On Component Order Edge Reliability and the Existence of Uniformly Most Reliable Unicycles

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Abstract: Let G be a graph with n nodes and e edges, where the nodes are perfectly reliable and the edges fail independently with equal probability ρ. A failure state exists if the surviving edges induce a graph having all components of order less than a preassigned threshold value k. The unreliability of G,$U_k(G;\rho)$, is the probability of a failure state and a graph G is *k-uniformly most reliable (k-UMR) over a class of graphs if and only if $U_k(G;\rho) \leq U_k(H;\rho)$ for all $0 < \rho < 1$ and all H in the same class as G. If $U_k(G;\rho) \geq U_k(H;\rho)$ for all $0 < \rho < 1$ and all H in the same class as G then G is k- uniformly least reliable (k- ULR). In this paper we show that $K_{1,n-1}$ is the unique tree that is k-UMR over all trees with $2 \leq k \leq n$. We also show that the unicycle U_s^x, i.e. $K_{1,n-1}$ with an added edge, is uniquely 3-UMR over all graphs having $n \geq 5$ nodes and $e = n$ edges. We extend this study for $4 \leq k \leq \frac{n}{2}$ and prove that U_s^x is the unique k-UMR unicycle. In the last two sections we give the necessary conditions for a graph to be k-UMR and show that there exists a range of values of k for which a k-UMR unicycle does not exist.*

Key–Words: Component Order edge reliability, uniformly most reliable, unreliability, unicycle

1 Introduction

In [1] we introduced the component order edge reliability model. We repeat some of the elementary notions pertinent to this model for the sake of completeness as well as discuss other fundamentals. Unless otherwise noted, we follow the notation of [2]. Consider a scenario of a network which is modeled by an undirected graph G having n nodes and e edges in which nodes are perfectly reliable but edges fail independently of one another, all with the same probability $\rho \in (0,1)$. A threshold value $2 < k < n$ is given and the surviving subgraph remaining after the failure of edges is said to be an **operating state** provided it contains at least one component of order k or more. It is a **failure state** provided each component has order at most $k-1$ and the subset of failing edges that produce the failure state is referred to as a **failure set**. The probability that, at a snapshot of time, the network is in an operating state is referred to as the **k-component order edge reliability** and is denoted by $R_k(G;\rho)$ while $U_k(G;\rho) = 1 - R_k(G;\rho)$, the **k-component order edge unreliability**, is just the probability that the network is in a failure state.

As a consequence of the underlying assumption of independent failure of edges, all with the same probability $\rho \in (0,1)$, the probability of a failure set for which a specific set of i edges fail is just $\rho^i(1 -$

$\rho)^{e-i}$. Thus if $f_i(G)$ denotes the number of failure sets of size i, then $U_k(G;\rho) = \sum_i f_i(G)\rho^i(1-\rho)^{e-i}$.

Now the smallest number of edges required to fail in order to produce a failure state is referred to as the **k-component order edge connectivity of G** and is denoted by $\lambda_c^{(k)}(G)$ (see [3],[4],[5],[6] for studies of $\lambda_c^{(k)}(G)$). Thus $f_i(G) = 0$ for all $i < \lambda_c^{(k)}(G)$. Furthermore, assuming $e \geq k-2$, a set of edges of size $i \geq e-(k-2)$, leaves a surviving subgraph having at most $k-2$ edges upon failure, and hence a failure state, so

$$f_i(G) = \binom{e}{i} \text{for} i \geq e-(k-2).$$

Thus, $\lambda_c^{(k)}(G) \leq e-(k-2)$ and we may write

$$U_k(G;\rho) = \sum_{i=\lambda_c^{(k)}(G)}^{e} f_i(G)\rho^i(1-\rho)^{e-i},$$

where $f_i(G) = \binom{e}{i}$ for all $e-(k-2) \leq i \leq e$. Of course, if $\lambda_c^k(G)$, then $U_k(G;\rho) = \sum_{i=e-(k-2)}^{e} \binom{e}{i}\rho^i(1-\rho)^{e-i}$. Thus any such graph will have the minimum value of the k-component order edge reliability among all graphs on n nodes and e edges for all $\rho \in (0,1)$.

Now it is clear that if ρ is held fixed then, as the collection of all graphs on n nodes and e edges is finite, there is a graph on n nodes and e edges having minimum k-component order edge reliability among all those in the collection. However, as has been shown for $k = n$, there may be no one graph in a collection that minimizes the unreliability for all $\rho \in (0,1)$[7]. If a graph exists in a specified collection of graphs, that minimizes $U_k(G;\rho)$ for all G in the collection and all $\rho \in (0,1)$, it is referred to as a **k-uniformly most reliable graph (k-UMR)** in the collection. It has been shown that if $e \geq \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$, then K_n minus a matching is $n-UMR$ over the class of all graphs having n nodes and e edges [8, 9]. If $e = n-1$, then all trees have the same n-component order edge unreliability, i.e., $1-(1-\rho)^{n-1}$. On the other hand it was shown that if $e = n$, then C_n is the unique unicycle which is $n-UMR$ over all unicycles on n [10].

In this work we shall show that $K_{1,n-1}$ is the unique tree that is $k-UMR$ over all trees and all $2 \leq k \leq n$. We shall also show that there exists a range of values of k for which a $k-UMR$ unicycle exists and a range for which no $k-UMR$ unicycle exists.

2 Graphs with $\lambda_c^k = e-(k-2)$

As was noted above, any graph G in the collection of all graphs having n nodes and e edges, where $n \geq k \geq 2$ and $e \geq k-2$; having $\lambda_c^{(k)} = e-(k-2)$ has k-component order edge unreliability equal to $U_k(G;\rho) = \sum_{i=e-(k-2)}^{e} \binom{e}{i}\rho^i(1-\rho)^{e-i}$

Now any graph on n nodes and e edges with $\lambda_c^{(k)} \leq e-(k-1)$ has strictly larger unreliability than G so we have the following theorem:

Theorem 1 *If $n \geq k \geq 2$, $e \geq k-2$ and $\lambda_c^{(k)} \leq e-(k-1)$, then G is k-UMR over all graphs having n nodes and e edges.*

In the remainder of this section we determine those cases for which $\lambda_c^{(k)} = e-(k-2)$. First note that if $e = k-2$, then $\lambda_c^k = 0 = e-(k-2)$ and $U_k(G;\rho) = 1$, so all such G on n nodes and e edges, where $n \geq k \geq 2$, have the same k-component order reliability. Another trivial situation occurs when $k = 2$. In this case $\lambda_c^{(k)} = e = e-(k-2)$ and $U_k(G;\rho) = \rho^e$ for all G on n nodes and e edges where $n > k = 2$.

