Non-collocated Feedback Stabilization of a Kind of System Described by Wave Equations

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Abstract: In this paper, we study the stabilization problem of a three-edge network system described by variable coefficients wave equations. With the root node fixed and a tip mass attached on the common vertex, we design two non-collocated controllers. Then we show that the closed-loop system is well-posed and satisfies spectrum-determined growth condition while the feedback gain constants fulfill some requirements. Moreover, we prove that the system is exponentially stable by applying Riesz basis method and utilizing some tricks of inequalities.

Key–Words: wave equation, variable coefficient, non-collocated control, exponential stabilization

1 Introduction

A collocated system means that the actuators and sensors are placed at the same location. This is the preferred method of sensors and actuators placement because, for collocated measurement, the transfer function is passive and hence it is easy to stabilize the system. The stabilization of such systems, especially one dimensional multi-link flexible systems with collocated boundary controls, has been the object of intensive research in past decades. See for instance [5, 13, 24, 1, 15, 11, 26, 23] and the references therein. However, in many real life mechanical systems, collocation is simply not possible and this presents some unique problems for system control [6, 17].

In recent years, researchers pay their attention on the stabilization of non-collocated system gradually. But as the zeros in such a system are much more sensitive to perturbations in the system parameters and boundary conditions, small increment of feedback controller gains can result in the closed-loop instability [18, 22]. The controller design for the stabilization of non-collocated systems is much harder than that of the collocated ones. Several articles considered non-collocated control for specific systems by using simulation and experiments [3, 21, 14, 20]. Theoretical work about such controllers has been done quite few. The first effort was made in [8] where an observer based compensator for a string system with a non-collocated actuator/sensor configuration was constructed. The authors proved that the observer is exponentially convergent and the closed-loop system is indeed exponentially stable. This work was then applied to the Euler-Bernoulli beam equation [9] and was generalized to the two-collected strings system [10]. Recently, observing that the complementary system used to stabilize the closed-loop system is usually more complex than the original one, [4] designed a non-collocated feedback controller to stabilize the system by using the technique of spectral analysis and Riesz basis method. For the multi-link flexible structures, [12] discussed the spectrum and the dynamical behavior of a star-shaped network of non-uniform strings with non-collocated feedbacks. However, the stabilization of the closed-loop system was not addressed.

The aim of this paper is to obtain the exponential stabilization of a three-edge network of non-uniform strings system with a boundary vertex (called the root node) fixed and a tip mass attached on the common vertex. Two suitable non-collocated feedback controllers are designed to get the well-posedness of the closed-loop system with the perturbation theory of C_0 semigroup. Moreover, the spectrum distribution and the exponential stabilization of the closed-loop system are obtained by using the technique of asymptotic spectral analysis, Riesz basis method and some tricks of inequalities.

Let us begin with some notations. Let $G = (V, E)$ be a simply connected graph, where $V = \{a_0, a_1, a_2, a_3\}$ is the set of the vertices, and $E = \{e_1, e_2, e_3\}$ is the set of the edges. a_0 is the common vertex of the graph G, and a_1, a_2 and a_3, each of them receiving only one edge, are called boundary nodes of the graph G. Let one of the boundary nodes, say a_1, be fixed and the others be free. Suppose that each of the edges $e_i(i=1,2,3)$ has finite arc length ℓ_i, which can be parameterized by its arc length by means of the function π_i defined by

$$\pi_i : [0,\ell_i] \longrightarrow e_i, i=1,2,3.$$

So that e_i can be identified as a real interval $[0,\ell_i](i=1,2,3)$.

Now let the strings be expanded on G and coincide with G at rest. Denote by $w_i(x,t)(i=1,2,3)$ the displacement function of i_{th} string departing from the equilibrium position in position $\pi_i(x)\in e_i$ at time t. The dynamic behavior of the network of strings can be described by the following partial differential equations

$$\begin{cases} \rho_i(x)\frac{\partial^2 w_i(x,t)}{\partial t^2} - \frac{\partial}{\partial x}[\sigma_i(x)\frac{\partial w_i(x,t)}{\partial x}] = 0, \\ \quad x\in(0,\ell_i), t>0,\ i=1,2,3; \\ w_1(0,t)=0; \\ w_1(\ell_1,t)=w_2(\ell_2,t)=w_3(\ell_3,t); \\ \sum_{i=1}^3 \sigma_i(\ell_i)\frac{\partial w_i}{\partial x}(\ell_i,t) + M\frac{\partial^2 w_1}{\partial t^2}(\ell_1,t)=0; \\ \sigma_i(0)\frac{\partial w_i}{\partial x}(0,t)=u_i(t),\ i=2,3; \\ w_i(x,0)=w_i^0(x),\ \frac{\partial w_i}{\partial t}(x,0)=w_i^1(x), \\ \ x\in(0,\ell_i),\ i=1,2,3, \end{cases} \tag{1}$$

where $\rho_i(x)>0,\ \sigma_i(x)>0$ are the mass density and the elasticity modulus of the i_{th} string, respectively; M is the mass of the tip body attached and $u_i(t)$ is the control input, $i=2,3$.

We design the following feedback controllers

$$u_i(t)=\alpha_i\frac{\partial w_i}{\partial t}(0,t)+\beta_i\frac{\partial w_i}{\partial t}(\ell_i,t),\ i=2,3,$$

where $\frac{\partial w_i}{\partial t}(0,t), \frac{\partial w_i}{\partial t}(\ell_i,t)$ are the outputs. The feedback gain constants $\alpha_i\ge 0, \beta_i>0, i=2,3$.

We organize the rest of this paper as follows. In section 2, we shall rewrite system (1) as an equivalent evolutionary equation in a suitable Hilbert space and show the well-posedness of the closed-loop system (4). In section 3, we shall study the eigenvalue problem of system (4) and manifest the spectral distribution of the system. In section 4, we shall show that the spectrum determined growth condition holds for the closed-loop system by using its Riesz basis property. After that, we will discuss the stabilization of system (4) and obtain that the closed-loop system is indeed exponentially stable. Finally, we shall make some conclusions of this paper.

2 Well-posedness of the System

Let $\mathbf{R}$, $\mathbf{C}$ be the sets of real numbers and complex numbers, respectively. Let $H^k(0,\ell_i)(i=1,2,3;k=1,2)$ be the usual Sobolev space and $L^2(0,\ell_i)(i=1,2,3)$ be the usual Hilbert space.

Set $\mathcal{X} := \{W=(w_i)_{i=1}^3 \in \Pi_{i=1}^3 H^1(0,\ell_i) : w_1(0)=0; w_1(\ell_1)=w_2(\ell_2)=w_3(\ell_3)\}$ endowed with the inner product

$$\begin{aligned} <W,V>_{\mathcal{X}} &= \sum_{i=1}^{3}\int_0^{\ell_i}\Big(\sigma_i(x)w_{ix}(x)\overline{v_{ix}(x)} \\ &\quad +\rho_i(x)w_i(x)\overline{v_i(x)}\Big)\,dx, \end{aligned}$$

for $W=(w_1,w_2,w_3), V=(v_1,v_2,v_3)\in\mathcal{X}$, where $w_{ix}(x)$ means the derivative of $w_i(x)$ with respect to x.

