i'),$$

such that, $t_c = t_c', t_d = s_d$, and $s_e = s_e'$. Clearly, (c) is a disjoint union of two critical triangles, (d) is a disjoint union of a critical triangle and a critical star, and (e) shows a disjoint union of two critical stars, (f) is $2nK_2$. Let $\mathcal{G}$ denote the family of multigraphs in Figure 1 satisfying the conditions above.

![Figure 1: the multigraphs set $\mathcal{G}$](image)

Lemma 22 Let $M$ be a multigraph of size $2m (m \geq 1)$. Suppose that $M \notin \mathcal{G}$ (described in Figure 1). Then $M$ can be decomposed into one copy of $P_3$ or $C_2$ and $m - 1$ copies of $M_2$ if and only if $\omega_1(M) \leq m + 1$.

Proof: At first, it is not hard to note that the lemma holds for $m = 1, 2$. Then we assume that $m \geq 3$.

Suppose that $M$ has been decomposed into one copy of $P_3$ or $C_2$ and $m - 1$ copies of $M_2$. If $\omega_1(M) \leq m + 1$ is not true, we consider for $\omega_1(M) \geq m + 2$. There are two cases:

Case 1: $M$ can be decomposed into one copy of $P_3$ and $m - 1$ copies of $M_2$. Consider two subcases.

Subcase 1: $\omega_1(M) = \Delta(M)$.

Then $M$ contains a vertex $v$ of degree $\Delta(M) \geq m + 2$. If we delete a copy of $P_3$, which contains $v$ as an end-vertex, there must exist $m + 1$ edges incident with $v$ which belong to different $M_2$. If we delete a copy of $P_3$ with $v$ as the center, there must exist $m$ edges incident with $v$ which belong to different $M_2$. So at least $m$ edges incident with $v$ belong to different $M_2$. Then $e(M) \geq n + 2 + n > 2n$. This is a contradiction, so $\omega_1(M) \leq m + 1$.

Subcase 2: $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3])$.

Then $M$ contains a submultigraph $M[K_3]$, say $T$, with size $e(T) \geq m + 2$. Clearly, $\Delta(M) \leq e(T)$. We take any two edges of $T$, forming a copy of $P_3$, the remaining $m$ edges of $T$ must belong to different $2K_2$. Similar to case 1, we get a contradiction.

Case 2: $M$ can be decomposed into one copy of $C_2$ and $m - 1$ copies of $M_2$, as the case 1, there are two cases.

Subcase 1: $\omega_1(M) = \Delta(M) \geq m + 2$.

Let $v$ be a vertex with the biggest degree, then $v$ must be the end-vertex of $C_2$. As case 1, delete the copy of $C_2$, there are $m$ edges incident with $v$ which belong to different $M_2$. A contradiction.

Subcase 2: $\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3])$.

Then $M$ contains a submultigraph $M[K_3]$, say $T$, with size $e(T) \geq m + 2$. So $T$’s any two multiple
edges between any two vertices of \( T \), form a copy of \( C_2 \). As case 1, get a contradiction.

Conversely, suppose that \( M \notin \mathcal{G} \) and \( \omega_1(M) \leq m + 1 \), then we have three cases:

Case 1: \( \omega_1(M) \leq m - 1 \).

Since \( M \notin \mathcal{G} \), so \( M \neq 2mK_2 \), then \( M \) must contain \( P_3 \) or \( C_2 \), whose removal results in a submultigraph \( M' \), which has \( 2(m - 1) \) edges, and \( \omega_1(M') \leq \omega_1(M) \leq m - 1 \). By Theorem 4, \( M' \) has an \( M_2 \)-decomposition. So \( M \) can be decomposed into one copy of \( P_3 \) or \( C_2 \) and \( m - 1 \) copies of \( M_2 \).

Case 2: \( \omega_1(M) = m \).

Then \( M \) contains either a vertex with degree \( m \) or a \( M[K_3] \) with size \( m \). Since \( M \notin \mathcal{G} \), \( M \) is not a disjoint union of two critical triangles or two critical stars, \( M \) is not a disjoint union of a critical triangle and a critical star, either. Then \( M \) at most contains a union of two critical stars or two critical triangles with a common vertex, or, contains a union of a critical stars and a critical triangle with a common vertex. Let \( u \) denote the common vertex. We can delete such a \( P_3 \) (i.e., \( K_{1,2} \)) with \( u \) as the center vertex and its two edges belong to the two critical submultigraphs, respectively. Then we get the remaining submultigraph \( M' \), which has \( 2(m - 1) \) edges and \( \omega_1(M') \leq \omega_1(M) \leq m - 1 \). By Theorem 4, \( M \) can be decomposed into one copy of \( P_3 \) and \( m - 1 \) copies of \( M_2 \).