Next suppose $e = k-1 \geq 2$. Note that if

$$G = T_k \cup (n-k)K_1$$

where T_k is a tree on k nodes, then

$$\lambda_c^{(k)} = 1 = (k-1)-(k-2) = e-(k-2).$$

Otherwise, $\lambda_c^{(k)} = 0 = e-(k-1)$. Hence, every

$$G = T_k \cup (n-k)K_1$$

where T_k is a tree on k nodes has $\lambda_c^{(k)} = e-(k-2)$ and no other graph on $n \geq k \geq 3$ nodes and $e = k-1$ edges does. Note that when $n = k$ this subsumes the fact that trees on n nodes have

$$\lambda_c^{(k)}(T) = 1 = e-(k-2).$$

The next logical case is $e = k \geq 3$. Suppose G has n nodes and e edges where $n \geq k = e \geq 3$ and assume $\lambda_c^{(k)} = e-(k-2)$. Then $f_{e-(k-1)} = 0$ and every subset of $k-1$ edges must induce a tree on k nodes. Choose one, say T_k, and let the remaining edge be denoted by x. Now at least one endpoint of x must lie in T_k or else the removal of an edge in T_k and the addition of x yields a disconnected subgraph with $k-1$ edges. Suppose the addition of x to T_k yields another tree. If that tree has a path with three edges, the removal of an internal edge of that path yields a disconnected subgraph having $k-1$ edges. Hence, if T_k with x added is a tree, then it must be $K_{1,k}$. Otherwise, consider the possibility that the addition of x to T_k yields a unicycle. If that unicycle contains a pendant edge its removal yields a subgraph of order $k-1$ containing $k-1$ edges, which contradicts $f_{e-(k-1)} = 0$.

Thus, in this case, T_k with x added is just C_k. In summary, if $\lambda_c^{(k)} = e - (k-2) = k - (k-2) = 2$, then $G = K_{1,k} \cup (n-k)K_1$ or $G = C_k \cup (n-1)K_1$. Observe that, if $n = k+1$, then in the first case $G = K_{1,n-1}$, while in the second case $G = C_n$.

Finally, suppose $e \geq k+1 \geq 4$ and assume $\lambda_c^{(k)} = e - (k-2)$. As in the previous case, we choose T_k, a tree on k nodes, from the set of e edges. Suppose T_k has two independent edges so that the addition of an edge to T_k cannot produce $K_{1,k}$.Thus each of the remaining edges when added to must produce C_k. As there are at least two additional edges, this is impossible. Thus, $T_k = K_{1,k}$ and each of the additional edges, when added to T_k,must yield $K_{1,k}$. Hence $G = K_{1,e}$ follows. Observe that in this case if $n = e+1$, then $G = K_{1,n-1}$. We summarize those findings in our next theorem.

Theorem 2

Consider $n \geq k \geq 2$, and let G be a graph on n nodes and e edges.

1. *If $e \leq k-2$, then $\lambda_c^{(k)}(G) = 0$, and equals $e - (k-2)$ when $e = k-2$. Also, $U_k(G;\rho) = 1$ for all $\rho \in (0,1)$.*

2. *If $k = 2$, then $\lambda_c^{(k)}(G) = e = e - (k-2)$, and $U_k(G;\rho) = \rho^e$ for all $\rho \in (0,1)$.*

3. *If $e \geq k-1 \geq 2$, then $\lambda_c^{(k)}(G) = 1 = e - (k-2)$ if and only if $G = T_k \cup (n-k)K_1$, where T_k is a tree on k nodes. In the event that $n = k$, then every tree on n nodes has $\lambda_c^k(T) = 1$ and $U_k(T;\rho) = 1 - (1-\rho)^{n-1}$.*

4. *If $e = k \geq 3$,then $\lambda_c^{(k)}(G) = 2 = k - (k-2)$ if and only if $G = K_{1,k} \cup (n-(k+1))K_1$ or $G = C_k \cup (n-k)K_1$. Also, $U_k(G;\rho) = 1 - (1-\rho)^k - k(\rho)(1-\rho)^k$. In the event that $n = k, G = C_n$ is the unique graph on n nodes with $e = n$ edges having $\lambda_c^n = 2$ and therefore, the unique such graph which is n-UMR. If $n = k+1$ (i.e. $k = n-1$) , then $K_{1,n-1}$ is the unique graph on n nodes with $e = n-1$ edges having $\lambda_c^{(n)} = 2$ and therefore, the unique such graph which is n-UMR.*

5. *If $e \geq k+1 \geq 4$,then $\lambda_c^k(G) = e - (k-2)$ if and only if $n \geq e+1$ and $G = K_{1,e} \cup [(n-(e+1))K_1]$. Also as was observed previously, $U_k(G;\rho) = \sum_{i=e-(k-2)}^{e} \binom{e}{i} \rho^i (1-\rho)^{e-i}$. In the event that $e = n-1 \geq k+1 \geq 4$, the unique graph having $\lambda_c^{(k)}(G) = e - (k-2) = n-k+1$ is $K_{1,n-1}$. It is, therefore, the unique k-UMR graph having n nodes and $e = n-1$ edges, and $U_k(G;\rho) = \sum_{i=n-k+1}^{n-1} \binom{n-1}{i} \rho^i (1-\rho)^{e-i}$.*

We conclude this section with three corollaries of the last theorem.

Corollary 3

1. *If $e = n-1, n \geq k$ and $k \geq 2$, then, $K_{1,n-1}$ is k-UMR over all graphs with n nodes and $e = n-1$ edges.*

2. *If $e = n-1 = k$, then $K_{1,n-1}$ is the unique k-UMR tree on n nodes.*

3. *If $e = n-1 \geq k+1 \geq 4$, then $K_{1,n-1}$ is k-UMR over all graphs with n nodes and $e = n-1$ edges.*

Corollary 4 *If $e = n-1 \geq k+1 \geq 4$, then $\lambda_c^{(k)}(G) \leq e-(k-1)$.*

Proof: By Theorem 2(5), $\lambda_c^{(k)}(G) = e-(k-2)$ forces $e \leq n-1$. ■

Corollary 5 *If $e = n = k$, then C_n is the unique graph in the class of all graphs having n nodes and $e = n$ edges with $\lambda_c^{(k)}(G) = 2 = e-(k-2)$ and is therefore also the unique n-UMR graph in this class.*

3 k-UMR Unicycles exist for $3 \leq k \leq \frac{n}{2}$

We begin with preliminary observations. First observe that by Corollary 4, $\lambda_c^{(k)}(U) \leq e-(k-1) = n-(k-1)$ for each unicycle U on n nodes. With $n = n_0 + k$, we obtain $\lambda_c^{(k)}(U) \leq n_0 + 1$. Next, it is easy to see that $\lambda_c^{(k)}(U_s^x) = n_0 + 1$ for $k \geq 3$, where U_s^x denotes the star $K_{1,n-1}$ with edge x added between two leaf nodes. Since $f_i(U) = \binom{n}{i}$ for each $i \geq n-(k-2)$, for each U it follows that we can prove U_s^x to be k-UMR by establishing the inequality $f_{n_0+1}(U_s^x) \leq f_{n_0+1}(U)$ for all unicycles $U \neq U_s^x$. Furthermore, if either $\lambda_c^{(k)}(U) < \lambda_c^{(k)}(U_s^x)$ and $f_{n_0+1}(U_s^x) \leq f_{n_0+1}(U)$ or $\lambda_c^{(k)}(U) \leq \lambda_c^{(k)}(U_s^x)$ and $f_{n_0+1}(U_s^x) < f_{n_0+1}(U)$ for each unicycle $U \neq U_s^x$, then U_s^x is seen to be uniquely k-UMR over the class of unicycles on n nodes. The final result of a preliminary nature is given in our first proposition.