Set $\mathcal{H} := \mathcal{X}\times\Pi_{i=1}^3 L^2(0,\ell_i)\times\mathbf{C}$ equipped with the norm

$$\begin{aligned} &\|(W,V,p)\|^2 \\ &:= \sum_{i=1}^{3}\int_0^{\ell_i}\Big[\sigma_i(x)|w_{ix}(x)|^2+\rho_i(x)|v_i(x)|^2\Big]\,dx \\ &\quad +M|p|^2 \end{aligned}$$

for any $(W,V,p)\in\mathcal{H}$. Then $(\mathcal{H},\|\cdot\|)$ is a Hilbert space.

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} W \\ V \\ p \end{pmatrix} = \begin{pmatrix} V \\ \left(\frac{1}{\rho_i(x)}[\sigma_i(x)w_{ix}(x)]_x\right)_{i=1}^3 \\ -\frac{1}{M}\sum_{i=1}^3\sigma_i(\ell_i)w_{ix}(\ell_i) \end{pmatrix} \tag{2}$$

with the domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (W,V,p)\in\mathcal{X}\cap\Pi_{i=1}^3 H^2(0,\ell_i) \\ \quad \times\mathcal{X}\times\mathbf{C}: \\ \sigma_i(0)w_{ix}(0)=\alpha_i v_i(0)+\beta_i v_i(\ell_i), \\ \quad i=2,3; \\ \quad p=v_1(\ell_1) \end{array} \right\}. \tag{3}$$

Then, we rewrite system (1) as an equivalent evolutionary equation

$$\begin{cases} \frac{d\widetilde{W}(t)}{dt} = \mathcal{A}\widetilde{W}(t), \quad t>0, \\ \widetilde{W}(0)=\widetilde{W}_0, \end{cases} \tag{4}$$

where $\widetilde{W}(t)=(W(\cdot,t),\frac{\partial W}{\partial t}(\cdot,t),\frac{dw_1}{dt}(\ell_1,t))^\tau$, $\widetilde{W}(0)=(W_0,W_1,p)^\tau\in\mathcal{H}$, $W_0=(w_i^0)_{i=1}^3, W_1=(w_i^1)_{i=1}^3$ are given.

We need the following characteristic features of the operator $\mathcal{A}$.

Theorem 1 *Let $\mathcal{A}$ be defined by (2)-(3). Then $\mathcal{A}$ is a closed and densely defined linear operator in $\mathcal{H}$. $\mathcal{A}-KI$ is a dissipative operator for $K\ge\frac{1}{M}\max\{\frac{\beta_2}{\gamma_2},\frac{\beta_3}{\gamma_3}\}$, where $\gamma_i,\ i=2,3$ are any positive real numbers satisfying $\gamma_i\le\frac{2\alpha_i}{\beta_i},\ i=2,3$.*

Proof: The proof of that $\mathcal{A}$ is a densely defined and closed linear operator is a routine work. Here we only prove the other assertion.

For any $(W,V,p)\in D(\mathcal{A})$, we have

$$\begin{aligned}
&\Re\langle(\mathcal{A}-KI)(W,V,p),(W,V,p)\rangle\\
=\;&\Re\sum_{i=1}^{3}\int_0^{\ell_i}\left[\sigma_i(x)v_{ix}(x)\overline{w_{ix}(x)}dx\right]\\
&+\Re\sum_{i=1}^{3}\int_0^{\ell_i}\overline{v_i(x)}\left[\sigma_i(x)w_{ix}(x)\right]_x dx\\
&-\Re\sum_{i=1}^{3}\sigma_i(\ell_i)w_{ix}(\ell_i)\overline{p}-K\|(W,V,p)\|^2\\
\le\;&\Re\sum_{i=1}^{3}\sigma_i(x)w_{ix}(x)\overline{v_i(x)}\Big|_0^{\ell_i}\\
&-\Re\sum_{i=1}^{3}\sigma_i(\ell_i)w_{ix}(\ell_i)\overline{p}-KM|p|^2\\
=\;&\Re\sum_{i=1}^{3}\sigma_i(\ell_i)w_{ix}(\ell_i)\overline{v_i(\ell_i)}\\
&-\Re\sum_{i=2}^{3}\sigma_i(0)w_{ix}(0)\overline{v_i(0)}\\
&-\Re\sum_{i=1}^{3}\sigma_i(\ell_i)w_{ix}(\ell_i)\overline{v_i(\ell_i)}-KM|p|^2\\
=\;&-\Re\sum_{i=2}^{3}[\alpha_i v_i(0)+\beta_i v_i(\ell_i)]\overline{v_i}(0)-KM|p|^2\\
\le\;&\sum_{i=2}^{3}-\alpha_i|v_i(0)|^2-\frac{KM}{2}\sum_{i=2}^{3}|v_i(\ell_i)|^2\\
&+\frac{1}{2}\sum_{i=2}^{3}[\beta_i\gamma_i|v_i(0)|^2+\frac{\beta_i}{\gamma_i}|v_i(\ell_i)|^2]\\
=\;&\sum_{i=2}^{3}(\frac{\beta_i\gamma_i}{2}-\alpha_i)|v_i(0)|^2\\
&+\sum_{i=2}^{3}\frac{1}{2}(\frac{\beta_i}{\gamma_i}-MK)|v_i(\ell_i)|^2.
\end{aligned}$$

Notice that $\frac{\beta_i\gamma_i}{2}-\alpha_i\le 0$ and $\frac{\beta_i}{\gamma_i}-MK\le 0$ as $\gamma_i\le\frac{2\alpha_i}{\beta_i}$ and $K\ge\frac{1}{M}\max\{\frac{\beta_2}{\gamma_2},\ \frac{\beta_3}{\gamma_3}\}$, which implies that

$$\Re\langle(\mathcal{A}-KI)(W,V,p),(W,V,p)\rangle\le 0.$$

That is, $\mathcal{A}-KI$ is dissipative. □

Theorem 2 *Let $\mathcal{A}$ be defined by (2) and (3). Then $0\in\rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact. Hence, $\mathcal{A}$ is discrete (there exists a number λ in its resolvent set for which $R(\lambda,\mathcal{A})=(\lambda I-\mathcal{A})^{-1}$ is compact), and $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues of finite algebraic multiplicity only.*

Proof: For any $(F,G,c)\in\mathcal{H}$, where $F=(f_i(x))_{i=1}^3$, $G=(g_i(x))_{i=1}^3$, we consider the solvability of the equation $\mathcal{A}(W,V,p)=(F,G,c)$, $(W,V,p)\in D(\mathcal{A})$.

From the definition of $\mathcal{A}$, we get that

$$v_i(x)=f_i(x),\ x\in(0,\ell_i),i=1,2,3; \tag{5}$$
$$p=v_1(\ell_1)=v_2(\ell_2)=v_3(\ell_3), \tag{6}$$

and $W=(w_i(x))_{i=1}^3$ satisfies the following equations

$$[\sigma_i(x)w_{ix}(x)]_x=\rho_i(x)g_i(x),x\in(0,\ell_i),\quad i=1,2,3; \tag{7}$$
$$w_1(0)=0; \tag{8}$$
$$w_1(\ell_1)=w_2(\ell_2)=w_3(\ell_3); \tag{9}$$
$$Mc+\sum_{i=1}^{3}\sigma_i(\ell_i)w_{ix}(\ell_i)=0; \tag{10}$$
$$\sigma_i(0)w_{ix}(0)=\alpha_i f_i(0)+\beta_i f_i(\ell_i),\quad i=2,3. \tag{11}$$

