A new type of sequence space of non-absolute type and matrix transformation

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Abstract: In this paper, we introduce the space \( r^q(\triangle u^p) \), where

\[
r^q(\triangle u^p) = \{ x = (x_k) \in \omega : (\triangle x_k) \in r^q(u, p) \};
\]

where \( r^q(u, p) \) has recently been introduced and studied by Neyaz and Hamid (Acta Math. Acad. Paeda. Nyreg., 28, 2012, pp. 47-58). We show its completeness property, prove that the space \( r^q(\triangle u^p) \) and \( l(p) \) are linearly isomorphic and compute \( \alpha, \beta \) and \( \gamma \)-duals of \( r^q(\triangle u^p) \). Moreover, we construct the basis of \( r^q(\triangle u^p) \). Finally, we characterize some matrix class.

Key-Words: Sequence space of non-absolute type; paranormed sequence space; \( \alpha, \beta \) and \( \gamma \)-duals; matrix transformations.

1 Introduction

We denote the set of all sequences with complex terms by \( \omega \). It is a routine verification that \( \omega \) is a linear space with respect to the co-ordinatewise addition and scalar multiplication of sequences which are defined, as usual, by

\[
x + y = (x_k) + (y_k) = (x_k + y_k)
\]

and

\[
\alpha x = \alpha (x_k) = (\alpha x_k),
\]

respectively; where \( x = (x_k), y = (y_k) \in \omega \) and \( \alpha \in \mathbb{C} \). By sequence space we understand a linear subspace of \( \omega \) i.e. the sequence space is the set of scalar sequences (real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper \( N, R \) and \( C \) denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let \( l_\infty, c \) and \( c_0 \), respectively, denotes the space of all bounded sequences, the space of all convergent sequences and the sequences converging to zero. Also, by \( l_1, l(p), cs \) and \( bs \) we denote the spaces of all absolutely convergent, \( p \)-absolutely convergent, convergent and bounded series, respectively.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [1] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices. The theory of matrix transformations is a wide field in summability; it deals with the characterizations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

Let \( X, Y \) be two sequence spaces and let \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers \( a_{nk} \), where \( n, k \in N \). Then, the matrix \( A \) defines the \( A \)-transformation from \( X \) into \( Y \), if for every sequence \( x = (x_k) \in X \) the sequence \( Ax = \{ (Ax)_n \} \), the \( A \)-transform of \( x \) exists and is in \( Y \); where \( (Ax)_n = \sum_k a_{nk} x_k \). For simplicity in notation, here and in what follows, the summation without limits runs from 0 to \( \infty \). By \( A \in (X : Y) \) we mean the characterizations of matrices from \( X \) to \( Y \) i.e., \( A : X \rightarrow Y \). A sequence \( x \) is said to be \( A \)-summable to \( l \) if \( Ax \) converges to \( l \) which is called as the \( A \)-limit of \( x \).

We denote by \( (A) \) the set of all sequences which are summable \( A \). The set \( (A) \) is called summability field of the matrix \( A \). Thus, if \( Ax = \{ (Ax)_n \} \), then \( (A) = \{ x : Ax \in c \} \), where \( c \) is the set of convergent sequences. For example, \( (I) = c \).

For a sequence space \( X \), the matrix domain \( X_A \)
of an infinite matrix $A$ is defined as

$$X_A = \{ x = (x_k) : Ax \in X \}. \quad (1)$$

A infinite matrix $A = (a_{nk})$ is said to be regular [2, 3] if and only if the following conditions hold:

(i) $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1$,

(ii) $\lim_{n \to \infty} a_{nk} = 0$, $(k = 0, 1, 2, \ldots)$,

(iii) $\sum_{k=0}^{\infty} |a_{nk}| < M$, $(M > 0, n = 0, 1, 2, \ldots)$.