Case 3: \( \omega_1(M) = m + 1 \).

Then \( M \) contains only one vertex \( u \) with degree \( m + 1 \) or one \( M[K_3] \), say \( M[u,v,w] \) of size \( m + 1 \). For the former, delete two edges \( uv \) and \( uw \) from \( M \), where \( u \) and \( w \) have the two largest degrees among those vertices adjacent to \( u \). For the latter, delete two edges \( uv \) and \( uw \) from \( T \), where \( u \) has the largest degree in \( M \) among \( x, y, z \). Since \( \omega_1(M - uw - uw) \leq m - 1 \), we proceed as before to obtain the decomposition of \( M \).

\[ \square \]

**Theorem 23** Let \( M \) be a multigraph of size \( 2m(m \geq 1) \), then \( M \) is randomly \( M_2 \)-decomposable if and only if \( M \notin \mathcal{G} \).

**Proof:** If \( M \in \mathcal{G} \), clearly, \( M \) is randomly \( M_2 \)-decomposible. Conversely, let \( M \) is randomly \( M_2 \)-decomposible. We claim that \( M \notin \mathcal{G} \). On the contrary, suppose that \( M \notin \mathcal{G} \). By Theorem 20, since \( M \) is \( M_2 \)-decomposable, \( \omega_1(M) \leq m \). It follows that \( M \) can be decomposed into one copy of \( P_3 \) or \( C_2 \) and \( m - 1 \) copies of \( M_2 \), by Lemma 22. These \( m - 1 \) copies of \( M_2 \) do not belong to any \( M_2 \)-decomposition of \( M \), contradicting to the fact that \( M \) is randomly \( M_2 \)-decomposable. Therefore, \( M \notin \mathcal{G} \).

\[ \square \]

### 3 Some notes about randomly \( M_t \)-decomposable multigraphs

The following lemma is useful to our work.

**Lemma 24** ([11]) If \( G \) is a randomly \( H \)-decomposable graph (or multigraph) and \( F \) is an \( H \)-decomposable subgraph (or submultigraph),

1. If \( G \) is randomly \( H \)-decomposable, so is \( F \);
2. If \( F \) is \( H \)-forbidden, then \( G \) isn’t randomly \( H \)-decomposable.

**Lemma 25** A graph \( M \) is randomly \( M_t \)-decomposable, then the following conditions are necessary:

1. \( t|e(M) \),
2. \( \omega_1(M) \leq \frac{e(M)}{t} \).

The conditions above are not sufficient, for example, \( C_2 \cup P_4 \cup K_2 \) satisfies both the two conditions for \( t = 3 \), however it isn’t randomly \( M_3 \)-decomposable.

**Theorem 26** ([6]) A path \( P_n \) is \( M_t \)-equipackable if and only if \( n = kt \) \( (k \in N, k \geq 2) \).

**Theorem 27** ([6]) A circle \( C_n \) is \( M_t \)-equipackable if and only if \( 2t \leq n \leq 3t - 2 \), or \( n = kt + t - 1 \) \( (k \in N, k \geq 2) \).

It is easy to prove the lemma below by the Definition 3 and Definition 4.

**Lemma 28** If graph \( G \) if \( H \)-decomposable, then \( G \) is randomly \( H \)-decomposable if and only if \( G \) is \( H \)-equipackable.

By Theorem 26, Theorem 27 and Lemma 28, get the following corollary.

**Corollary 29** Only \( C_{2t} \) is a randomly \( M_t \)-decomposable path or circle.

**Theorem 30** Let a multigraph \( M \) be a connected graph which doesn’t contain submultigraphs isomorphic to \( K_{3}, K_{1,3} \) or \( S^{1,2} \), then \( M \) is \( M_t \)-decomposable if and only if \( M \cong C_{2t} \).

**Proof:** If \( M \) is a connected graph satisfying the conditions, then \( M \) is a simple path or circle. So by Corollary 29, \( M \cong C_{2t} \).

\[ \square \]

**Theorem 31** If a multigraph \( M \) is both randomly \( M_p \)-decomposable and randomly \( M_q \)-decomposable, and \( p(M : M_p) = m \), \( p(M : M_q) = l \), where \( p, q, m, l \in N, p \neq q \), and \( p \ast m = q \ast l = k \). Then \( M \cong M_k \).
Proof: Without losing of generality, let $p > q$, so $m < l$.