Integrating (7) from 0 to x for any $x\in(0,\ell_i)$ and in view of (11), we have

$$\begin{aligned}
\sigma_i(x)w_{ix}(x)&=\sigma_i(0)w_{ix}(0)+\int_0^x\rho_i(s)g_i(s)ds\\
&=\alpha_i f_i(0)+\beta_i f_i(\ell_i)+\int_0^x\rho_i(s)g_i(s)ds\\
&=:\phi_i(x),\ i=2,3.
\end{aligned}\tag{12}$$

Especially, for $i=1$, we have

$$\sigma_1(x)w_{1x}(x)=\sigma_1(0)w_{1x}(0)+\int_0^x\rho_1(s)g_1(s)ds. \tag{13}$$

(12) and (13) together with (10) yield that

$$\sigma_1(\ell_1)w_{1x}(\ell_1)=-Mc-\sum_{i=2}^{3}\phi_i(\ell_i). \tag{14}$$

So,

$$\begin{aligned}
&\sigma_1(0)w_{1x}(0)\\
&=\sigma_1(\ell_1)w_{1x}(\ell_1)-\int_0^{\ell_1}\rho_1(s)g_1(s)ds\\
&=-Mc-\sum_{i=2}^{3}\phi_i(\ell_i)-\int_0^{\ell_1}\rho_1(s)g_1(s)ds
\end{aligned}$$

$$=: \phi_0. \tag{15}$$

Let us insert (15) into (13), then

$$\sigma_1(x)w_{1x}(x) = \phi_0 + \int_0^x \rho_1(s)g_1(s)ds =: \phi_1(x),$$

which implies

$$w_1(x) = \int_0^x \frac{\phi_1(s)}{\sigma_1(s)}ds. \tag{16}$$

We now integrate (12) from x to ℓ_i for any $x \in (0, \ell_i),\ i = 2, 3$ and employ (9) and (16) to find

$$\begin{aligned} w_i(x) &= w_i(\ell_i) - \int_x^{\ell_i} \frac{\phi_i(s)}{\sigma_i(s)}ds \\ &= \int_0^{\ell_1} \frac{\phi_1(s)}{\sigma_1(s)}ds - \int_x^{\ell_i} \frac{\phi_i(s)}{\sigma_i(s)}ds. \end{aligned} \tag{17}$$

Denote by $W = (w_i(x))_{i=1}^3,\ V = (v_i(x))_{i=1}^3$, then $(W, V, p) \in D(\mathcal{A})$ is the unique solution of $\mathcal{A}(W, V, p) = (F, G, c)$. Consequently, $\mathcal{A}^{-1}$ is continuous and $0 \in \rho(\mathcal{A})$ by the inverse operator theorem.

Moreover, in view of $D(\mathcal{A}) \subset \mathcal{X} \cap \Pi_{i=1}^3 H^2(0, \ell_i) \times \mathcal{X} \times C$, $\mathcal{A}^{-1}$ is compact owing to the compact embedding theorem. Hence, $\mathcal{A}$ is discrete. Consequently, $\sigma(\mathcal{A})$ consists of isolated eigenvalues of finite algebraic multiplicity only ([7]). □

Theorem 2 together with Theorem 1 asserts the following result according to the Lumer-Phillips theorem ([19]).

Theorem 3 *$\mathcal{A}$ generates a C_0 semigroup of contraction on $\mathcal{H}$. Thereby, system (4) is well-posed in $\mathcal{H}$.*

3 Spectral Distribution of the System

In this section, we shall study the eigenvalue problem of the operator $\mathcal{A}$ to find the asymptotic distribution of its spectrum.

From $(\lambda I - \mathcal{A})(W, V, p) = 0$, it holds that

$$v_i(x) = \lambda w_i(x),\ x \in (0, \ell_i),\ i = 1, 2, 3,$$
$$p = v_1(\ell_1) = \lambda w_1(\ell_1),$$

and each component w_i of $W = (w_i)_{i=1}^3$ satisfies

$$\lambda^2 w_i(x) - \frac{1}{\rho_i(x)}\left[\sigma_i(x)w_{ix}(x)\right]_x = 0,$$
$$x \in (0, \ell_i),\ i = 1, 2, 3; \tag{18}$$
$$w_1(0) = 0; \tag{19}$$
$$w_1(\ell_1) = w_2(\ell_2) = w_3(\ell_3); \tag{20}$$
$$\sigma_i(0)w_{ix}(0) = \lambda[\alpha_i w_i(0) + \beta_i w_i(\ell_i)],$$
$$i = 2, 3; \tag{21}$$
$$\sum_{i=1}^3 \frac{\lambda^2 M w_i(\ell_i)}{3} + \sum_{i=1}^3 \sigma_i(\ell_i)w_{ix}(\ell_i) = 0. \tag{22}$$

Clearly, the eigenvalue problem $(\lambda I - \mathcal{A})(W, V, p) = 0$ can be solved by finding a solution $W = (w_i)_{i=1}^3$ to (18)–(22).

Let us consider the differential equation (18) at first. Set $\tilde{w}_i(x) := \sqrt{\sigma_i(x)}w_i(x),\ i = 1, 2, 3$ and substitute it into (18) to get

$$\tilde{w}_{ixx}(x) - \lambda^2 \frac{\rho_i(x)}{\sigma_i(x)}\tilde{w}_i(x) + b_i(x)\frac{\rho_i(x)}{\sigma_i(x)}\tilde{w}_i(x) = 0, \tag{23}$$

where

$$b_i(x) = \left[\frac{1}{4}\left(\frac{\sigma_{ix}(x)}{\sigma_i(x)}\right)^2 - \frac{1}{2}\left(\frac{\sigma_{ixx}(x)}{\sigma_i(x)}\right)\right]\frac{\sigma_i(x)}{\rho_i(x)},$$
$$i = 1, 2, 3. \tag{24}$$

Then (19)-(22) has the following form with $\tilde{w}_i$ replacing $w_i,\ i = 1, 2, 3$

$$\tilde{w}_1(0) = 0; \tag{25}$$
$$\frac{\tilde{w}_1(\ell_1)}{\sqrt{\sigma_1(\ell_1)}} = \frac{\tilde{w}_2(\ell_2)}{\sqrt{\sigma_2(\ell_2)}} = \frac{\tilde{w}_3(\ell_3)}{\sqrt{\sigma_3(\ell_3)}}; \tag{26}$$
$$\sigma_i(0)\tilde{w}_{ix}(0) - \frac{1}{2}\sigma_{ix}(0)\tilde{w}_i(0)$$
$$= \lambda\left[\alpha_i\tilde{w}_i(0) + \beta_i\sqrt{\frac{\sigma_i(0)}{\sigma_i(\ell_i)}}\tilde{w}_i(\ell_i)\right],$$
$$i = 2, 3; \tag{27}$$
$$\sum_{i=1}^3 \sqrt{\sigma_i(\ell_i)}\tilde{w}_{ix}(\ell_i)$$
$$= \sum_{i=1}^3 (-\frac{\lambda^2 M}{3} + \frac{1}{2}\sigma_{ix}(\ell_i))\frac{\tilde{w}_i(\ell_i)}{\sqrt{\sigma_i(\ell_i)}}. \tag{28}$$

Next, let $m_i := \int_0^{\ell_i}\sqrt{\frac{\rho_i(\theta)}{\sigma_i(\theta)}}d\theta$, and define a new independent variable

$$\xi_i(x) := \int_0^x \sqrt{\frac{\rho_i(\theta)}{\sigma_i(\theta)}}d\theta,\ x \in (0, \ell_i).$$

Then $\xi_i \in (0, m_i)$ for each $i = 1, 2, 3$. In addition, let

$$z_i(\xi_i) := \left[\frac{\rho_i(x(\xi_i))}{\sigma_i(x(\xi_i))}\right]^{1/4}\tilde{w}_i(x(\xi_i)); \tag{29}$$

$$\phi_i(\xi_i) := \left[\left(\frac{5}{16}\frac{\sigma_i}{\rho_i}(\frac{\rho_i}{\sigma_i})_x - \frac{1}{4}(\frac{\rho_i}{\sigma_i})_{xx}\right)(\frac{\rho_i}{\sigma_i})^{-2}\right](x(\xi_i)), \tag{30}$$

for $i = 1, 2, 3$, where $x(\xi_i)$ is the inverse function of $\xi_i(x)$.