Proposition 6 *If $U \neq U_s^x$ is a unicycle on n nodes, then U consists of two node disjoint trees T_1 and T_2 each having at least two nodes, joined by two edges x and y (see Figure 3.1).*

Proof: Let U be a unicycle different from U_s^x having unique cycle $v_1, v_2, ..., v_\ell, v_l$ so that U consists of the cycle together with ℓ node disjoint trees $T_1, T_2, ..., T_\ell$ rooted at $v_1, v_2, ..., v_\ell$, respectively (see Figure 3.2). If $\ell \geq 4$, then setting $x = v_1v_2$ and $y = v_3v_4$ establishes the claim. If $\ell = 3$, then, as $U \neq U_s^x$,at least two of the trees, say T_1 and T_2, have two or more nodes each. Then setting $x = v_1v_2$ and $y = v_1v_3$ establishes the claim.

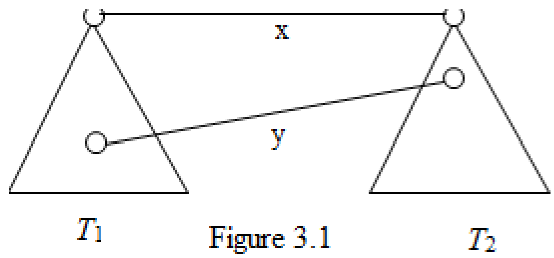


Figure 3.1

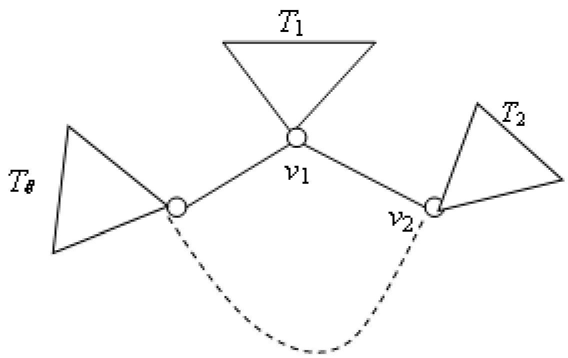


Figure 3.2

We proceed now to the case $k = 3$ and $n \geq 7$. In this case $n_0 + 1 = n - 2$ and we shall prove that U_s^x is uniquely 3-UMR not only over all the unicycles but also over all graphs having n nodes and $e = n$ edges provided $n \geq 7$.

Theorem 7 *The unicycle U_s^x is uniquely 3-UMR over all graphs having $n \geq 7$ nodes and $e = n$ edges.*

Proof: Consider a graph G on $n \geq 5$ nodes with $e = n$ edges and maximum degree $\Delta(G)$. If $G \neq U_s^x$ then $\Delta(G) \leq n-2$. Suppose $\Delta(G) \leq 2$; then, since the sum of the degrees of the nodes in G is $2n$, either $G = C_n$ or G is a disjoint union of cycles. Note that, in each case, for $n \geq 7$,

$$f_{n-2}(G) = \frac{n(n-3)}{2} > n - 3 = f_{n-2}(U_s^x).$$

Thus for each $n \geq 7, U(U_s^x;\rho) < U(G;\rho)$ for all $\rho \in (0,1)$. Next suppose that $\Delta(G) \geq 3$ and let node u have degree equal to $\Delta(G)$. Now each edge not incident at u forms pair of independent edges with at least $\Delta(G) - 2$ of the edges incident at u. Hence $f_{n-2}(G) \geq (n - \Delta(G))(\Delta(G) - 2)$. The parabola $y = (n - \Delta)(\Delta - 2)$ opens downward with axis of symmetry $\Delta = \frac{n}{2} + 1$. Also, the y values at $\Delta = 4$ and $\Delta = n - 2$ are equal to $2n - 8$. Thus, if $\Delta \geq 4$ then $f_{n-2}(G) \geq 2n - 8 > n - 3$, for $n \geq 6$. Now $\lambda_c^{(k)}(G) \leq n - 2 = \lambda_c^{(k)}(U_s^x)$ since the removal of $n-2$ edges from G leaving an independent pair yields a failure state. Thus when $\Delta \geq 4$ it follows that $U(U_s^x;\rho) < U(G;\rho)$ for all $\rho \in (0,1)$. Finally, we consider the case $\Delta(G) = 3$. Let node u have degree equal to $\Delta(G) = 3$ and $N(u) = v_1, v_2, v_3$. As there are at least seven edges in G, at least one of the n - 3 edges not incident at u is incident at most one node in $N(u)$. Each of the remaining n - 4 edges not incident at u forms a pair of independent edges with at least one edge incident at u. Thus

$$f_{n-2}(G) \geq n - 2 > n - 3 = \lambda_c^{(3)}(U_s^x).$$

Since $f_{n-2}(G) \geq n - 2$ it follows that $U(U_s^x;\rho) < U(G;\rho)$ for all $\rho \in (0,1)$ and the proof is complete. ■

To complete the case $k = 3$ we examine the situations $e = n = 5$ and $e = n = 6$ in turn. First, if $n = 5$ and $\Delta(G) = 4$ then $G = U_s^x$, and if $\Delta(G) = 2$, then $G = C_5$. Now $\lambda_c^{(k)}(C_5) = 3$ but $f_3(C_5) = 4 > 2 = f_3(U_s^x)$. If $\Delta(G) = 3$, let $deg(u) = 3$. If one of the two edges not incident at u is incident at only one node in N(u) then $f_3(G) \geq 3$. Otherwise $G = (K_4 - x) \cup K_1$ and has $\lambda_c^{(3)}(G) = 3, f_3(G) = 2$. Thus U_s^x and $(K_4 - x) \cup K_1$ are the 3-UMR graphs on $n = 5$ nodes. Next suppose $n = 6$. We know if $\Delta(G) \geq 4$ and $G \neq U_s^x$, then $U(U_s^x;\rho) < U(G;\rho)$ for all $\rho \in (0,1)$. If $\Delta(G) = 2$ then $G = C_6$ or $G = 2K_3$. If $G = C_6$, then $\lambda_c^{(k)}(C_6) = 3$ but $f_4(C_6) = 9 > 3 = f_4(U_s^x)$. If $G = 2K_3$, then $\lambda_c^{(k)}(2K_3) = 4$ but $f_4(2K_3) = 9 > 3 = f_4(U_s^x)$ If $\Delta(G) = 3$ then let $deg(u) = 3$ and observe that if one of three edges not incident at u is incident with at most one node in $N(u)$ then such an edge forms at least two pairs of independent edges with edges incident at u. The other two form at least one pair with an edge incident at u so $f_4(G) \geq 4 > 3 = f_4(U_s^x)$. Otherwise, $G = K_4 \cup 2K_1$, which has $\lambda_c^{(3)}(G) = 4, = \lambda_c^{(3)}(U_s^x)$ $f_3(G) = 2$. Thus U_s^x and $K_4 \cup 2K_1$ are the 3-UMR graphs with $e = n = 6$.

The preceding analysis leads to the following corollary.