Assume that $M$ isn’t isomorphic with $M_k$, then $M$ must contain $P_3$ or $C_2$ as submultigraphs. Let the two edges of the $P_3$ or $C_2$ are $e$ and $f$, respectively. Then $e$ and $f$ must belong to different $M_p$’s copies and $M_q$’s copies in $M$. Let $H$ be a submultigraph of $M$ composed of the copy of $M_p$ containing $e$, say $H_1$, and the copy of $M_p$ containing $f$, say $H_2$. As $M$ is randomly $M_p$-decomposable, by Lemma 24, $H$ is also randomly $M_p$-decomposable. Obviously, in $H_1$ there is a copy of $M_p$ containing $e$, say $F_1$, and in $H_2$ there is a copy of $M_q$ containing $f$, say $F_2$. Consider two cases:

Case 1: Consider subgraph $P_3$. There are two subcases.

Subcase 1: No edge except $e$ in $F_1$ has neighbor edge in $F_2$.

Then replace any edge in $F_1$, say $g$, with $f$, where $g \neq e$. Now $F_2$ is still isomorphic with $M_q$, while $F_1$ contains $P_3$. And $M - F_1 - F_2$ is the union of $l - 2$ copies of $M_q$. So there is a maximal $M_q$-packing in $M$, which doesn’t use up all edges of $M$, that means $M$ is not randomly $M_q$-decomposable. A contradiction.

Subcase 2: There exits at least one edge, which is adjacent to $F_2$, except $e$ in $F_1$. See Figure 2, where $g, e \in F_1$ and $f, h \in F_2$.

Figure 2

(1) and (2): $g$ is adjacent to $h$, but not adjacent to $f$.

If $F_1 \cup F_2$ is an union of $q$ copies of clusters with size of 2, mean $K_{1,2}$ or $C_2$. Then $M$ must be the union of $q$ clusters of size $l$. (Because otherwise, there exists a cluster with size $a$, where $a < l$. We just consider the worst case: $a = l - 1$. And by Lemma 25, $\omega_1(M) = l$. So $M$ is the disjoint union of $q - 1$ copies of $K_{1,l-1}$ and one copy of $K_2$. Then we can take the $K_2$ and the other $q - 1$ edges form the copy of $K_{1,l-1}$ and from $q - 2$ copies of $K_{1,1}$. These $q$ edges form a copy of $M_q$. Obviously the remaining graph has only $q - 2$ copies of $M_q$. So $M$ has $q - 1$ copies of $K_{1,1}$, a contradiction.) So go back to $M \cong qK_{1,1}$. In this condition, $\omega_1(M) = \frac{e(M)}{q} = l > m$. However, $M$ is also randomly $M_p$-decomposable, and $\omega_1(M) = \frac{e(M)}{q} = m$, a contradiction.

If $F_1 \cup F_2$ isn’t an union of $q$ copies of clusters with size of 2, then $F_1 \cup F_2$ may contain subgraphs as $P_3$ or $K_2$, but not clusters with size of 3, for definitions of $F_1$ and $F_2$. So if there is a copy of $P_3$ or $K_2$, $e$ or $f$ can be replaced, by subcase 1, get a contradiction. So (1) and (2) are not correct.

(3): $g$ is adjacent to $h$ and $f$, and $h$ isn’t adjacent with $e$. So replace $h$ with $g$. As same as subcase 1, $F_1 \cong M_p$, but $F_2$ has $P_3$ as a subgraph, so $F_1 \cup F_2$ is not randomly $M_q$-decomposable. A contradiction.

(4): $e, f, g$ and $h$ form a copy of $C_4$. Because of Lemma 25, the copy of $C_4$ is independent of other edges of $H$. Except $e, f, g$ and $h$, if in $F_1$ (or $F_2$) there exists an edge, say $e$, which isn’t adjacent to any of edges in $F_2$ (or $F_1$), then replaced $h$ (or $g$) with $e$, as above get a contradiction. Otherwise, all edges in $F_1$ have adjacent edges in $F_2$ and all edges in $F_2$ have adjacent edges in $F_1$, and note that by Lemma 25, $\Delta(F_1 \cup F_2) \leq 2$. Then $F_1 \cup F_2$ is the union of paths and cycles. Obviously, if odd cycles or odd paths are contained in $F_1 \cup F_2$, then $F_1 \cup F_2$ is not randomly $M_p$-decomposable. So $F_1 \cup F_2$ is the union of even paths and even cycles. If $F_1 \cup F_2$ contains copies of $P_3$, we can take the two end-edges of a copy of $P_3$ as a $M_2$ belonging to a copy of $M_q$, then the remaining edges can’t form a copy of $M_q$. A contradiction. So $F_1 \cup F_2$ contains no paths with size larger than 2. It’s easy to see that if $F_1 \cup F_2$ is the union of even cycles and $P_3$ or $C_2$, it is randomly $M_q$-decomposable. As a result, $H$ is also the union of even cycles, $P_3$ or $C_2$ by the same analytical procedure. Now, because $p > q$, so at least there is a component in $H$ not belonging to $F_1 \cup F_2$. This component would be a even cycle, $P_3$ or $C_2$, whatever, we can replace one edge of the components with $e$ or $f$, then new $F_1 \cup F_2$ is not randomly $M_q$-decomposable, contradict to Lemma 24. So that $M$ doesn’t contain $P_3$ as a subgraph has been proved. Now consider the case 2.