Let $(q_k)$ be a sequence of positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in N$. Then the matrix $R^q = (r_{nk}^q)$ of the Riesz mean $(R, q_n)$ is given by

$$r_{nk}^q = \left\{ \begin{array}{cl} \frac{q_k}{Q_n}, & 0 \leq k \leq n, \\ 0, & k > n \end{array} \right.$$  

The Riesz mean $(R, q_n)$ is regular if and only if $Q_n \to \infty$ as $n \to \infty$ [3].

Kizmaz [4] defined the difference sequence spaces $Z(\Delta)$ as follows

$$Z(\Delta) = \{ x = (x_k) \in \omega : (\Delta x_k) \in Z \}$$

where, $Z \in \{l_\infty, c, c_0\}$ and $\Delta x_k = x_k - x_{k+1}$.

Başar and Altay [5] has studied the sequence space as

$$bv_p = \{ x = (x_k) \in \omega : \sum_{k} |x_k - x_{k-1}|^p < \infty \},$$

where $1 \leq p < \infty$. With the notation of (1), the space $bv_p$ can be redefined as

$$bv_p = (l_p)_\Delta, 1 \leq p < \infty$$

where, $\Delta$ denotes the matrix $\Delta = (\Delta_{nk})$ defined as

$$\Delta_{nk} = \left\{ \begin{array}{cl} (-1)^{n-k}, & if \ n - 1 \leq k \leq n, \\ 0, & if \ k < n - 1 \ or \ k > n \end{array} \right.$$  

This space was further studied by Başar, Altay and Mursaleen [6] and have introduced $bv(u, p)$ and $bv_\infty(u, p)$ which are defined as follows:

$$bv(u, p) = \{ x = (x_k) \in \omega : \sum_{k} |u_k \Delta x_k|^{p_k} < \infty \},$$

where $0 \leq p_k < \infty$ and

$$bv_\infty(u, p) = \{ x = (x_k) \in \omega : \sup_{k} |u_k \Delta x_k|^{p_k} < \infty \}.$$

With the notation of (1), the space $bv(u, p)$ and $bv_\infty(u, p)$ can be redefined as

$$bv(u, p) = [l(p)]_{\Delta^{\infty}} \ \text{and} \ \bv_\infty(u, p) = [l_\infty(p)]_{\Delta^{\infty}}$$

where, $\Delta^{\infty}$ denotes the matrix $\Delta = (\Delta_{nk}^{\infty})$ defined as:

$$\Delta_{nk}^{\infty} = \left\{ \begin{array}{cl} (-1)^{n-k}u_k, & if \ n - 1 \leq k \leq n, \\ 0, & if \ k < n - 1 \ or \ k > n. \end{array} \right.$$  

for all $n, k \in N$.

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors [5-30]. They introduced the sequence spaces

$$l_p = (l_p)^F, (l_\infty)_{c0}, (l_\infty)_{c0} \ [7], \ \text{and} \ \{c, c_0\} \ [8],$$

$$c_{00} = (c_0)^F, (c_\infty)_{c0} = (c_\infty)^F \ [9], \ \text{and} \ \{c, c_0\} \ [10],$$

$$r_p = (r_p)^F, (r_\infty)_{c0} = (r_\infty)^F \ [11], \ \text{and} \ \{c, c_0\} \ [12].$$

The Riesz Sequence space $r^q(\Delta^p_u)$ of non-absolute type

In this section, we define the Riesz sequence space $r^q(\Delta^p_u)$, prove that the space $r^q(\Delta^p_u)$ is complete paranormed linear space and it is shown to be linearly isomorphic to the space $l(p)$.

A linear Topological space $X$ over the field of real numbers $R$ is said to be a paranormed space if there is a sub-additive function $h : X \to R$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|ax| - |a| \to 0 \ and \ h(x-y) \to 0 \ imply \ h(ax, x-y) \to 0 \ for \ all \ a' \ in \ R \ and \ x \ in \ X$, where $\theta$ is a zero vector in the linear space $X$. Assume here and after that $(p_k)$ be
a bounded sequence of strictly positive real numbers with \( \sup_k p_k = H \) and \( M = \max \{1, H\} \). Then, the linear spaces \( l(p) \) and \( l_\infty(p) \) were defined by Maddox [2] (see also, [25,26]) as follows:

\[
l(p) = \{ x = (x_k) : \sum_k |x_k|^{p_k} < \infty \}
\]

and

\[
l_\infty(p) = \{ x = (x_k) : \sup_k |x_k|^{p_k} < \infty \}
\]

which are complete spaces paranormed by

\[
h_1(x) = \left[ \sum_k |x_k|^{p_k} \right]^{1/M}
\]

and

\[
h_2(x) = \sup_k |x_k|^{p_k/M}
\]

iff \( \inf p_k > 0 \).