Corollary 8 *The graph U_s^x is the unique 3-UMR unicycle for $n \geq 5$.*

Next we proceed to the general case:

$$4 \leq k \leq n_0; n_0 \geq 2.$$

We begin by proving an important vulnerability result in the context of unicycles. We shall show that in all instances, save one, U_s^x is the unique unicycle with $\lambda_c^{(k)} = n_0 + 1$. All others have $\lambda_c^{(k)} = n_0$ or less, thereby indicating that U_s^x is the unique most invulnerable unicycle subject to system failure. Initially, we deal with the case where the cycle of the unicycle $U \neq U_s^x$ has length $\ell \geq 4$ and in preparation for that result we require the next lemma.

Lemma 9 *If $n_0 \geq 2$ and $k \geq 4$, then $\lambda_c^{(k)}(C_n) \leq n_0$.*

Proof: We know that $\lambda_c^{(k)}(C_n) = \lceil \frac{n}{k-1} \rceil = \lceil \frac{n_0+1}{k-1} \rceil + 1$ [5]. But $\lceil \frac{n_0+1}{k-1} \rceil + 1 \leq n_0$ if and only if $\frac{k}{k-2} \leq n_0$, which is the case when $k \geq 4$.■

Theorem 10 *If $k \geq 4, n_0 \geq 2$ and U is a unicycle with cycle length $\ell \geq 4$, then $\lambda_c^{(k)}(U) \leq n_0$.*

Proof: We shall employ induction on n_0 beginning with base case $n_0 = 2$. First, if $\ell = n$, so that $U = C_n$, the result follows by Lemma 9. If $4 \leq \ell \leq n-1$, then referring to Proposition 6 there is a T_i, say, T_1, with order at least two. Removal of the edges $v_1 v_\ell$ and $v_2 v_3$ leaves two components, one consisting of T_1 and T_2 together with edge $v_1 v_2$ and the other consisting of $T_3, ..., T_\ell$ together with the path $v_3, v_4, ..., v_\ell$. The first component contains at least two edges, so if the second does as well, each will contain at most $k-2$ edges, as the number of remaining edges is k. Hence in this event $\lambda_c^{(k)}(U) = 2 = n_0$. Suppose the second component consists of one edge $v_3 v_4$, i.e. $\ell = 4$, (see Figure 3.3). Since $n = k + n_0 \geq 6, |V(T_1)| + |V(T_2)| \geq 4$. In the event that $|V(T_1)| \geq 3$ but $|V(T_2)| = 1$, removal of $v_1 v_2$ and $v_1 v_4$ yields a failure state while if $|V(T_1)|, |V(T_2)| \geq 2$, removal of $v_1 v_2$ and $v_3 v_4$ yields a failure state. Hence if $n_0 = 2$ and $k \geq 4, \lambda_c^{(k)}(U) = 2 = n_0$. Our induction hypothesis is that if $n_0 = m-1$, where $m \geq 3$, and U has cycle length l where $4 \leq \ell \leq n$, then $\lambda_c^{(k)}(U) \leq n_0 = m-1$, for $k \geq 4$. Now consider a unicycle with $n_0 = m$ and cycle length l where $4 \leq \ell \leq n$ and suppose $U = C_n$; then $\lambda_c^{(k)}(U) \leq n_0$ by Lemma 9. If $U \neq C_n$ then U has a pendant node and edge. Remove the pendant node and edge obtaining a unicycle $\hat{U}$ with $n - 1 = m - 1 + k$ nodes and cycle length l. The induction hypothesis forces $\lambda_c^{(k)}(\hat{U}) \leq m-1$ so $\lambda_c^{(k)}(U) \leq m = n_0$ for $k \geq 4$.

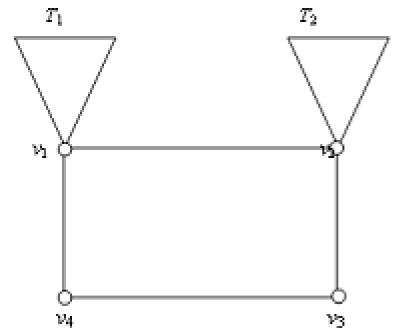


Figure 3.3

Our next theorem deals with the case $l = 3$ and includes the one exceptional case previously mentioned.

Theorem 11 *Consider the class of all unicycles on $n = n_0 + k$ nodes when $n_0 \geq 2$ and $l = 3, k \geq 4$. If U_6 is the unicycle where $|V(T_1)| = |V(T_2)| = |V(T_3)| = 2$ (see Figure 3.3), then $\lambda_c^{(4)}(U_6) = 3 = n_0 + 1$. Otherwise, i.e., if $U \neq U_s^x, U_6$ then $\lambda_c^{(k)}(U) \leq n_0$.*

Proof: First observe that if U has six nodes but is not equal to U_s^x or U, then, without loss of generality, $|V(T_1)| = 3, |V(T_2)| = 2, |V(T_3)| = 1$(see Figure 3.4).

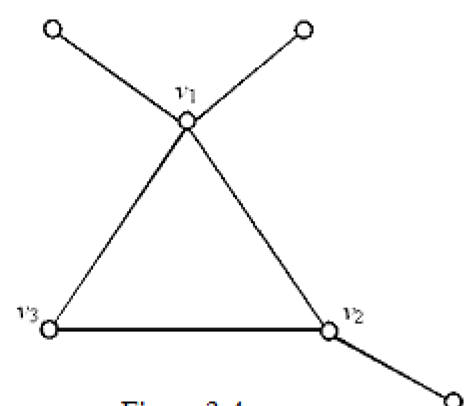


Figure 3.4

The set $\{v_1 v_2, v_1 v_3\}$ is a failure set so $\lambda_c^{(4)}(U) = 2 = n_0$. We complete the proof by induction on n_0 starting with the base case $n_0 = 2$. Since we have already considered $n_0 = 2, k = 4$, it remains to assume $n \geq 7$ so that by the Pigeonhole Principle, some

$|V(T_i)|$, say $|V(T_1)| \geq 3$. But $U \neq U_s^x$ implies, without loss of generality, that $|V(T_2)| \geq 2$ and it follows that $\{v_1v_2, v_1v_3\}$ is a failure set since each of the two components it leaves upon removal has at most $k-2$ edges. Our induction hypothesis simply states that if U is a unicycle on n nodes, different from U_s^x with cycle length $l = 3$ and $n_0 = m-1 \geq 2$, then $\lambda_c^{(k)}(U) \leq n_0$ where $k \geq 4$. Now consider a unicycle U on n nodes with cycle length $l = 3$ and $n_0 = m \geq 3$. The Pigeonhole Principle forces one $|V(T_i)|$, say $|V(T_1)| \geq 3$ and, since $U \neq U_s^x$, another, say $|V(T_2)| \geq 2$. Again, as $n \geq 7$, either

1. $|V(T_1)| \geq 3, |V(T_2)| \geq 2$, and $|V(T_3)| \geq 2$ or

2. $|V(T_1)| \geq 3, |V(T_2)| \geq 3$, and $|V(T_3)| = 1$ or

3. $|V(T_1)| \geq 4, |V(T_2)| \geq 2$, and $|V(T_3)| = 1$.