Case 2: Consider subgraph $C_2$. As case 1, there are also two subcases.
Subcase 1: No edge except $e$ in $F_1$ has neighbor edge in $F_2$.

As same as subcase 1 in case 1, get a contradiction.

Subcase 2: There exits at least one edge, which is adjacent to $F_2$, except $e$ in $F_1$. By Lemma 25, see Figure 3, where $g, e \in F_1$ and $f, h \in F_2$.

![Figure 3](image)

In (1) and (2) it is easy to get a contradict by the same thought process of subcase 2 in case 1.

Above all, it has been proved that $M$ doesn’t contain $P_3$ or $C_2$, so $M \cong M_k$.

The lemmas given below is obvious.

**Lemma 32** $ntK_2$ is randomly $M_t$-decomposable, where $n, t \in N$ and $n \geq 1$.

**Lemma 33** The disjoint union of $t$ clusters, whose sizes are equal to each other, is randomly $M_t$-decomposable, where $t \in N$.

**Lemma 34** Let multigraph $F_1$ is randomly $M_p$-decomposable with $n$ copies of $M_p$, and multigraph $F_2$ is randomly $M_q$-decomposable with $n$ copies of $M_q$, where $n \in N$. Then the disjoint union of $F_1$ and $F_2$ is randomly $M_{p+q}$-decomposable with $n$ copies of $M_{p+q}$.

**Proof:** Let $F_1 + F_2$ denote the disjoint union of $F_1$ and $F_2$, and let $t = p + q$.

Assume that $F_1 + F_2$ isn’t randomly $M_t$-decomposable. Then, a maximal $M_t$-packing of $F_1 + F_2$ doesn’t use all of edges of $F_1 + F_2$, that is $p(F_1 + F_2 : M_t) = m < n$. So there are at least $t$ edges that cannot form a copy of $M_t$. As a result $F_1 + F_2$ must contain copies of $P_3$ or $C_2$ as subgraphs. Consider $P_3$ and $C_2$ in two cases respectively.

Case 1: At least one copy of $P_3$ is contained in $F_1 + F_2$.

So the copy of $P_3$ belongs to either $F_1$ or $F_2$. Without loss of generality, let the copy of $P_3 \subseteq F_1$. Let in $F_1 - P_3$ there be $k$ copies of $M_p$, then $n - 2 \leq k \leq n - 1$.

Subcase 1: $k = n - 1$.

Then $e(F_1) - ke(M_p) = ne(M_p) - (n - 1)e(M_p) = e(M_p) = p\ (e(G) = \text{the number of the edges of } G, \text{where } G \text{ is a graph})$, it means that the copy of $P_3$ is in the $p$ edges. This is contradict to that $F_1$ is randomly $M_p$-decomposable.

Subcase 2: $k = n - 2$.

Then $e(F_1) - ke(M_p) = ne(M_p) - (n - 2)e(M_p) = 2e(M_p) = 2p$. Different from subcase 1, the two edges of $P_3$ belongs to the $2$ copies of $M_p$ separately. So the remaining $2p$ edges are union of $p$ clusters of size $2$ or even cycles. (Otherwise, we can choose a $M_p$, but remain $p$ edges which cannot form a $M_p$. So it is not randomly $M_p$-decomposable.) If so, there is no copy of $M_p$ containing $P_3$. This is contradict to the assumption.

Case 2: At least one copy of $C_2$ is contained in $F_1 + F_2$. By case 1, easy to get a contradict.

All above, we prove the lemma.

**Theorem 35** Let $M$ is a non-connected multigraph with $k$ components, when $t \geq 2$, $M$ is randomly $M_t$-decomposable and $p(M : M_t) = n$ (n $\geq 1$, n $\in N$), if and only if $M \in \Psi$, where $\Psi = \{\sum_j=1^k F_j \mid F_j \in RD(M_{\sigma(j)}, n), \sigma(j) \in N, \sum \sigma(j) = t\}$, where $F_j \in RD(M_{\sigma(j)}, n)$ means graph $F_j(2 \leq j \leq k)$ is randomly $M_{\sigma(j)}$-decomposable and $p(F_j : M_{\sigma(j)}) = n$. Especially, $ntK_2 \in \Psi$ and $t$ clusters with size of $n$ belong to $\Psi$.

**Proof:** By Lemma 32, Lemma 33 and Lemma 34, sufficiency of the theorem is obvious, here only prove its necessity.

Let $M \in RD(M_t, n)$, and $M$ is non-connected. Consider number of components of $M$, denoted by $w(M)$, in three cases.