Accordingly, (23), (25) and (28) can be changed into

$$z_i''(\xi_i) - \lambda^2 z_i(\xi_i) + [b_i(x(\xi_i)) + \phi_i(\xi_i)]z_i(\xi_i),\quad \xi_i \in (0, m_i), i = 1, 2, 3; \tag{31}$$

$$z_1(0) = 0; \tag{32}$$

$$\frac{z_1(m_1)}{\sqrt[4]{\rho_1(\ell_1)\sigma_1(\ell_1)}} = \frac{z_2(m_2)}{\sqrt[4]{\rho_2(\ell_2)\sigma_2(\ell_2)}} = \frac{z_3(m_3)}{\sqrt[4]{\rho_3(\ell_3)\sigma_3(\ell_3)}}; \tag{33}$$

$$\sqrt[4]{\sigma_i(0)\rho_i(0)}z_i'(0) + c_i z_i(0) = \lambda[\frac{\alpha_i}{\sqrt[4]{\sigma_i(0)\rho_i(0)}}z_i(0) + \frac{\beta_i}{\sqrt[4]{\sigma_i(\ell_i)\rho_i(\ell_i)}}z_i(m_i)]\quad i = 2, 3; \tag{34}$$

$$\sum_{i=1}^{3} \sqrt[4]{\sigma_i(\ell_i)\rho_i(\ell_i)}z_i'(m_i) = \sum_{i=1}^{3}[\frac{-\lambda^2 M}{3\sqrt[4]{\sigma_i(\ell_i)\rho_i(\ell_i)}} + d_i]z_i(m_i), \tag{35}$$

where $b_i(x(\xi_i))$ and $\phi_i(\xi_i)$ are given by (24), (30), respectively, and

$$c_i := -\frac{1}{4}\sigma_i(0)[\rho_i(0)\sigma_i(0)]^{-5/4}(\frac{\rho_i}{\sigma_i})_x(0) - \frac{1}{2}\sigma_{ix}(0)(\sigma_i(0)\rho_i(0))^{-1/4}; \tag{36}$$

$$d_i := \frac{\sqrt{\sigma_i(\ell_i)}}{4}(\frac{\rho_i(\ell_i)}{\sigma_i(\ell_i)})^{-5/4}(\frac{\rho_i}{\sigma_i})_x(\ell_i) + \frac{\sigma_{ix}(\ell_i)}{2\sqrt[4]{\rho_i(\ell_i)\sigma_i(\ell_i)}}, i = 1, 2, 3. \tag{37}$$

Herein and afterwards, the prime always denotes the derivative with respect to the independent variable ξ_i.

In what follows, we shall get the asymptotic expression of the solution to (31)-(35). According to the theory of ordinary differential equation, there exist two linear independent solutions $\widetilde{\varphi}_i(\xi_i; \lambda)$ and $\widetilde{\psi}_i(\xi_i; \lambda)$ to (31) for $\lambda \in \mathbf{C}$ with $|\lambda| \geq \delta > 0$. Furthermore,

$$\widetilde{\varphi}_i(\xi_i; \lambda) = e^{\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \tag{38}$$

$$\widetilde{\psi}_i(\xi_i; \lambda) = e^{-\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right]; \tag{39}$$

$$\widetilde{\varphi}_i'(\xi_i; \lambda) = \lambda e^{\lambda\xi_i}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \tag{40}$$

$$\widetilde{\psi}_i'(\xi_i; \lambda) = \lambda e^{-\lambda\xi_i}\left[-1 + O\left(\frac{1}{\lambda}\right)\right], \tag{41}$$

where $O\left(\frac{1}{\lambda}\right)$ means some function like $\frac{f(x,\lambda)}{\lambda}$ and there exist constants C, R_0 such that $|f(x, \lambda)| \leq C$ for any $x \in [0, m_i]$ whenever $|\lambda| > R_0$ (see, [16, Theorem 1, pp. 49]). Hence, the general solution to (31) can be expressed as

$$z_i(\xi_i) = A_i(\lambda)e^{\lambda\xi_i}[1]_1 + B_i(\lambda)e^{-\lambda\xi_i}[1]_1, \tag{42}$$

where $A_i(\lambda),\ B_i(\lambda)$ are constants dependent on λ and the notation $[a]_1 := a + O\left(\frac{1}{\lambda}\right)$ is used for simplification. From (42), it is obvious that

$$z_i'(\xi_i) = A_i(\lambda)\lambda e^{\lambda\xi_i}[1]_1 - B_i(\lambda)\lambda e^{-\lambda\xi_i}[1]_1. \tag{43}$$

By denoting

$$k_i = \sqrt[4]{\rho_i(\ell_i)\sigma_i(\ell_i)}, \tag{44}$$

$$\widetilde{k}_i = \sqrt[4]{\rho_i(0)\sigma_i(0)}, \tag{45}$$

and inserting (42)-(43) into (32)-(35), we get

$$\Delta(\lambda)\cdot(A_1(\lambda)\ A_2(\lambda)\ A_3(\lambda)\ B_1(\lambda)\ B_2(\lambda)\ B_3(\lambda))^\tau = \vec{0},$$

where

$$\Delta(\lambda) = \begin{pmatrix} [1]_1 & 0 & 0 & [1]_1 & 0 & 0 \\ \frac{e^{\lambda m_1}}{k_1}[1]_1 & -\frac{e^{\lambda m_2}}{k_2}[1]_1 & 0 & \frac{e^{-\lambda m_1}}{k_1}[1]_1 & -\frac{e^{-\lambda m_2}}{k_2}[1]_1 & 0 \\ 0 & \frac{e^{\lambda m_2}}{k_2}[1]_1 & -\frac{e^{\lambda m_3}}{k_3}[1]_1 & 0 & \frac{e^{-\lambda m_2}}{k_2}[1]_1 & -\frac{e^{-\lambda m_3}}{k_3}[1]_1 \\ 0 & a_{42} & 0 & 0 & a_{45} & 0 \\ 0 & 0 & a_{53} & 0 & 0 & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix} \tag{46}$$

with

$$a_{42} = \lambda(\widetilde{k}_2 - \frac{\alpha_2}{\widetilde{k}_2} - \frac{\beta_2}{k_2}e^{\lambda m_2})[1]_1 + c_2[1]_1;$$
$$a_{45} = -\lambda(\widetilde{k}_2 + \frac{\alpha_2}{\widetilde{k}_2} + \frac{\beta_2}{k_2}e^{-\lambda m_2})[1]_1 + c_2[1]_1;$$
$$a_{53} = \lambda(\widetilde{k}_3 - \frac{\alpha_3}{\widetilde{k}_3} - \frac{\beta_3}{k_3}e^{\lambda m_3})[1]_1 + c_3[1]_1;$$
$$a_{56} = -\lambda(\widetilde{k}_3 + \frac{\alpha_3}{\widetilde{k}_3} + \frac{\beta_3}{k_3}e^{-\lambda m_3})[1]_1 + c_3[1]_1;$$
$$a_{6i} = (\frac{\lambda^2 M}{3k_i} + \lambda k_i - d_i)e^{\lambda m_i}[1]_1,\ i = 1, 2, 3;$$
$$a_{6j} = (\frac{\lambda^2 M}{3k_i} - \lambda k_i - d_i)e^{-\lambda m_i}[1]_1,\quad j = 4, 5, 6, i = j - 3.$$

Clearly, $\lambda \in \mathbf{C} \setminus \{0\}$ is an eigenvalue of $\mathcal{A}$ if and only if λ is a zero point of the determinant of the matrix $\Delta(\lambda)$.