In the first case remove a pendant node and its edge from T_3, in the second case from T_2 and in the third case from T_1, thereby arriving at a unicycle $\hat{U}$ on $n-1 = m-1+k$ nodes with cycle length $l = 3$ and different from U_s^x, U_6. Hence $\lambda_c^{(k)}(U) \leq \lambda_c^{(k)}(\hat{U})+1 \leq m-1+1 = m = n_0$ and the proof is complete. ■

In the remainder of this section we prove that U_s^x is the unique k-UMR unicycle for $4 \leq k \leq \frac{n}{2}$. Since $\lambda_c^{(k)}(U) \leq \lambda_c^{(k)}(U_s^x)$ for all unicycles with only one exception, i.e. $n = 6, n_0 = 2$, $k = 4$, and $e-(k-2) = n_0+2$, it is only necessary to prove that $f_3(U_6) > f_3(U_s^x)$ in the exceptional case and that $f_{n_0+1}(U_6) \geq f_{n_0+1}(U_s^x)$ for the other cases. Let's consider $n = 6, n_0 = 2$ and $k = 4$ to begin with. The unicycles in this case are shown in Figure 3.3. It is easy to see that U_1, U_2, U_3, U_4 and U_5 all have $\lambda_c^{(4)} = 2 = n_0$ and $f_3 \geq 4$ while $f_3(U_s^x) = 4$. On the other hand, $f_3(U_6) > 4 = f_3(U_s^x)$.

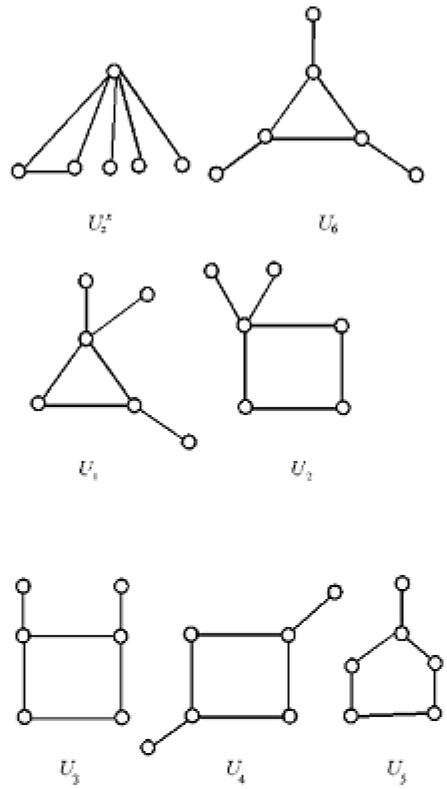


Figure 3.5

Now, for the remaining cases, we begin with the observation that if $U \neq U_s^x$ then by Proposition 6 we can represent U by two trees T_1 and T_2 of orders $n_1 \geq 2$ and $n_2 \geq 2$, respectively, joined by two edges x and y (See Figure 2.1). We begin the analysis of this case under the assumption that $n_1 \geq k$ and $n_2 \geq k$ and establish a lower bound on the failure sets of size n_0+1 which include x and y. Of course, the removal of n_0+1 edges leaves a total of $k-1$ edges so that if x and y are removed and n_0-1 additional edges are removed so that at least one edge from T_1 and one edge from T_2 remain then a failure state is obtained. Thus, the number of failure sets of size n_0+1 which include x and y, denoted by $f_{n_0+1}^{x,y}$, satisfies

$$f_{n_0+1}^{x,y} \geq \sum_{i=1}^{k-2} \binom{n_1-1}{i}\binom{n_2-1}{k-1-i} =$$

$$\binom{n_1+n_2-2}{k-1} - \binom{n_1-1}{k-1} - \binom{n_2-1}{k-1}.$$

But $n_1, n_2 \geq k$ implies that $\binom{n_1-1}{k-1}+\binom{n_2-1}{k-1}$ is maximized when $n_1 = k$ and $n_2 = n_0$ or vice versa. Thus

$$f_{n_0+1}^{x,y}(U) \geq \binom{n-2}{k-1} - \binom{n_0-1}{k-1} - 1.$$

Next we prove that there exist $\binom{n-4}{k-1}$ additional failure sets that include at most one of the edges x and

y. There are two scenarios to consider dependent on $deg(u_2)$, where $x = u_1u_2$. Assume without loss of generality that $y = uv$ where $u \in V(T_1), v \in V(T_2)$ but $v \neq u_2$. Now if $deg(u_2) \geq 3$ let w be the edge incident on u_2 that lies on the unique path of T_2 from u_2 to v and let z be any other edge incident on u_2 in T_2. Now if x and w are removed together with $n_0 - 1$ additional edges not including y and z a failure state results, since y and z lie in separate components (see Figure 3.6(a)). As there are $\binom{n-4}{n_0-1}$ of these sets the claim is established in this case. The other scenario involves the case where $deg(u_2) = 2$. Let w be the edge of T_2 with endpoint u_2 and consider removing $n_0 - 1$ edges including y and w but not x or z where z is an arbitrary but fixed edge of T_2 different from w. Realize edge z exists since $n_2 \geq k \geq 4$ (see Figure 3.6(b)). A failure state results since x and z lie in separate components. Here too there are $\binom{n-4}{n_0-1}$ such sets. Thus

$$f_{n_0+1}(U) \geq \binom{n-4}{k-1} + \binom{n-4}{n_0-1} - \binom{n_0-1}{k-1} - 1.$$

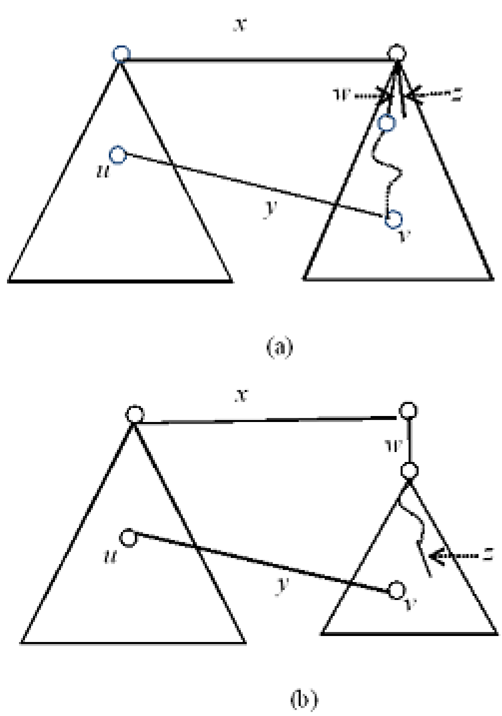


Figure 3.6

Consider

$$D = f_{n_0+1}(U) - f_{n_0+1}(U_s^x) =$$

$$\binom{n-2}{k-1}+\binom{n_0-1}{k-1}+\binom{n-4}{n_0-1}-\binom{n-3}{n_0-1}-\binom{n-3}{n_0+1}-1 =$$

$$\frac{(n-2)!}{(k-1)!(n_0-1)!}+\frac{(n-4)!}{(k-3)!(n_0-1)!}-\frac{(n-3)!}{(k-2)!(n_0-1)!}-\frac{(n-3)!}{(k-4)!(n_0+1)!}-[\binom{n_0-1}{k-1}+1] = \frac{(n-4)!}{(k-1)!(n_0+1)!}-$$