We shall assume throughout the text that \( p_k^{-1} + \{p'_k\}^{-1} = 1 \) provided \( 1 < \inf p_k \leq H < \infty \) and we denote the collection of all finite subsets of \( N \) by \( F \), where \( N = \{0, 1, 2, \ldots\} \).

Neyaz and Hamid [18] have recently introduced \( r^q(u, p) \) which is defined as:

\[
r^q(u, p) = \left\{ x = (x_k) \in \omega : \sum_k \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k q_j x_j \right\}
\]

where, \( 0 < p_k \leq H < \infty \).

With the notation of (1) we redefine \( r^q(u, p) \) as:

\[
r^q(u, p) = \{ l(p) \} R^q_u.
\]

Following Başar and Altay [5], Mursaleen et al [17, 23], Hamid et al [18, 23, 24], Basarir [27], Choudhary and Mishra [28], Gross Erdmann [30], Tripathy [31], we define the Reisz sequence space \( r^q(\triangle^p_u) \) as the set of all sequences such that \( R^q \triangle \)-transform of it is in the space \( l(p) \), that is,

\[
r^q(\triangle^p_u) = \left\{ x = (x_k) \in \omega : \sum_k \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k q_j \triangle x_j \right\}
\]

where, \( 0 < p_k \leq H < \infty \).

\section*{Remark 1}

If we take \( (u_k) = e = (1, 1, \ldots) \) in \( r^q(\triangle^p_u) \), we get the results obtained in [27].

With the notation of (1) we redefine \( r^q(\triangle^p_u) \) as:

\[
r^q(\triangle^p_u) = \{ l(p) \} _R^q_u.
\]

Define the sequence \( y = (y_k) \), which will be used, by the \( R^q \triangle \)-transform of a sequence \( x = (x_k) \), i.e.,

\[
y_k = \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k q_j \triangle x_j.
\]

Now, we begin with the following theorem which is essential in the text.

\section*{Theorem 2}

\( r^q(\triangle^p_u) \) is a complete linear metric space paranormed by \( h_\triangle \), defined as:

\[
h_\triangle(x) = \left[ \sum_k \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k (q_j - q_{j+1}) x_j + \frac{q_k u_k x_k}{Q_k} \right]^{1 \over 2}
\]

with \( 0 < p_k \leq H < \infty \).

\section*{Proof:}

The linearity of \( r^q(\triangle^p_u) \) with respect to the co-ordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for \( z, x \in r^q(\triangle^p_u) \) [2]

\[
\left[ \sum_k \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k (q_j - q_{j+1}) (x_j + z_j) + \frac{q_k u_k x_k}{Q_k} \right]^{1 \over 2} \\
\leq \left[ \sum_k \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k (q_j - q_{j+1}) x_j + \frac{q_k u_k x_k}{Q_k} \right]^{1 \over 2} \\
+ \left[ \sum_k \frac{1}{Q_k} \sum_{j=0}^{k-1} u_k (q_j - q_{j+1}) z_j + \frac{q_k u_k z_k}{Q_k} \right]^{1 \over 2}
\]

and for any \( \alpha \in \mathbb{R} \) [32]

\[
|\alpha|^{p_k} \leq \max(1, |\alpha|^M).
\]

It is clear that, \( h_\triangle(\theta) = 0 \) and \( h_\triangle(x) = h_\triangle(-x) \) for all \( x \in r^q(\triangle^p_u) \). Again the inequality (3) and (4), yield the subadditivity of \( h_\triangle \) and

\[
h_\triangle(\alpha x) \leq \max(1, |\alpha|) h_\triangle(x).
\]