Case 1: $w(M) > t$.

Now $M$ must contain more than two copies of $M_t$. Assume that $M \neq ntK_2$, then at least one component of $M$ contains $P_3$ or $C_2$. There is always a subgraph $H$ composed of two copies of $M_t$, such that $H$ contains the copy of $P_3$ or $C_2$, and each of the two edges of $P_3$ or $C_2$ belongs to the two copies of $M_t$ separately. If $H$ is not union of clusters with size of $2$ (means $P_3$ or $C_2$) and even cycles, then $H$ is $M_t$-forbidden. If $H$ is the union of clusters with size of $2$ and even cycles, then arbitrarily replace one edge of other components of $M$ except these $t$ components with one edge of $H$. Easy to see, $H$ is $M_t$-forbidden. So, $M \cong ntK_2$.

Case 2: $w(M) = t$.

Because that randomly $M_t$-decomposable, so a subgraph say $H$ of $M$, composed by any two copies of
$M_t$ is still randomly $M_t$-decomposable by Lemma 24. Let $W_i$ denote the $i$th ($1 \leq i \leq t$) component of $M$ and let $e(W_1)$ is largest. Assume that exists one of the components in $M$ is not a cluster with size of $n$. And by Lemma 25, $\omega_1(M) \leq n$, then there must be a copy of $P_3$ as a subgraph.

Claim that in $W_2, \ldots, W_t$, at least one of them contains $P_3$ or $C_2$.

(The existing of $P_3$ or $C_2$ is true. Otherwise, let $M = M' + (t - 1)K_2$, where $M'$ is $W_1$ and other components are $K_2$. So, in $W_1$ there must be a copy of $M_t$, and because $W_1$ is connected, any two edges of the copy of $M_t$ must be in a same path. Then this copy of $M_t$ and any common neighbor edge of it's two edges, with other $(t - 1)K_2$ form a subgraph $H'$ of $M$, such a subgraph is $M_t$-forbidden. So $M \neq M' + (t - 1)K_2$)

Let $W_2 \supseteq P_3$ or $W_2 \supseteq C_2$, and let $P_4 \subseteq W_1$, and $P_4 = v_1e_2f_3gv_4$, where $e, f$ and $g$ are edges of $P_4$. Take $e$ and $g$ from $W_1$, and take one edge each from $W_2, \ldots, W_{t-1}$, the $t$ edges form a copy of $M_t$. And take $f$, with $t - 1$ edges from $W_2, \ldots, W_t$, form a copy of $M_t$, where make sure at least one edge from $W_2$. And the two copies of $M_t$ form a subgraph $H$ of $M$ and the two edges taken from $W_2$ is adjacent, that is $P_3$ or $C_2$. From $H$, take edges each from $W_3, \ldots, W_t$ with $e$ and $f$ form a copy of $M_t$. Obviously, the $t$ edges left can't form a copy of $M_t$. So $H$ is $M_t$-forbidden. A contradiction. So all components of $M$ are clusters with same size of $\omega_1(M) = \omega_1$. Case 3: $2 \leq w(M) < t$. Let the components of $M$ is $W_1, \ldots, W_k(2 \leq k < t)$. $W_1 \cup \ldots \cup W_k$ is randomly $M_t$-decomposable, then $W_j$ must be randomly $M_{\sigma(j)}$-decomposable, where $1 \leq j \leq k, \sigma(j) \in N$ and $\sum_{j=1}^{k} \sigma(j) = t$. Otherwise, assume that $W_1$ is not randomly $M_{\sigma(1)}$-decomposable. Then in $W_1$, there is a maximal $M_{\sigma(1)}$-packing which doesn’t use up all edges in $W_1$, with other maximal $M_{\sigma(j)}$-packing of $W_j(2 \leq j \leq k)$ form a $M_t$-packing is till maximal and doesn’t use all edges of $M$. A contradiction.

The non-connected randomly $M_t$-decomposable multigraphs have been characterized. Frankly speaking, it is hard to characterize connected randomly $M_t$-decomposable multigraphs. If once they are characterized, then Theorem 35 would be rewrite more clearly.

4 $M_2$-equipackable multigraphs

Firstly, we prove a lemma.

Lemma 36 Let $M$ be a multigraph with size $n$, and cluster number $\omega_1(M) = \omega_1$. If $\omega_1 > [\frac{n}{2}]$, then the number of $M_2$ in the maximum $M_2$-packing of $M$ is $n - \omega_1$, i.e., $p(M : M_2) = n - \omega_1$.

Proof: Since $\omega_1(M) = \max\{\Delta(M), \max_{K_2 \subseteq M} e(M[K_3])\}$, we have two cases:

Case 1: $\omega_1(M) = \Delta(M) = \omega_1 > [\frac{n}{2}]$.