$$[(n-2)(n-3)(n_0+1)(n_0)+(k-1)(k-2)(n_0+1)(n_0)-(n-3)(k-1)(n_0+1)(n_0)-(n-3)(k-1)(k-2)(k-3)]-[\binom{n_0-1}{k-1}+1].$$ Now,

$$\binom{n_0-1}{k-1} = \frac{(n_0-1)!}{(n_0-k)!(k-1)!} =$$

$$\frac{(n-4)!}{(n_0+1)!(k-1)!}\frac{(n_0+1)!(n_0)\cdots(n_0-k+1)!}{(n-4)(n-5)\cdots(n_0)} \leq$$

$$\frac{(n-4)!}{(n_0+1)!(k-1)!}(n_0+1)(n_0)(n_0-1)(n_0-2)$$

and it follows that

$$D \geq \frac{(n-4)!}{(n_0+1)!(k-1)!}[(n_0+1)(n_0)(n_0-2)(n_0-3)+$$

$$(k-1)(k-2)(n_0+1)(n_0)-(k-1)(n-3)(n_0+1)(n_0)-$$

$$(n-3)(k-1)(k-2)(k-3)-(n_0+1)(n_0)(n_0-1)(n_0-2)]-$$

$$1 = \frac{(n-4)!}{(n_0+1)!(k-1)!}[(n-3)(k-1)((n_0+1)(n_0)-$$

$$(k-2)(k-3))]-1 = \binom{n-3}{n_0-1} - \binom{n-3}{n_0+1} - 1 =$$

$$\binom{n-2}{k-2} - \binom{n-3}{k-4} - 1 > 0$$

because the binomial coefficients strictly increase as the subset size increases toward the midpoint of the set size.

Next we consider the case where $3 \leq n_1 \leq k-1$, $n_2 \geq n_1$. As in the previous case, $f_{n_0+1}^{x,y}(U) \geq \binom{n_0+k-2}{n_0-1} - \binom{n_2-1}{k-1}$. But here, $n_2 \leq n-3$ so $f_{n_0+1}^{x,y}(U) \geq \binom{n_0+k-2}{n_0-1} - \binom{n_0+k-4}{k-1}$. Also, as the argument used in the previous case applies whether y is incident at either u_1 or u_2, we can show that there exists $\binom{n_0+k-4}{k-1}$ additional failure sets so that $f_{n_0+1}^{x,y}(U) \geq \binom{n_0+k-2}{n_0-1}$. Thus, $D = f_{n_0+1}^{x,y}(U) - f_{n_0+1}^{x,y}(U_x^s) \geq \binom{n_0+k-2}{n_0-1} - \binom{n_0+k-3}{n_0-1} - \binom{n_0+k-3}{n_0+1} = \binom{n-3}{k-1} - \binom{n-3}{k-4} > 0$ for basically the same reason as in the previous case.

Finally, suppose that $n_1 = 2$. In this case, $n_2 = n-2$ so $f_{n_0+1}^{x,y}(U) \geq \binom{n_0+k-3}{n_0-1}$. Here we claim that there are at least $2\binom{n_0+k-4}{n_0-1}$ additional failure sets for all possible scenarios but one (see Figure 3.7). Indeed, if this is the case then

$$f_{n_0+1}^{x,y}(U) - f_{n_0+1}^{x,y}(U_x^s) \geq$$

$$\binom{n_0+k-3}{n_0-1} + 2\binom{n_0+k-4}{n_0-1} - \binom{n_0+k-3}{n_0-1} -$$

$$\binom{n_0+k-3}{n_0+1} = 2\binom{n_0+k-4}{n_0-1} - \binom{n_0+k-3}{n_0+1} =$$

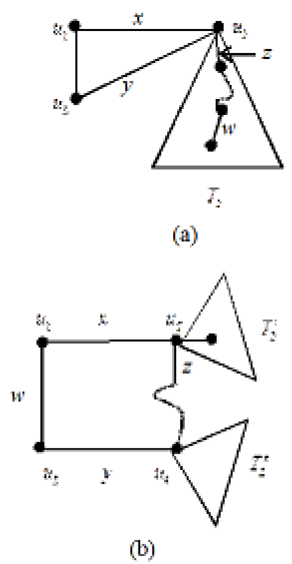
$$\frac{(n_0+k-4)!}{(k-3)!(n_0+1)!}[2{n_0}^2 - (k-5)n_0 - (k-3)^2] >$$

0 if $n_0 \geq k$.

To prove the claim we consider all possible scenarios as shown below in Figure 3.7 and show that for (a) through (d) the claim holds. We verify $f^{x,y}_{n_0+1}(U) - f^{x,y}_{n_0+1}(U^s_x) \geq 0$ directly for the scenario shown in (e).

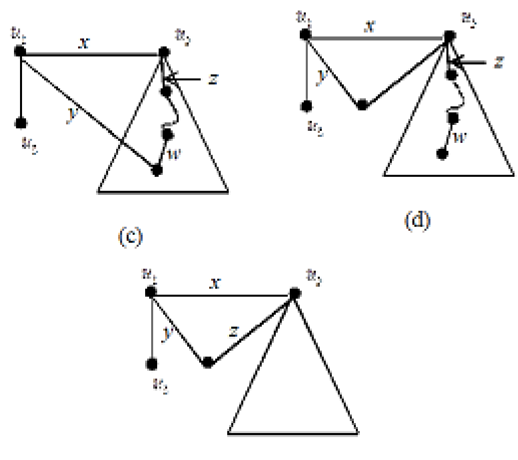
In (a) T_2 must contain a path from u_2 to some node u_2 of length at least two, since $U \neq U^s_x$. Then by choosing failure sets, including x and z but not y and w and failure sets including y and z but not x and w we obtain $2\binom{n_0+k-4}{n_0-1}$ additional failure sets. In (b) realize that either T_2' or T_2'' contains an edge, say T_2'. Then failure sets containing x and z but not y and w together with failure sets containing w and z but not x and y yield an additional failure sets.

In (c) we may consider failure sets including x and w but not y and z and failure sets including y and z but not x and w to obtain the claim. As regards (d), failure sets (i) including x and z but not y and w and (ii) including y and z but not x and w establish the claim. As for (e) observe that if either x and y are included but z isn't or if x and z are included but y isn't, then the number of failure sets is at least $2\binom{n_0+k-4}{n_0-1}$. Thus $f^{x,y}_{n_0+1}(U) - f^{x,y}_{n_0+1}(U^s_x) = \binom{n-3}{n_0-1} - \binom{n-3}{n_0+1} = \binom{n-3}{k-2} - \binom{n-3}{k-4} > 0$ and the proof is complete.



(a)

(b)



(c)

(d)

(e)

Figure 3.7

4 Necessary conditions for uniform optimality

Throughout this section the overriding assumption is that $4 \le k+1 \le n \le e$ so that by Corollary 4, $\lambda_c^{(k)}(G) \le e-(k-1)$ and we may write $U_k(G;\rho) = \sum_{i=\lambda_c^{(k)}(G)}^{e-(k-1)} f_i(G)\rho^i(1-\rho)^{e-i} + \sum_{i=e-(k-2)}^{e} \binom{e}{i}\rho^i(1-\rho)^{e-i}$. . Observe that $e - f_{e-k-1}$ is the number of subtrees of G with k nodes.