Let \( \{x^n\} \) be any sequence of points of the space \( r^q(\triangle^p_u) \) such that \( h_\triangle(x^n - x) \to 0 \) and \( (\alpha_n) \) is a sequence of scalars such that \( \alpha_n \to \alpha \). Then, since the inequality,

\[
h_\triangle(x^n) \leq h_\triangle(x) + h_\triangle(x^n - x)
\]
holds by subadditivity of $h_\triangle$, \( \{h_\triangle(x^n)\} \) is bounded and we thus have
\[
h_\triangle(\alpha_n x^n - \alpha x) = \\
\left[ \sum_k \frac{1}{Q_k} \left| \sum_{j=0}^{k} u_k(q_j - q_{j+1}) (\alpha_n x_j^n - \alpha x_j) \right|^p \right]^{\frac{1}{p}} \\
\leq |\alpha_n - \alpha| \frac{1}{M} h_\triangle(x^n) + |\alpha| \frac{1}{M} h_\triangle(x^n - x)
\]
which tends to zero as \( n \to \infty \). That is to say that the scalar multiplication is continuous. Hence, \( h_\triangle \) is paranorm on the space \( r^q(\Delta_p^\infty) \).

It remains to prove the completeness of the space \( r^q(\Delta_p^\infty) \). Let \( \{x^i\} \) be any Cauchy sequence in the space \( r^q(\Delta_p^\infty) \), where \( x^i = \{x_0^i, x_1^i, \ldots\} \), then for a given \( \epsilon > 0 \) there exists a positive integer \( n_0(\epsilon) \) such that
\[
h_\triangle(x^i - x^j) < \epsilon \quad \text{(5)}
\]
for all \( i, j \geq n_0(\epsilon) \). Using definition of \( h_\triangle \) and for each fixed \( k \in N \) that
\[
\left| (R^q \Delta x^i)_k - (R^q \Delta x^j)_k \right| \\
\leq \sum_k \left| (R^q \Delta x^i)_k - (R^q \Delta x^j)_k \right|^p \frac{1}{p} \\
< \epsilon
\]
for \( i, j \geq n_0(\epsilon) \), which leads us to the fact that \( \{(R^q \Delta x^0)_k, (R^q \Delta x^1)_k, \ldots\} \) is a Cauchy sequence of real numbers for every fixed \( k \in N \). Since \( R \) is complete, it converges, say, \( (R^q \Delta x^i)_k \to ((R^q \Delta x)_k \) as \( i \to \infty \). Using these infinitely many limits \( (R^q \Delta x^0)_k, (R^q \Delta x^1)_k, \ldots \), we define the sequence \( \{(R^q \Delta x)_0, (R^q \Delta x)_1, \ldots\} \). From (5) for each \( m \in N \) and \( i, j \geq n_0(\epsilon) \),
\[
\sum_{k=0}^{m} \left| (R^q \Delta x^i)_k - (R^q \Delta x^j)_k \right|^p \\
\leq h_\triangle(x^i - x^j)^M < \epsilon^M \quad \text{(6)}
\]
Take any \( i, j \geq n_0(\epsilon) \). First, let \( j \to \infty \) in (6) and then \( m \to \infty \), we obtain
\[
h_\triangle(x^i - x) \leq \epsilon.
\]
Finally, taking \( \epsilon = 1 \) in (6) and letting \( i \geq n_0(1) \), we have by Minkowski’s inequality for each \( m \in N \) that
\[
\left[ \sum_{k=0}^{m} \left| (R^q x)_k \right|^p \right]^{\frac{1}{p}} \\
\leq h_\triangle(x^i - x) + h_\triangle(x^i) \leq 1 + h_\triangle(x^i)
\]
which implies that \( x \in r^q(\Delta_p^\infty) \). Since \( h_\triangle(x-x^i) \leq \epsilon \)
for all \( i \geq n_0(\epsilon) \), it follows that \( x^i \to x \) as \( i \to \infty \), hence we have shown that \( r^q(\Delta_p^\infty) \) is complete, hence the proof. \( \square \)

Note that one can easily see the absolute property does not hold on the spaces \( r^q(\Delta_p^\infty) \), that is \( h_\triangle(x) \not= h_\triangle(|x|) \) for atleast one sequence in the space \( r^q(\Delta_p^\infty) \) and this says that \( r^q(\Delta_p^\infty) \) is a sequence space of non-
absolute type.