Assume that $v$ is a vertex with degree $\omega_1$ and has $k$ neighbor vertices. Let the set of $v$’s neighbor vertices is $\{v_i\}_{i=1}^{k} n(v, v_i) = \omega_1, i = 1, 2, \ldots, k$.

Let $E_1$ be the edges set which are adjacent to $v$, and $E_2 = E(M) - E_1 = \{e_1, e_2, \ldots, e_{n-\omega_1}\}$. $|E_1| = \omega_1, |E_2| = n - \omega_1 < [\frac{n}{2}]$. It’s obvious that each edge of $E_2$ is adjacent to at most two edges of $E_1$. Let $n(v_i, v_j) = l, i \neq j$. We have $n(v, v_i) + n(v, v_j) + l \leq \omega_1$. Hence, $l \leq \omega_1 - n(v, v_i) - n(v, v_j)$. That is, for any $e_i \in E_2$, there exists an edge $e_i' \in E_1$ such that $e_i \sim e_i'$. Remove $\{e_i, e_i'\}$ forms a copy of $M_2$. Remove $\{e_i, e_i'\}$.

Let $E_1(1) = E_1 - \{e_i, e_i'\}$ and $E_2(1) = E_2 - \{e_i\}$. In the same way, each edge in $E_2(1)$ has at most two neighbors in $E_2(1)$, we can get another copy of $M_2$. We remove it and repeat this process $(n - \omega_1)$ times, thus removing all the edges of $E_2$ each of which along with one edge of $E_1$ forms a copy of $M_2$, while the remaining edges in $E_1$ contains no $M_2$. So the removed $n - \omega_1$ copies of $M_2$ form a maximum $M_2$-packing of $M$. Because all edges in $E_1$ are adjacent and there are no edges in $E_2$, this $M_2$-packing is maximum.

Case 2: $\omega_1(M) = \max_{K_2 \subseteq M} e(M[K_3]) = \omega_1$.

Let $T$ be a $M[K_3]$ of size $\omega_1$, $E_1 = E(T), |E_1| = \omega_1, V(T) = \{x, y, z\}$, $E_2 = E(M) - E_1 = n - \omega_1$. Obviously, any edge in $E_2$ is adjacent to at most one of $\{x, y, z\}$. Without loss of generality, we take any vertex of $T$, say $x$, let $e_x$ be the edge set adjacent to $x$ in $E_2$ and $T_x$ be edges between $y$ and $z$, which are not adjacent to $x$ in $T$, that is, $T_x = \{y, z\}$. We have $e_x \leq \Delta(M) - n(x, y) - n(x, z) \leq \omega_1(M) - \omega_1 = \omega_1 - (\omega_1 - T_x) = T_x$. Hence, for any $e \in E_2$, there exists an edge $e_i \in E_1$ such that $e_i, e_i'$ forms a copy of $M_2$. The proof is completed similar to Case 1. □

Theorem 37 Let $M$ be a multigraph of size $n$, $\omega_1 > [\frac{n}{2}]$, then $M$ is $M_2$-equipackable if and only if $M$ satisfies one of the following:

(1) $\omega_1(M) = \Delta(M) = \omega_1 > [\frac{n}{2}]$, and for any vertex $v$ whose degree is $\omega_1$, the submultigraph $M[M - v]$ must be a member of the multigraphs family $M[K_{1,r}]$ or $M[K_3]$, where $e(M[K_{1,r}]) = e(M[K_3]) = n - \omega_1$.
\begin{align*}
(2) \, \omega_1(M) &= \max_{K_3 \subseteq M} e(M[K_3]) = \omega_1 > \left[ \frac{n}{2} \right], \\
\text{and for any submultigraph } T = M[K_3] \text{ of size } \omega_1, \text{ the submultigraph } M[M - T] \text{ must be a member of the multigraphs family } M[K_{1,r}] \text{ or } M[K_3], \text{ where } \\
e(M[K_{1,r}]) &= e(M[K_3]) = n - \omega_1.
\end{align*}

**Proof:** We can easily verify that the multigraphs satisfying (1) or (2) are all \(M_2\)-equipackable.

Conversely, let \(M\) be an \(M_2\)-equipackable multigraph.