The theorem of this section describes conditions on the coefficients in the unreliability expression given above for determining when $U_k(G_1;\rho) < U_k(G_2;\rho)$ for all sufficiently small ρ and also when the inequality holds for all sufficiently large ρ.

Theorem 12 *Suppose G_1 and G_2 have n nodes and e edges where $4 \le k+1 \le n \le e$. Then*

1. *if $\lambda_c^{(k)}(G_1) > \lambda_c^{(k)}(G_2)$, then there exists $\rho_0 \in (0,1)$ such that $U_k(G_1;\rho) < U_k(G_2;\rho)$ for all $\rho \in (0,\rho_0)$;*
2. *if $\lambda_c^{(k)}(G_1) = \lambda_c^{(k)}(G_2)$, and i_0 is the smallest index such that $f_{i_0}(G_1) \ne f_{i_0}(G_2)$, then $f_{i_0}(G_1) < f_{i_0}(G_2)$ implies there exists $\rho_0 \in (0,1)$ such that $U_k(G_1;\rho) < U_k(G_2;\rho)$ for all $\rho \in (0,\rho_0)$;*
3. *if i_1 is the largest index, necessarily at most $e-(k-1)$, such that $f_{i_1}(G_1) \ne f_{i_1}(G_2)$, then $f_{i_1}(G_1) < f_{i_1}(G_2)$ implies there exists $\rho_1 \in (0,1)$ such that $U_k(G_1;\rho) < U_k(G_2;\rho)$ for all $\rho \in (\rho_1, 1)$.*

Proof:

1. Observe that

$$U_k(G_2;\rho) - U_k(G_1;\rho) =$$

$$f_{\lambda_c^{(k)}(G_2)}\rho^{\lambda_c^{(k)}(G_2)}(1-\rho)^{e-\lambda_c^{(k)}(G_2)}$$

$$+ \sum_{i=\lambda_c^{(k)}(G_2)+1}^{e} (f_i(G_2) - f_i(G_1))\rho^i(1-\rho)^{e-i}.$$

Set $p = \frac{\rho}{1-\rho}$ so that $\rho^j(1-\rho)^{e-j} = (1-\rho)^e p^j$ for all j. Thus

$$U_k(G_2;\rho) - U_k(G_1;\rho) =$$

$$(1-\rho)^e p^{\lambda_c^{(k)}(G_2)}[f_{\lambda_c^{(k)}(G_2)}(G_2)+$$

$$\sum_{i=\lambda_c^{(k)}(G_2)+1}^{e} (f_i(G_2) - f_i(G_1))p^{i-\lambda_c^{(k)}(G_2)}].$$

Hence there exists a p_0 such that if $p \in (0,p_0)$ then the quantity in the brackets is positive. But $\rho = \frac{p}{1+p_0}$ is an increasing function of p so there exists ρ_0 such that $\rho \in (0,\rho_0)$ implies $p \in (0,p_0)$ and the result follows.

2. In this case $U_k(G_2;\rho) - U_k(G_1;\rho) = (1-\rho)^e p^{i_0}[f_{i_0}(G_2) - f_{i_0}(G_1) + \sum_{i=i_0+1}^{e}(f_i(G_2) - f_i(G_1))p^{i-i_0}]$, and the result follows as in the previous argument.

3. Observe that $U_k(G_2;\rho) - U_k(G_1;\rho) = \sum_{i=0}^{i_1}(f_i(G_2) - f_i(G_1))\rho^i(1-\rho)^{e-i} = \rho^{i_1}(1-\rho)^{e-i_1}\sum_{i=0}^{i_1}[(f_i(G_2)-f_i(G_1))\rho^{i-i_1}(1-\rho)^{i_1-i}] = \rho^{i_1}(1-\rho)^{e-i_1}\sum_{i=0}^{i_1}[(f_i(G_2)-f_i(G_1))(\frac{1-\rho}{\rho})^{i-i_1}]$. Set $p = \frac{1-\rho}{\rho}$ so that $U_k(G_2;\rho) - U_k(G_1;\rho) = \rho^{i_1}(1-\rho)^{e-i_1}\sum_{i=0}^{i_1}[(f_i(G_2) - f_i(G_1))p^{i_1-i}]$. Now if $i_1 = 0$ then $f_0(G_2) = 1$ and $f_0(G_1) = 0$ and $U_k(G_2;\rho) - U_k(G_1;\rho) = \rho^{i_1}(1-\rho)^{e-i_1}((f_{i_1}(G_2) - f_{i_1}(G_1) + \sum_{i=0}^{i_1-1}[(f_i(G_2) - f_i(G_1))\rho)^{i-i_1}(1-\rho)^{i-i_1}])$. Thus there exists $p_1 \le 1$ such that if $p \in (0,p_1)$ then the quantity in the brackets is positive. As $\rho = \frac{1}{1+p}$ is decreasing on $(0,p_1]$ there exists ρ_1 such that $\rho \in (\rho_1,1)$ implies $p \in (0,p_1)$ and the result follows. ■

The following corollary yields necessary conditions for a graph to be UMR over all graphs in a given collection (necessarily having the same numbers of nodes and edges).

Corollary 13 *If C is a collection of graphs, all with the same number of nodes n and the same number of edges e, where $4 \le k+1 \le n \le e$, and G is k-UMR over C then*

1. *$\lambda_c^{(k)}(G)$ is maximum over C;*
2. *$f_{\lambda_c^{(k)}(G)}$ is minimum over all $H \in C$ having $\lambda_c^{(k)}(H) = \lambda_c^{(k)}(G)$;*
3. *G has the minimum value of $f_{e-(k-1)}$ over C (or equivalently, has the maximum number of subtrees of order k).*

Proof:

1. $\lambda_c^{(k)}(H) > \lambda_c^{(k)}(G)$ where $H \in C$, then, by Theorem 12(1), there exists $\rho_0 \in (0,1]$ such that $U_k(H;\rho) < U_k(G;\rho)$ for all $\rho \in (0,\rho_0)$, which contradicts G being k-UMR over C.
2. If $\lambda_c^{(k)}(H) = \lambda_c^{(k)}(G) = \lambda$ but $f_\lambda(H) < f_\lambda(G)$ then, by Theorem 12(2), there exists $\rho_0 \in$

$(0,1]$ such that $\rho \in (0,\rho_0)$ implies $U_k(H;\rho) < U_k(G;\rho)$, which contradicts G being k-UMR over C.

3. If $f_{e-(k-1)}(H) < f_{e-(k-1)}(G)$, then as $i_1 = e-(k-1)$, we obtain a contradiction to G being k-UMR by Theorem 12(3). ■

We apply this Corollary in our next section in showing that there exists a range of values of k for which no k-UMR graph on n nodes with $e = n$ exists.

5 On the Existence of k-UMR Unicycles for Large Values of k Relative to n

As we have shown in Sections 2 and 3, k-UMR unicycles exist whenever $3 \le k \le \frac{n}{2}$. In this section we note that the cycle C_n is n-UMR and prove that (n - 1)-UMR unicycles exist. The somewhat surprising result that if

$$\frac{2n_0 + 5 + \sqrt{8n_0^2 + 1}}{2} < k \le n-2,$$

then k-UMR unicycles *do not exist* is also established here.