**Theorem 3** The Riesz sequence space \( r^q(\Delta_p^\infty) \) of non-
absolute type is linearly isomorphic to the space \( l(p) \), where \( 0 < p_k \leq H < \infty \).

**Proof:** To prove the theorem, we will show the existence of a linear bijection between the spaces \( r^q(\Delta_p^\infty) \) and \( l(p) \), where \( 0 < p_k \leq H < \infty \). With the not-
tion of (3), define the transformation \( T \) from \( r^q(\Delta_p^\infty) \) to \( l(p) \) by \( x \to y = Tx \). The linearity of \( T \) is trivial. Further, it is obvious that \( x = \theta \) whenever \( Tx = \theta \) and hence \( T \) is injective.

Let \( y \in l(p) \) and define the sequence \( x = (x_k) \) by
\[
x_k = \sum_{n=0}^{k-1} \left( \frac{1}{q_n} - \frac{1}{q_{n+1}} \right) u_k^{-1} Q_k y_k + u_k^{-1} Q_k y_k,
\]
for \( k \in N \). Then,
\[
h_\triangle(x) = \left[ \sum_k \left| \sum_{j=0}^{k-1} u_k(q_j - q_{j+1}) x_j + \frac{q_k u_k}{Q_k} x_k \right|^p \right]^{\frac{1}{p}} \\
= \left[ \sum_k \left| \sum_{j=0}^{k} \delta_{kj} y_j \right|^p \right]^{\frac{1}{p}} \\
= \left[ \sum_k |y_k|^p \right]^{\frac{1}{p}} \\
= h_1(y) < \infty,
\]
where,
\[
\delta_{kj} = \begin{cases} 
1, & \text{if } k = j, \\
0, & \text{if } k \not= j.
\end{cases}
\]

Thus, we have \( x \in r^q(\Delta_p^\infty) \). Consequently, \( T \) is surjective and is paranorm preserving. Hence, \( T \) is a linear bijection and this proves that the spaces \( r^q(\Delta_p^\infty) \) and \( l(p) \) are linearly isomorphic, hence the proof.
3 Basis and $\alpha$, $\beta$- and $\gamma$-duals of the space $r^q(\Delta_u^p)$

In this section, we compute $\alpha$, $\beta$, and $\gamma$-duals of the space $r^q(\Delta_u^p)$ and finally in this section we give the basis for the space $r^q(\Delta_u^p)$.

For the sequence space $X$ and $Y$, define the set

$$S(X : Y) = \{z = (z_k) : xz = (x_kz_k) \in Y\}.$$  \hfill (7)

With the notation of (7), the $\alpha$, $\beta$, and $\gamma$-duals of a sequence space $X$, which are respectively denoted by $X^\alpha$, $X^\beta$ and $X^\gamma$ and are defined by

$$X^\alpha = S(X : l_1), \quad X^\beta = S(X : cs) \text{ and } X^\gamma = S(X : bs).$$

If a sequence space $X$ paranormed by $h$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence of scalars $(\alpha_n)$ such that

$$\lim_{n \to \infty} h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0,$$

then $(b_n)$ is called a Schauder basis (or briefly basis) for $X$. The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$ and written as $x = \sum \alpha_k b_k$.

First we state some lemmas which are needed in proving our theorems.