Case 1: \(\omega_1(M) = \Delta(M) = \omega_1 > \left[ \frac{n}{2} \right].\)

By Lemma 36, the number of \(M_2\) in the maximum \(M_2\)-packing in \(M\) is \(n - \omega_1\). Let \(v\) be a vertex with maximum degree. If there are two edges (say \(\{e, f\}\)) in \(M - v\) which are not adjacent, then after removing \(\{e, f\}\) (we denote \(M - \{e, f\}\)) by \(M_1\), \(\Delta(M_1) = \omega_1 > \left[ \frac{n}{2} \right] > \left[ \frac{n - \omega_1}{2} \right].\) The graph \(M_1\) also satisfies Lemma 36. So we can get a maximum \(M_2\)-packing in \(M_1\) with \(n - 2 - \omega_1\) copies of \(M_2\) which along with \(\{e, f\}\) form a maximal \(M_2\)-packing of \(M\). Obviously this resulting maximal packing with only \(n - 1 - \omega_1\) copies of \(M_2\) is not maximum, which contradicts that \(M\) is \(M_2\)-equipackable. So all edges in \(M - v\) are mutually adjacent. That is, the submultigraph \(M[M - v]\) must be a member of the multigraphs family \(M[K_{1,r}]\) or \(M[K_3]\), where \(e(M[K_{1,r}]) = e(M[K_3]) = n - \omega_1\).

Case 2: \(\omega_1(M) = \max_{K_3 \subseteq M} e(M[K_3]) = \omega_1 > \left[ \frac{n}{2} \right].\)

Let \(T\) be a submultigraph \(M[K_3]\) of size \(\omega_1\). Similar to Case 1, suppose there are two edges (say \(\{e, f\}\)) in \(M - T\) which are not adjacent, by Lemma 36, we can get a maximal, but not a maximum \(M_2\)-packing, which contradicts to the fact that \(M\) is \(M_2\)-equipackable. Hence, all edges in \(M - T\) are mutually adjacent. \(\square\)

Now, we shall consider the multigraphs \(M\) of size \(n\), with \(\omega_1 \leq \left[ \frac{n}{2} \right].\)

**Theorem 38** Let \(M\) be a multigraph of size \(2m\), and \(\omega_1(M) \leq m\). Then \(M\) is \(M_2\)-equipackable if and only if \(M \in \mathcal{G}\).

**Proof:** Obviously, any multigraph \(M \in \mathcal{G}\) is \(M_2\)-equipackable.

Conversely, let \(M\) be an \(M_2\)-equipackable multigraph with size \(2m\), then by Theorem 20, \(M\) is \(M_2\)-decomposable. So \(p(M : M_2) = m\). If \(M\) is not randomly \(M_2\)-decomposable, then there exists a maximal \(M_2\)-packing which does not use all edges in \(M\) and consequently which is not maximum. It contradicts to the fact that \(M\) is \(M_2\)-equipackable. So \(M\) must be randomly \(M_2\)-decomposable. By Theorem 23, \(M \in \mathcal{G}\). \(\square\)

**Lemma 39** Let \(M\) be a multigraph with size \(2m + 1\), and \(F\) be a maximal \(M_2\)-packing. If \(F\) satisfies one of the following:

1. \(F\) omits all the edges of a subgraph \(K_3;\)
2. \(F\) contains a copy (say \(\{e, f\}\)) of \(M_2\) such that neither \(e\) nor \(f\) is incident with the center of the star \(K_{1,3};\)
3. \(F\) contains a copy (say \(\{e, f\}\)) of \(M_2\) such that neither \(e\) nor \(f\) is incident with the center of the submultigraph \(S^{1,2}\). Then the multigraph \(M\) is not \(M_2\)-equipackable.

**Proof:** We just prove (3). Now we assume that \(F\) satisfies (3).

Without loss of generality, we denote the edges of the submultigraph by \(S^{1,2}\) by \(vv_1, vv_2\) and \(vv_1\), where \(n(v, v_1) = 1\) and \(n(v, v_2) = 2\). Neither \(e\) nor \(f\) is incident with \(v_2\). For the edge \(e\), there are three subcases:

Subcase 1: The edge \(e\) is not incident with any vertex of \(\{v_1, v_2\}\).

Since there are at most two vertices of \(\{v_1, v_2\}\) which are incident with \(f\), say \(v_1\), we can replace \(\{e, f\}\) with \(\{f, vv_2\}\) (or \(\{f, vv_2\}\)) and \(\{e, v_1\}\) to get a maximum \(M_2\)-packing whose size is larger than that of the given maximal \(M_2\)-packing. So \(M\) is not \(M_2\)-equipackable.

Subcase 2: The edge \(e\) is incident with \(v_1\).

Then \(f\) is not incident with \(v_1\) since \(e\) and \(f\) are independent. We can replace \(\{e, f\}\) with \(\{vv_1, f\}\) and \(\{e, vv_2\}\) (or \(\{e, vv_2\}\)) to get another larger maximal \(M_2\)-packing which is maximum. So \(M\) is not \(M_2\)-equipackable.

Subcase 3: The edge \(e\) is incident with \(v_2\).