To begin, the fact that C_n is the unique n-UMR unicycle was established in [10] as $\lambda_c^{(n)} = \lambda$, the line-connectivity. The case $k = n-1$ is a bit more complicated and is the subject of our next theorem.

Theorem 14 *If $e = n = k+1 \ge 5$, then*

1. *when k is even, the unicycle U_k consisting of $\frac{k}{2}$ pendant edges all attached to a single node of the cycle $C_{\frac{k}{2}+1}$ is the unique (n - 1)-UMR unicycle on n nodes (see Figure 5.1(a));*
2. *when k is odd, the unicycle U'_k consisting of $\frac{k+1}{2}$ pendant edges all incident on a single node of the cycle $C_{\frac{k+1}{2}}$ and the unicycle U''_k consisting of $\frac{k-1}{2}$ pendant edges all attached to a single node of the cycle $C_{\frac{k+3}{2}}$ are the only two (n - 1)-UMR unicycles on n nodes (see Figure 5.1(b)).*

Proof: Suppose that U is a unicycle on n nodes with cycle length ℓ and let ℓ_2 be the number of nodes on the cycle of degree equal to two. We claim that $\lambda_c^{(n-1)}(U) \le 2$. Indeed, if $\ell = 3$ then there exists a node on the cycle C_3 of degree at least three, so removal of the two edges of C_3 adjacent to such a node yields a failure state. If $\ell \ge 4$, then removal of two independent edges of C_ℓ yields a failure state. Next realize that $f_2(U) \ge \binom{\ell}{2} - \ell_2 + \binom{k+1-\ell}{2}$ since every pair of edges of C_ℓ except for those adjacent to a node on the cycle of degree two and every pair of edges not on the cycle are failure sets. Consider the parabolic function $f(x) = \binom{x}{2} - (x-1) + \binom{k+1-x}{2} = \frac{2x^2-(2k+4)x+k^2+k+2}{2} = x^2-(k-2)x+\frac{k^2+k+2}{2}$ (where the binomial coefficients have the obvious interpretation when x not an integer) which has a unique minimum at $x = \frac{k}{2}+1$. Since U_k is the only unicycle with $f_2(U) = f(\frac{k}{2}+1)$ when k is even, the result in (1) follows immediately. In the event that k is odd consider the problem of minimizing $f(x)$ when x is constrained to be an integer. Then the minimum value of $f(x)$ occurs only when $x = \frac{k+1}{2}$ or $x = \frac{k+3}{2}$. As $f_2(U'_k) = f(\frac{k+1}{2}) = f(\frac{k+3}{2}) = f_2(U''_k)$, (2) follows and the proof is complete. ■

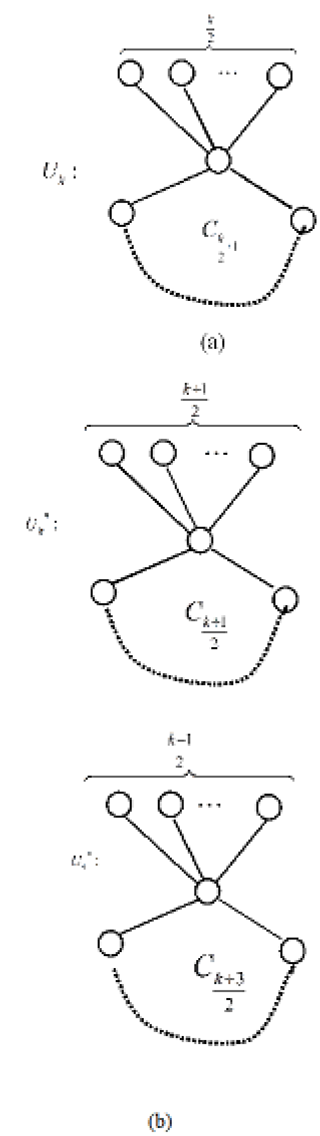


Figure 5.1

The final result of this section establishes the non-existence of k-UMR unicycles for a range of large values of k.

Theorem 15 *If $e = n = n_0 + k, n_0 \geq 2$ and $k \geq 4$, then a k-UMR unicycle does not exist when*

$$k > \frac{2n_0 + 5 + \sqrt{8n_0^2 + 1}}{2}.$$

Proof: First recall from Theorem 10 and Theorem 11 that $\lambda_c^{(k)}(U) \leq n_0$ for $U \neq U_s^x$ with the sole exception of $n = 6, k = 4$ and $n_0 = 2$, where $\lambda_c^{(k)}(U_6) = 3 = n_0+1$. As $\lambda_c^{(k)}(U_s^x) = n_0+1$, it follows that $\lambda_c^{(k)}(U) < \lambda_c^{(k)}(U_s^x)$ whenever $n \geq 4, k \geq 4$ and $n_0 \geq 2$. Hence if a k-UMR unicycle exists it must be U_s^x, by Corollary 13(1), and in this case, $f_{n_0+1}(U) \geq f_{n_0+1}(U_s^x)$ for all unicycles U on n nodes, by Corollary 13(3). But we shall see that if k satisfies the condition of the theorem the unicycle U^4, having $n - 4$ pendant edges all incident on a single node of C_4 (see Figure 5.2) has a smaller value of f_{n_0+1} than U_s^x, thereby proving that a k-UMR unicycle doesn't exist. Indeed

$$f_{n_0+1}(U_s^x) = \binom{n_0 + k - 4}{n_0 + 1} + 3\binom{n_0 + k - 4}{n_0 - 1}$$
$$+2\binom{n_0 + k - 4}{n_0 - 2}$$

so that

$$f_{n_0+1}(U_s^x) - f_{n_0+1}(U^4) =$$
$$\frac{1}{n_0!}[(n_0 + k - 4)(n_0 + k - 5)\cdots(k - 1)]$$
$$[(k - 2)(k - 3) + n_0(n_0 - 2k + 3) - 2n_0(n_0 + 1)]$$
$$\frac{(n_0 + k - 4)!}{n_0!(k - 2)!}(k^2 - (2n_0 + 5)k + 6 + 5n_0 - {n_0}^2).$$

Now the expression in the second set of brackets is positive if

$$k > \frac{2n_0 + 5 + \sqrt{8n_0^2 + 1}}{2}.$$

Of course if $n_0 \geq 2$ then the condition requires that $k > \frac{9+\sqrt{33}}{2} > 7$ which is consistent with the proviso that $n \geq 7$. This concludes the proof. ■

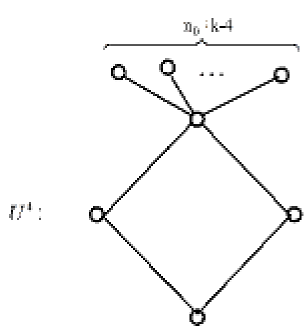


Figure 5.2

Corollary 16 follows immediately from Theorem 15 after direct substitution of numerical values.

Corollary 16 *If $n_0 = 2$, then when $k \geq 8$, no k-UMR unicycle exists.*

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