**Lemma 4** \cite{33}

(i) Let $1 < p_k \leq H < \infty$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{K \in F} \left| \sum_{n \in K} a_{nk} b_k \right| < \infty.$$  \hfill (8)

(ii) Let $0 < p_k \leq 1$. Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{K \in F} \left| \sum_{n \in K} a_{nk} b_k \right| < \infty.$$  \hfill (9)

**Lemma 5** \cite{33}

(i) Let $1 < p_k \leq H < \infty$. Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{n} \left| \sum_{k=0}^{n-1} a_{nk} b_k \right| < \infty.$$  \hfill (10)

(ii) Let $0 < p_k \leq 1$ for every $k \in N$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} \left| \sum_{i=k+1}^{n} a_{nk} b_k \right| < \infty.$$  \hfill (11)

**Lemma 6** \cite{33} Let $0 < p_k \leq H < \infty$ for every $k \in N$. Then $A \in (l(p) : cs)$ if and only if

$$\lim_{n \to \infty} a_{nk} = 0 \text{ for } k \in N.$$  \hfill (12)

**Theorem 7** Let $1 < p_k \leq H < \infty$ for every $k \in N$. Define the sets $D_1(u, p)$ and $D_2(u, p)$ as follows

$$D_1(u, p) = \bigcup_{B > 1} \{a = (a_k) \in \omega :$$

$$\sup_{K \in F} \left| \sum_{n \in K} x_n \right| \leq \infty \}$$

and

$$D_2(u, p) = \bigcup_{B > 1} \{a = (a_k) \in \omega :$$

$$\sup_{K \in F} \left| \sum_{n \in K} x_n \right| \leq \infty \}$$

Then,

$$[r^q(\Delta_u^p)]^\alpha = D_1(u, p)$$

and

$$[r^q(\Delta_u^p)]^\beta = [r^q(\Delta_u^p)]^\gamma = D_2(u, p) \cap cs.$$}

**Proof:** Let us take any $a = (a_k) \in \omega$. We can easily derive with (2) that

$$a_{nk} x_n = n \sum_{k=0}^{n-1} \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) u_{k}^{-1} a_{n} Q_{k} y_{k} + a_{nk} u_{k}^{-1} Q_{n} y_{n}$$

$$= (Cy)_n$$  \hfill (11)

where, $C = (c_{nk})$ is defined as

$$c_{nk} = \begin{cases} \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) u_{k}^{-1} a_{n} Q_{k}, & \text{if } 0 \leq k \leq n - 1, \\ 0, & \text{if } k > n, \end{cases}$$

for all $n, k \in N$. Thus we observe by combining (11) with (i) of Lemma 4 that $a x = (a_{nk} x_n) \in l_1$ whenever $x = (x_n) \in r^q(\Delta_u^p)$ if and only if $C y \in l_1$ whenever $y \in l(p)$. This shows that $[r^q(\Delta_u^p)]^\alpha = D_1(u, p)$. 

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Further, consider the equation,
\[
\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \frac{a_k}{q_k} + \frac{1}{q_k} - \frac{1}{q_{k+1}} \right] \sum_{i=k+1}^{n} a_i u^{-1}_k Q_k \cdot y_k = (Dy)_n
\]
where, \( D = (d_{nk}) \) is defined as
\[
d_{nk} = \begin{cases} \left( \frac{a_k}{q_k} + \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i u^{-1}_k Q_k, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}
\]
Thus we deduce from Lemma 6 with (12) that \( ax \in \omega \cap \text{cs} \) whenever \( x = (x_n) \in r^q(\Delta^p_u) \) if and only if \( Dy \in \omega \cap \text{cs} \). Therefore, we derive from (8) that
\[
\sum_{k} \left| \left( \frac{a_k}{q_k} + \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i u^{-1}_k Q_k \cdot B^{-1} \right| \leq \infty,
\]
and \( \lim d_{nk} \) exists and hence shows that \( [r^q(\Delta^p_u)]^\alpha = D_2(u,p) \cap \text{cs} \). As proved above, from Lemma 5 together with (12) that \( ax = (a_k x_k) \in b_s \) whenever \( x = (x_n) \in r^q(\Delta^p_u) \) if and only if \( Dy \in l_\infty \) whenever \( y = (y_k) \in l(p) \). Therefore, we again obtain the condition (13) which means that \( [r^q(\Delta^p_u)]^\gamma = D_2(u,p) \cap \text{cs} \) and the proof of the theorem is complete. \( \square \)