Now, the spectral distribution of the system operator $\mathcal{A}$ can be found in the following theorem.

Theorem 4 *Let $\mathcal{A}$ be defined by (2) and (3). Assume that $\alpha_i \neq \sqrt{\rho_i(0)\sigma_i(0)} =: \widetilde{k}_i^2,\ i = 2, 3$. Then the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. Moreover, $\sigma(\mathcal{A})$ is a union of finitely separable sets.*

Proof: By a straightforward computation, we obtain

$$\begin{aligned} &\lim_{\Re\lambda\to+\infty} \frac{|\det\Delta(\lambda)|}{|\lambda^4|e^{\lambda\sum_{i=1}^3 m_i}} = \frac{M}{k_1k_2k_3}\prod_{i=2}^{3}(\widetilde{k}_i + \frac{\alpha_i}{\widetilde{k}_i}) \\ &\quad > 0; \\ &\lim_{\Re\lambda\to-\infty} \frac{|\det\Delta(\lambda)|}{|\lambda^4|e^{-\lambda\sum_{i=1}^3 m_i}} = \frac{M}{k_1k_2k_3}\prod_{i=2}^{3}|\widetilde{k}_i - \frac{\alpha_i}{\widetilde{k}_i}| \\ &\quad > 0, \end{aligned} \tag{47}$$

which implies that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. In other words, $|\det\Delta(\lambda)|$ is a sine-type function in λ. Then the conclusion holds by Levin lemma [2]. □

4 The Stabilization of System (4)

In this section, we will establish the exponential stabilization of system (4). The Riesz basis property of system (4) will be proved at first. The exponential stabilization of the system is then obtained.

Theorem 5 *Suppose $\alpha_i \neq \sqrt{\rho_i(0)\sigma_i(0)} =: \widetilde{k}_i^2,\ i = 2, 3$. The sequence of generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$.*

Proof: To begin with, we introduce an auxiliary operator $\mathcal{A}_0$ defined by $\mathcal{A}_0(W,V,p) = \mathcal{A}(W,V,p)$ for any $(W,V,p) = ((w_i)_{i=1}^3, (v_i)_{i=1}^3, p) \in D(\mathcal{A}_0)$ with domain

$$D(\mathcal{A}_0) = \left\{ \begin{array}{l} (W,V,p) \in \mathcal{H}, \mathcal{A}_0(W,V,p) \in \mathcal{H} \\ p = v_1(\ell_1), \sigma_i(0)w_{ix}(0) = 0,\ i = 2,3 \end{array} \right\}.$$

Then $\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ by the definition and hence $\|R(\lambda,\mathcal{A}_0)\| \le \dfrac{1}{|\lambda|},\ \ \forall\lambda \in \mathbf{R}\backslash\{0\}$.

The completeness of the sequence of generalized eigenvectors of $\mathcal{A}$ means that $Sp(\mathcal{A}) = \mathcal{H}$, or equivalently, $Sp(\mathcal{A})^\perp = \{0\}$, where $Sp(\mathcal{A})$ is the closed subspace spanned by all generalized eigenvectors of $\mathcal{A}$. Taking $(W_0, V_0, p_0) \in Sp(\mathcal{A})^\perp$, $R(\lambda,\mathcal{A})(W_0,V_0,p_0)$ is an entire function of λ as $\mathcal{A}$ is discrete. The same is true for $R(\lambda,\mathcal{A})^*(W_0,V_0,p_0)$.

Denote by

$$R(\lambda) := \langle (F,G,c), R(\lambda,\mathcal{A})^*(W_0,V_0,p_0)\rangle,\ \lambda \in \mathbf{C}$$

for any $(F,G,c) = ((f_i(x))_{i=1}^3, (g_i(x))_{i=1}^3, c) \in \mathcal{H}$. Clearly, $R(\lambda)$ is an entire function of λ, and $\lim_{\lambda\to+\infty} R(\lambda) = 0$ since $\mathcal{A}$ generates a C_0 semigroup.

We shall certify $R(\lambda) \equiv 0, \forall\lambda \in \mathbf{C}$ so that we can get $Sp(\mathcal{A}) = \mathcal{H}$. To this end, let us consider the following equations for $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbf{R}^-$.

$$\begin{aligned} &(\lambda - \mathcal{A})(S_1,T_1,r_1) = (F,G,c), \\ &(\lambda - \mathcal{A}_0)(S_2,T_2,r_2) = (F,G,c), \end{aligned} \tag{48}$$

where

$$\begin{aligned} &(S_1,T_1,r_1) = ((s_{1i}(x))_{i=1}^3, (t_{1i}(x))_{i=1}^3, r_1), \\ &(S_2,T_2,r_2) = ((s_{2i}(x))_{i=1}^3, (t_{2i}(x))_{i=1}^3, r_2). \end{aligned}$$

Set

$$\begin{aligned} &((s_i(x))_{i=1}^3, (t_i(x))_{i=1}^3, r) =: (S,T,r) \\ &= (S_1,T_1,r_1) - (S_2,T_2,r_2). \end{aligned} \tag{49}$$

Then

$$\begin{aligned} &\|R(\lambda,\mathcal{A})(F,G,c)\| = \|(S_1,T_1,r_1)\| \\ &= \|(S,T,r) + (S_2,T_2,r_2)\| \\ &= \|(S,T,r) + R(\lambda,\mathcal{A}_0)(F,G,c)\| \\ &\le \|(S,T,r)\| + \frac{1}{|\lambda|}\|(F,G,c)\|. \end{aligned} \tag{50}$$

To evaluate $\|(S,T,r)\|$, we use (48), (49) to give that

$$\begin{aligned} &t_{1i}(x) = \lambda s_{1i}(x) - f_i(x), \\ &t_{2i}(x) = \lambda s_{2i}(x) - f_i(x), \\ &t_i(x) = t_{1i}(x) - t_{2i}(x) = \lambda s_i(x), \\ &r = t_{11}(\ell_1) - t_{21}(\ell_1) = t_1(\ell_1) = \lambda s_1(\ell_1), \end{aligned}$$

and $s_i(x),\ i = 1,2,3$ satisfy

$$\left\{ \begin{array}{l} \lambda^2\rho_i(x)s_i(x) - [\sigma_i(x)s_{ix}(x)]_x = 0, \\ \quad x \in (0,\ell_i), i = 1,2,3; \\ s_1(0) = 0; \\ s_1(\ell_1) = s_2(\ell_2) = s_3(\ell_3); \\ \lambda^2 M s_1(\ell_1) + \sum_{i=1}^3 \sigma_i(\ell_i)s_{ix}(\ell_i) = 0; \\ \sigma_i(0)s_{ix}(0) - \lambda[\alpha_i s_i(0) + \beta_i s_i(\ell_i)] \\ = \lambda[\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)] - \alpha_i f_i(0) - \beta_i f_i(\ell_i), \\ \quad i = 2,3. \end{array} \right. \tag{51}$$

Similar to the analysis in the third section, we denote by

$$\xi_i(x) := \int_0^x \sqrt{\frac{\rho_i(t)}{\sigma_i(t)}}dt,$$
$$m_i = \int_0^{\ell_i} \sqrt{\frac{\rho_i(t)}{\sigma_i(t)}}dt,$$

and set

$$\widetilde{s}_i(\xi_i) := \sqrt[4]{\rho_i(x(\xi_i))\sigma_i(x(\xi_i))}s_i(x(\xi_i)), \quad \text{for } \xi \in (0, m_i),\ i = 1, 2, 3, \tag{52}$$

where $x(\xi_i)$ is the inverse function of $\xi_i(x)$.