Then \(v_2\) is not incident with \(f\) since \(e\) and \(f\) are independent. We can replace \(\{e, f\}\) with \(\{f, vv_2\}\) (or \(\{f, vv_2\}\)) and \(\{e, v_1\}\) to get another larger maximal \(M_2\)-packing. So \(M\) is not \(M_2\)-equipackable. \(\square\)

**Lemma 40** Let \(M\) be a multigraph with size \(2m + 1\) and cluster number \(\omega_1(M) = \omega_1 > 2, \omega_1 \leq m + 1\). If \(M\) is \(M_2\)-equipackable, then for any submultigraph \(H\) of \(M\) which is isomorphic to \(K_3, K_{1,3}\), or \(S^{1,2}\), then \(G - H\) is not \(M_2\)-decomposable.

**Proof:** Here we just give the proof about \(S^{1,2}\). Assume that the Lemma is not true, that is, \(M - H\) is \(M_2\)-decomposable, where \(H\) is isomorphic to \(S^{1,2}\). So \(M - H\) can be the union of \((m - 1)\) copies of \(M_2\). There must exists a copy (say \(\{e, f\}\)) of \(M_2\) in \(M - H\) such that neither \(e\) nor \(f\) is incident with the center of \(H\). Otherwise, each copy of \(M_2\) in \(M - H\) has an edge incident with the center (say \(v\)) of \(H\), then \(\omega_1 \geq \Delta(M) \geq d_M(v) \geq m - 1 + 3 = m + 2\), which contradicts to the fact that \(\omega_1 \leq m + 1\). By Lemma
39, $M$ is not $M_2$–equipackable, a contradiction. Similarly, the other two cases for $K_3$ and $K_{1,3}$ can also lead contradictions, respectively. □

**Theorem 41** Let $M$ be a multigraph with size $2m + 1 > 1$ and $\omega_1(M) = \omega_1 \leq m + 1$, then $M$ is $M_2$–equipackable if and only if $M$ satisfies one of the following:

1. None of $K_3$, $K_{1,3}$ or $S^{1,2}$ is contained in $M$ as a submultigraph;

2. $M$ is isomorphic to $M[K_{1,r}] \cup M[K_{1,r}] \cup e$, where $e$ is not incident with both of the two centers, or $M$ is isomorphic to $M[K_{1,r}] \cup M[K_3] \cup e$, where $e$ is neither incident with the star’s center nor incident with two vertices of the triangle at the same time, or $M$ is isomorphic to $M[K_3] \cup M[K_3] \cup e$, where $e$ is not incident with any two vertices of the triangle at the same time;

3. $M$ is isomorphic to $M[K_{1,r}] \cup M[K_{1,s}]$, or $M$ is isomorphic to $M[K_{1,r}] \cup M[K_3]$, or $M$ is isomorphic to $M[K_3] \cup M[K_3]$, or $M$ is isomorphic to $M[K_3] \cup M[K_3]$. Above all, they satisfies $1 \leq r_1, r_2 \leq m$, $1 \leq s \leq m + 1$, $e(M[K_{1,r_1}]) = e(M[K_{1,r_2}]) = e(M[K_3]) = e(M[K_3]) = e(M[K_{1,s}]) = e(M[K_3]) = m$, and $e(M[K_{1,s}]) = e(M[K_3]) = m + 1$.

**Proof:** We can easily verify that the multigraphs described in the theorem are all $M_2$–equipackable.

Let $M$ be $M_2$–equipackable, then we have two cases:

Case 1: None of $K_3$, $K_{1,3}$ or $S^{1,2}$ is contained in $M$ as a submultigraph.

That is, $M$ is a union of simple odd paths, simple even paths, simple odd circles or simple even circles, in which the number of all odd circles and odd paths is odd. Then for any edge of $M$, say $e$, there does always exist an edge, say $f$, and the two edges form an $M_2$. Delete them, and go on taking the same action, until only disjoint edges of odd number or a $P_4$ are remained in $M$. Obviously, $M$ is $M_2$–equipackable.

Case 2: At least one copy of $K_3$, $K_{1,3}$ or $S^{1,2}$ is contained in $M$ as a submultigraph.

By Lemma 39, for any submultigraph $H$ which is isomorphic to $K_3$, $K_{1,3}$ or $S^{1,2}$, $M - H$ is not $M_2$–decomposable, so by Theorem 20, $\omega_1(M - H) > m - 1$. Then we have $m \leq \omega_1(M - H) \leq \omega_1(M) \leq m + 1$. So, $\omega_1(M)$ has two possible values. When $\omega_1(M) = m$, we get the condition (2). When $\omega_1(M) = m + 1$, we get the condition (3). □

Hence, we have characterized all the $M_2$–equipackable multigraphs by Theorem 37, Theorem 38 and Theorem 41.

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**References:**