**Theorem 8** Let \( 0 < p_k < 1 \) for every \( k \in N \). Define the sets \( D_3(u,p) \) and \( D_4(u,p) \) as follows
\[
D_3(u,p) = \{ a = (a_k) \in \omega : \sup_{k \in F} \left| \sum_{n \in K} \left[ \frac{1}{q_k} - \frac{1}{q_{k+1}} \right] u^{-1}_n Q_k \right| < \infty \}
\]
and
\[
D_4(u,p) = \{ a = (a_k) \in \omega : \sup_{k} \left| \left( \frac{a_k}{q_k} + \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i u^{-1}_k Q_k \cdot B^{-1} \right| \leq \infty \}
\]
Then, \( [r^q(\Delta^p_u)]^\alpha = D_3(u,p) \) and
\[
[r^q(\Delta^p_u)]^\beta = [r^q(\Delta^p_u)]^\gamma = D_4(u,p) \cap \text{cs}.
\]

**Proof:** The proof follows easily from Theorem 7 (above) by using second parts of Lemmas 4, 5 and 6 instead of the first parts. \( \square \)

**Theorem 9** Define the sequence \( b^{(k)}(q) = \{ b^{(k)}_n(q) \} \) of the elements of the space \( r^q(\Delta^p_u) \) for every fixed \( k \in N \) by
\[
b^{(k)}_n(q) = \begin{cases} \left( \frac{1}{q_n} - \frac{1}{q_{n+1}} \right) u^{-1}_k Q_n + u^{-1}_k Q_k, & \text{if } 0 \leq n \leq k - 1, \\ 0, & \text{if } n > k - 1. \end{cases}
\]
Then, the sequence \( \{ b^{(k)}_n(q) \} \) is a basis for the space \( r^q(\Delta^p_u) \) and any \( x \in r^q(\Delta^p_u) \) has a unique representation of
\[
x = \sum_{k} \lambda_k(q) b^{(k)}(q)
\]
where, \( \lambda_k(q) = (R^q \Delta x)_k \) for all \( k \in N \) and \( 0 < p_k \leq H < \infty \).

**Proof:** It is clear that \( b^{(k)}(q) \subset r^q(\Delta^p_u) \), since
\[
R^q \Delta b^{(k)}(q) = e^{(k)} \in l(p) \text{ for } k \in N
\]
and \( 0 < p_k \leq H < \infty \), where \( e^{(k)} \) is the sequence whose only non-zero term is 1 in \( k^{\text{th}} \) place for each \( k \in N \).

Let \( x \in r^q(\Delta^p_u) \) be given. For every non-negative integer \( m \), we put
\[
x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q).
\]
Then, we obtain by applying \( R^q \Delta \) to (16) with (15) that
\[
R^q \Delta x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) R^q \Delta b^{(k)}(q)
\]
and
\[
\left( R^q \Delta \left( x - x^{[m]} \right) \right)_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m, \\ (R^q \Delta x)_i, & \text{if } i > m \end{cases}
\]
where \( i, m \in N \). Given \( \varepsilon > 0 \), there exists an integer \( m_0 \) such that
\[
\left( \sum_{i=m}^{\infty} |(R^q \Delta x)_i|^{p_k} \right)^{\frac{1}{p_k}} < \frac{\varepsilon}{2},
\]
for all $m \geq m_0$. Hence,
\[
\begin{align*}
    h_{\triangle}(x - x^{[m]}) = & \left(\frac{\sum_{i=m}^{\infty} |(R^q \triangle x)_i|^{p_k}}{\sum_{i=m_0}^{\infty} |(R^q \triangle x)_i|^{p_k}}\right)^{\frac{1}{p_k}} \\
    \leq & \left(\frac{\sum_{i=m_0}^{\infty} |(R^q \triangle x)_i|^{p_k}}{\sum_{i=m}^{\infty} |(R^q \triangle x)_i|^{p_k}}\right)^{\frac{1}{p_k}} \\
    \leq & \frac{\varepsilon}{2} < \varepsilon
\end{align*}
\]
for all $m \geq m_0$, which proves that $x \in r^q(\Delta^p_u)$ is represented as (14).