Then (51) can be rewritten as

$$\left\{\begin{array}{l} \widetilde{s}_i''(\xi_i) - \lambda^2\widetilde{s}_i(\xi_i) = \widetilde{b}_i((x(\xi_i))\widetilde{s}_i(\xi_i), \\ \quad \xi_i \in (0, m_i),\ i = 1, 2, 3; \\ \widetilde{s}_1(0) = 0; \\ [\rho_1(\ell_1)\sigma_1(\ell_1)]^{-1/4}\widetilde{s}_1(m_1) \\ = [\rho_2(\ell_2)\sigma_2(\ell_2)]^{-1/4}\widetilde{s}_2(m_2) \\ = [\rho_3(\ell_3)\sigma_3(\ell_3)]^{-1/4}\widetilde{s}_3(m_3); \\ \sqrt[4]{\rho_i(0)\sigma_i(0)}\widetilde{s}_i'(0) - [\frac{\lambda\alpha_i}{\sqrt[4]{\rho_i(0)\sigma_i(0)}} + \widetilde{c}_i(0)]\widetilde{s}_i(0) \\ \quad -\frac{\lambda\beta_i}{\sqrt[4]{\rho_i(\ell_i)\sigma_i(\ell_i)}}\widetilde{s}_i(m_i) = \lambda[\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)] \\ \quad -\alpha_i f_i(0) - \beta_i f_i(\ell_i),\ i = 2, 3; \\ \sum_{i=1}^3[\rho_i(\ell_i)\sigma_i(\ell_i)]^{1/4}\widetilde{s}_i'(m_i) \\ \quad + \sum_{i=1}^3[\frac{\lambda^2 M}{3\sqrt[4]{\rho_i(\ell_i)\sigma_i(\ell_i)}} - \widetilde{c}_i(\ell_i)]\widetilde{s}_i(m_i) = 0, \end{array}\right. \tag{53}$$

where

$$\widetilde{b}_i(x(\xi_i)) = [\tfrac{5}{16}\tfrac{(\rho_i\sigma_i)_x^2}{(\rho_i\sigma_i)^2} - \tfrac{1}{4}\tfrac{(\rho_i\sigma_i)_{xx}}{\rho_i\sigma_i}]\rho_i^{-1},$$
$$\widetilde{c}_i(x) = \tfrac{1}{4}(\rho_i\sigma_i)^{-5/4}(\rho_i\sigma_i)_x\sigma_i.$$

The general solution $\widetilde{s}_i(\xi_i)$ of the first equation in (53) has an asymptotic formula

$$\widetilde{s}_i(\xi_i) = A_i(\lambda)e^{\lambda\xi_i}[1]_1 + B_i(\lambda)e^{-\lambda\xi_i}[1]_1, \quad \xi_i \in (0, m_i),\ i = 1, 2, 3. \tag{54}$$

Here $[1]_1 = 1 + O(1/\lambda)$.

Substituting (54) into (53), and applying Cramers' Rule to the resulted identities, we obtain

$$A_2(\lambda) = \frac{\alpha_2 s_{22}(0) + \beta_2 s_{22}(\ell_2)}{\widetilde{k_2} - \frac{\alpha_2}{\widetilde{k_2}}} + o(1);$$
$$A_3(\lambda) = \frac{\alpha_3 s_{23}(0) + \beta_3 s_{23}(\ell_3)}{\widetilde{k_3} - \frac{\alpha_3}{\widetilde{k_3}}} + o(1);$$
$$B_2(\lambda) = O(e^{\lambda m_2});$$
$$B_3(\lambda) = O(e^{\lambda m_3}). \tag{55}$$

Thus, for each $i \in \{2, 3\}$,

$$\widetilde{s}_i(0) = \frac{\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)}{\widetilde{k_i} - \frac{\alpha_i}{\widetilde{k_i}}}[1]_1 + O(e^{\lambda m_i});$$
$$\widetilde{s}_i(m_i) = \frac{\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)}{\widetilde{k_i} - \frac{\alpha_i}{\widetilde{k_i}}}e^{\lambda m_i}[1]_1 + O(e^{\lambda m_i}).$$

By noticing that

$$|\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)| \le \mu\|(S_2, T_2, c)\|,$$

where

$$\mu = \max_{i=2,3}\left\{\frac{\alpha_i + \beta_i}{\min_{x\in[0,\ell_i]}\{\sigma_i(x)\}}\sqrt{\int_0^{\ell_i}\sigma_i(x)dx}\right\},$$

we have

$$\begin{aligned} |s_i(0)| &= \widetilde{k}_i^{-1}|\widetilde{s}_i(0)| \\ &= |\tfrac{\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)}{1 - \alpha_i/\widetilde{k_i}^2}[\tfrac{1}{\widetilde{k}_i^2}]_1 + O(e^{\lambda m_i})| \\ &\le \tfrac{\mu}{|1 - \alpha_i/\widetilde{k_i}^2|}\|(S_2, T_2, c)\|[\tfrac{1}{\widetilde{k}_i^2}]_1 + |O(e^{\lambda m_i})| \\ &\le \tfrac{\mu}{|\lambda|}|1 - \tfrac{\alpha_i}{\widetilde{k_i}^2}|^{-1}\|(F, G, c)\|[\tfrac{1}{\widetilde{k}_i^2}]_1 + |O(e^{\lambda m_i})| \\ &= [O(\tfrac{1}{|\lambda|}) + O(e^{\lambda m_i})]\|(F, G, c)\|, \end{aligned}$$

and

$$\begin{aligned} |s_i(\ell_i)| &= k_i^{-1}|\widetilde{s}_i(m_i)| \\ &= |\frac{\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)}{1 - \alpha_i/\widetilde{k_i}^2}e^{\lambda m_i}[\frac{1}{k_i\widetilde{k}_i}]_1 + O(e^{\lambda m_i})| \\ &\le \frac{\mu e^{\lambda m_i}}{|\lambda|}\left|\frac{[1/(k_i\widetilde{k}_i)]_1}{1 - \alpha_i/\widetilde{k_i}^2}\right|\|(F, G, c)\| + |O(e^{\lambda m_i})| \\ &= [O(\frac{1}{|\lambda|})e^{\lambda m_i} + O(e^{\lambda m_i})]\|(F, G, c)\|. \end{aligned}$$

Hence,

$$\begin{aligned} &\lambda\|(S, T, r)\|^2 \\ &= -\sum_{i=1}^3 \sigma_i(x)s_{ix}(x)\overline{t_i(x)}|_0^{\ell_i} + \sum_{i=1}^3[\sigma_i(\ell_i)s_{ix}(\ell_i)]\overline{r} \\ &= \sum_{i=1}^3 \sigma_i(0)s_{ix}(0)\overline{t_i(0)} \\ &= \sum_{i=2}^3[\alpha_i t_i(0) + \beta_i t_i(\ell_i) + \alpha_i t_{2i}(0) \\ &\quad +\beta_i t_{2i}(\ell_i)]\overline{t_i(0)} \\ &= \lambda^2\sum_{i=2}^3[\alpha_i|s_i(0)|^2 + \beta_i s_i(\ell_i)\overline{s_i(0)}] \\ &\quad +\lambda^2\sum_{i=2}^3[\alpha_i s_{2i}(0) + \beta_i s_{2i}(\ell_i)]\overline{s_i(0)} \\ &\quad -\lambda\sum_{i=2}^3[\alpha_i f_i(0) + \beta_i f_i(\ell_i)]\overline{s_i(0)}. \end{aligned}$$

Consequently,

$$\begin{aligned}&\|(S,T,r)\|^2\\ &\le |\lambda|\textstyle\sum_{i=2}^3[\alpha_i|s_i(0)|^2+\beta_i|s_i(\ell_i)||s_i(0)|]\\ &\quad +|\lambda|\textstyle\sum_{i=2}^3|\alpha_i s_{2i}(0)+\beta_i s_{2i}(\ell_i)||s_i(0)|\\ &\quad +\textstyle\sum_{i=2}^3|\alpha_i f_i(0)+\beta_i f_i(\ell_i)||s_i(0)|\\ &\le \textstyle\sum_{i=2}^3\alpha_i O(\frac{1}{|\lambda|})\|(F,G,c)\|^2\\ &\quad +\textstyle\sum_{i=2}^3\beta_i O(\frac{1}{\lambda})\|(F,G,c)\|^2\\ &\quad +\mu[O(\frac{1}{|\lambda|})+O(e^{\lambda m_i})]\|(F,G,c)\|^2\\ &= O(\tfrac{1}{|\lambda|})\|(F,G,c)\|^2.\end{aligned}$$

By using the above inequality and (50), we have the estimate

$$\begin{aligned}&\|R(\lambda,\mathcal{A})(F,G,c)\|\\ &\le \|(S,T,r)\|+\frac{1}{|\lambda|}\|(F,G,c)\|\\ &\le \left[O(\frac{1}{\sqrt{|\lambda|}})+\frac{1}{|\lambda|}\right]\|(F,G,c)\|, \end{aligned}\qquad (56)$$

which implies $\lim_{\Re\lambda\to-\infty}R(\lambda)=0$.

In addition, $R(\lambda)$ is uniformly bounded along the line $\Re\lambda=\gamma$ since $R(\lambda)$ is an entire function of finite exponential type. Then the Phragmén-Lindelöf theorem (see [27]) implies that $R(\lambda)$ is bounded in the complex plane. So $R(\lambda)\equiv 0,\forall\lambda\in\mathbf{C}$. Notice that $R(\lambda) =< (F,G,c), R(\lambda,\mathcal{A})^*(W_0,V_0,p_0)>,\ \forall(F,G,c)\in\mathcal{H}$. We conclude that $R(\lambda,\mathcal{A})^*(W_0,V_0,p_0)=0$, which means that $(W_0,V_0,p_0)\equiv 0$. That is, $Sp(\mathcal{A})=\mathcal{H}$. □

The following proposition ([25]) provides us with the sufficient conditions of that a sequence forms a subspace Riesz basis.

Proposition 6 *Let $\mathcal{A}$ be the generator of a C_0 semigroup $\{T(t):t\ge 0\}$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied*

1). The spectrum of $\mathcal{A}$ has a decomposition

$$\sigma(\mathcal{A})=\sigma_1(\mathcal{A})\bigcup\sigma_2(\mathcal{A});$$

2). There exists a real number $\alpha\in\mathbf{R}$ such that

$$\sup\{\Re\lambda|\lambda\in\sigma_1(\mathcal{A})\}\le\alpha\le\inf\{\Re\lambda|\lambda\in\sigma_2(\mathcal{A})\};$$

3). The set $\sigma_2(\mathcal{A})=\{\lambda_k\}_{k\in\mathbf{N}}$ consists of isolated eigenvalues of $\mathcal{A}$ and is a union of finitely separated sets.

Then there exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with

$$\mathcal{H}_1=\{f\in\mathcal{H}\mid E(\lambda,\mathcal{A})f=0,\forall\lambda\in\sigma_2(\mathcal{A})\},$$

$$\mathcal{H}_2=\overline{span}\{\sum_{k=1}^m E(\lambda_k,\mathcal{A})f:\lambda_k\in\sigma(\mathcal{A}),\ \forall m\in\mathbf{N},\ \forall f\in\mathcal{H}\}$$

such that $\mathcal{H}_1\bigcap\mathcal{H}_2=\{0\}$ with property that $\sigma(\mathcal{A}|_{\mathcal{H}_1})=\sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2})=\sigma_2(\mathcal{A})$.

Moreover, there exists a finite collection Ω_k of elements in $\sigma_2(\mathcal{A})$ such that $\{E(\Omega_k,\mathcal{A})\mathcal{H}_2\}_{k\in\mathbf{N}}$ forms a subspace Riesz basis for $\mathcal{H}_2$, where

$$E(\Omega_k,\mathcal{A})=\sum_{\lambda\in\Omega_k}E(\lambda,\mathcal{A})$$

is the Riesz projector corresponding to Ω_k .

Theorem 7 *Let $\mathcal{A}$ be defined by (2) and (3) and let $\alpha_i\ne\widetilde{k}_i^2,\ i=2,3$. Then there exists a sequence of generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$. Therefore, the closed-loop system (4) satisfies the spectrum determined growth condition.*

Proof: We take $\sigma_1(\mathcal{A})=-\infty$, $\sigma_2(\mathcal{A})=\sigma_p(\mathcal{A})$, then $\sigma(\mathcal{A})$ has a decomposition $\sigma(\mathcal{A})=\sigma_1(\mathcal{A})\bigcup\sigma_2(\mathcal{A})$. Thus, condition 1) is fulfilled. Besides, Conditions 2) and 3) also hold by Theorem 4 and Theorem 2. Hence, there exists $T(t)$-invariant closed subspace $\mathcal{H}_2$ such that the sequence of generalized eigenvectors of $\mathcal{A}$ forms a subspace Riesz basis (that is the Riesz basis with parentheses) for $\mathcal{H}_2$ by Proposition 6. Furthermore, Theorem 5 shows that the sequence of generalized eigenvectors of the system operator $\mathcal{A}$ is complete in $\mathcal{H}$, which implies that $\mathcal{H}_2=\mathcal{H}$. We complete the proof. □

Theorem 8 *Let $\mathcal{A}$ be defined by (2) and (3), and $0<\alpha_3\ne\widetilde{k}_3^2$. Then system (4) is exponentially stable provided that α_2,β_2 satisfy the following conditions:*

$$\alpha_2>5\widetilde{k}_2^2, \qquad (57)$$

$$\frac{\beta_2}{k_2}>\max\left\{\frac{\alpha_2^2-\widetilde{k}_2^4}{(\alpha_2-5\widetilde{k}_2^2)\widetilde{k}_2^2},\frac{2\alpha_2}{\widetilde{k}_2}\right\}, \qquad (58)$$

where $\widetilde{k}_i=\sqrt[4]{\rho_i(0)\sigma_i(0)}>0,\ i=2,3;\ k_2=\sqrt[4]{\rho_2(\ell_2)\sigma_2(\ell_2)}>0$.