Let us show the uniqueness of the representation for $x \in r^q(\Delta^p_u)$ given by (13). Suppose, on the contrary; that there exists a representation $x = \sum_k \mu_k(q)b^k(q)$. Since the linear transformation $T$ from $r^q(\Delta^p_u)$ to $l(p)$ used in the Theorem 3 is continuous we have
\[
(R^q \triangle x)_n = \sum_k \mu_k(q) \left(R^q \triangle b^k(q)\right)_n = \sum_k \mu_k(q)c_n^{(k)} = \mu_n(q)
\]
for $n \in \mathbb{N}$, which contradicts the fact that $(R^q \triangle x)_n = \lambda_n(q)$ for all $n \in \mathbb{N}$. Hence, the representation (14) is unique. This completes the proof. \hfill \Box

4 Matrix Mappings on the Space $r^q(\Delta^p_u)$

In this section, we characterize the matrix mappings from the space $r^q(\Delta^p_u)$ to the space $l_{\infty}$.

**Theorem 10** (i) : Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(\Delta^p_u) : l_{\infty})$ if and only if there exists an integer $B > 1$ such that
\[
C(B) = \sup_n \left| \left[ \frac{a_{nk}}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_{ni} \right] u_k^{-1}B^{-1}Q_k \right|^{p_k'} < \infty
\]
and $\{a_{nk}\}_{k \in \mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

(ii) : Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (r^q(\Delta^p_u) : l_{\infty})$ if and only if
\[
\sup_n \left| \left[ \frac{a_{nk}}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_{ni} \right] u_k^{-1}Q_k \right|^{p_k} < \infty,
\]
and $\{a_{nk}\}_{k \in \mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

**Proof**: We only prove the part (i) and (ii) follows in a similar fashion. So, let $A \in (r^q(\Delta^p_u) : l_{\infty})$ and $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $Ax$ exists for $x \in r^q(\Delta^p_u)$ and implies that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{r^q(\Delta^p_u)\}^{p_k}$ for each $n \in \mathbb{N}$. Hence necessity of (17) holds.

Conversely, suppose that the necessities (17) hold and $x \in r^q(\Delta^p_u)$, since $\{a_{nk}\}_{k \in \mathbb{N}} \in \{r^q(\Delta^p_u)\}^{p_k}$ for every fixed $n \in \mathbb{N}$, so the $A$-transform of $x$ exists. Consider the following equality obtained by using the relation (11) that
\[
\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m} \left[ \frac{a_{nk}}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{m} a_{ni} \right] u_k^{-1}Q_k y_k.
\]
(19)

Taking into account the assumptions we derive from (19) as $m \to \infty$ that
\[
\sum_{k} a_{nk}x_k = \sum_{k} \left[ \frac{a_{nk}}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{\infty} a_{ni} \right] u_k^{-1}Q_k y_k
\]
(20)

Now, by combining (20) and the inequality which holds for any $B > 0$ and any complex numbers $a$, $b$
\[
|ab| \leq B \left( |aB^{-1}|^{p_k'} + |b|^p \right)
\]
with $p^{-1} + p'^{-1} = 1$ (see [10]), one can easily see that
\[
\sup_{n \in \mathbb{N}} \left| \sum_{k} a_{nk}x_k \right| \leq \sup_{n \in \mathbb{N}} \sum_{k} \left[ \frac{a_{nk}}{q_k} + \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{\infty} a_{ni} \right] u_k^{-1}Q_k \left| y_k \right|
\]
\[
\leq B \left( C(B) + h_{\triangle}^{B+1}(y) \right) < \infty.
\]
This shows that $Ax \in l_{\infty}$ whenever $x \in r^q(\Delta^p_u)$. This completes the proof. \hfill \Box

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