Proof: Since the spectrum determined growth condition holds for system (4) (Theorem 7) and $\sigma(\mathcal{A})=\sigma_p(\mathcal{A})$ locates in the left half plane, the proof of the exponential stabilization of system (4) is equivalent to verify that the imaginary axis is not an asymptote of $\sigma(\mathcal{A})$. This can be done by showing that

$$\inf_{x\in\mathbf{R}\backslash\{0\}}|\det\Delta(ix)|>0. \qquad (59)$$

Let $\Delta_0(\lambda)$ be the main part of $\Delta(\lambda)$ (see (46)). Then (59) can be replaced by

$$\inf_{x\in\mathbf{R}\setminus\{0\}} |\det \Delta_0(ix)| > 0.$$

By a direct calculation, we have

$$\begin{aligned}
&\det \Delta_0(\lambda)\cdot(-\frac{3k_1k_2k_3}{2\lambda^4 M})\\
&= e^{\lambda m_1}\Pi_{i=2}^3[\widetilde{k}_i\cosh(\lambda m_i)+\frac{\alpha_i}{\widetilde{k}_i}\sinh(\lambda m_i)]\\
&\quad +2\sinh(\lambda m_1)[(\widetilde{k}_2+\frac{\alpha_2}{\widetilde{k}_2})e^{\lambda m_2}+\frac{\beta_2}{k_2}]\\
&\qquad \cdot[\widetilde{k}_3\cosh(\lambda m_3)+\frac{\alpha_3}{\widetilde{k}_3}\sinh(\lambda m_3)]\\
&\quad +2\sinh(\lambda m_1)[(\widetilde{k}_3+\frac{\alpha_3}{\widetilde{k}_3})e^{\lambda m_3}+\frac{\beta_3}{k_3}]\\
&\qquad \cdot[\widetilde{k}_2\cosh(\lambda m_2)+\frac{\alpha_2}{\widetilde{k}_2}\sinh(\lambda m_2)]\\
&-e^{-\lambda m_1}[(\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})e^{-\lambda m_2}-\frac{\beta_2}{k_2}]\\
&\cdot[\widetilde{k}_3\cosh(\lambda m_3)+\frac{\alpha_3}{\widetilde{k}_3}\sinh(\lambda m_3)]\\
&\quad +2\sinh(\lambda m_1)[(\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})e^{-\lambda m_2}-\frac{\beta_2}{k_2}]\\
&\qquad \cdot[\widetilde{k}_3\cosh(\lambda m_3)+\frac{\alpha_3}{\widetilde{k}_3}\sinh(\lambda m_3)]\\
&\quad +2\sinh(\lambda m_1)[(\widetilde{k}_3-\frac{\alpha_3}{\widetilde{k}_3})e^{-\lambda m_3}-\frac{\beta_3}{k_3}]\\
&\qquad \cdot[\widetilde{k}_2\cosh(\lambda m_2)+\frac{\alpha_2}{\widetilde{k}_2}\sinh(\lambda m_2)]\\
&= 8\sinh(\lambda m_1)\Pi_{i=2}^3[\widetilde{k}_i\cosh(\lambda m_i)+\frac{\alpha_i}{\widetilde{k}_i}\sinh(\lambda m_i)]\\
&+[\widetilde{k}_3\cosh(\lambda m_3)+\frac{\alpha_3}{\widetilde{k}_3}\sinh(\lambda m_3)]\\
&\cdot\{e^{\lambda m_1}[\widetilde{k}_2\cosh(\lambda m_2)+\frac{\alpha_2}{\widetilde{k}_2}\sinh(\lambda m_2)]\\
&-e^{-\lambda m_1}[(\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})e^{-\lambda m_2}-\frac{\beta_2}{k_2}]\}. \qquad (60)
\end{aligned}$$

For any $\lambda = ix, x\in\mathbf{R}\setminus\{0\}$, set

$$\begin{aligned}
N(x) &= \widetilde{k}_3\cosh(ixm_3)+\tfrac{\alpha_3}{\widetilde{k}_3}\sinh(ixm_3)\\
&= \widetilde{k}_3\cos(xm_3)+i\tfrac{\alpha_3}{\widetilde{k}_3}\sin(xm_3);\\
\widehat{N}(x) &= \tfrac{\alpha_2}{\widetilde{k}_2}\cos(xm_2)+\tfrac{\beta_2}{k_2}+i\widetilde{k}_2\sin(xm_2).
\end{aligned}$$

Then,

$$\begin{aligned}
&0\le \min\{\widetilde{k}_3,\frac{\alpha_3}{\widetilde{k}_3}\}\le |N(x)|\le \max\{\widetilde{k}_3,\frac{\alpha_3}{\widetilde{k}_3}\};\\
&|\widehat{N}(x)|\ge \min\{\widetilde{k}_2,\frac{\alpha_2}{\widetilde{k}_2}\}+\frac{\beta_2}{k_2}(\frac{\beta_2}{k_2}-\frac{2\alpha_2}{\widetilde{k}_2})>0;\\
&|\widehat{N}(x)|\le \max\{\widetilde{k}_2,\frac{\alpha_2}{\widetilde{k}_2}\}+\frac{\beta_2}{k_2}(\frac{\beta_2}{k_2}+\frac{2\alpha_2}{\widetilde{k}_2})\\
&=: Q. \qquad (61)
\end{aligned}$$

Thereout,

$$\begin{aligned}
&\frac{3k_1k_2k_3|\det\Delta_0(ix)|}{2x^4M|N(x)|}\\
&= \Big|8\sinh(ixm_1)[\widetilde{k}_2\cosh(ixm_2)+\frac{\alpha_2}{\widetilde{k}_2}\sinh(ixm_2)]\\
&\quad +e^{ixm_1}[\widetilde{k}_2\cosh(ixm_2)+\frac{\alpha_2}{\widetilde{k}_2}\sinh(ixm_2)]\\
&\ -e^{-ixm_1}[(\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})e^{-ixm_2}-\frac{\beta_2}{k_2}]\Big|\\
&= \Big|i8\sin(xm_1)[\widetilde{k}_2\cos(xm_2)+i\frac{\alpha_2}{\widetilde{k}_2}\sin(xm_2)]\\
&\quad +[\cos(xm_1)+i\sin(xm_1)]\\
&\qquad \cdot[\widetilde{k}_2\cos(xm_2)+i\frac{\alpha_2}{\widetilde{k}_2}\sin(xm_2)]\\
&\quad -[\cos(xm_1)-i\sin(xm_1)]\\
&\qquad \cdot\{(\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})[\cos(xm_2)-i\sin(xm_2)]-\frac{\beta_2}{k_2}\}\Big|\\
&= \Big|\sin(xm_1)\cdot\Big[(\widetilde{k}_2-\frac{10\alpha_2}{\widetilde{k}_2})\sin(xm_2)\\
&\qquad +i[(10\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})\cos(xm_2)-\frac{\beta_2}{k_2}]\Big]\\
&+\cos(xm_1)\Big[\frac{\alpha_2}{\widetilde{k}_2}\cos(xm_2)+\frac{\beta_2}{k_2}+i\widetilde{k}_2\sin(xm_2)\Big]\Big|\\
&= |\widehat{N}(x)|\Big|\frac{\sin(xm_1)}{|\widehat{N}(x)|^2}\cdot\Big[(\widetilde{k}_2-\frac{10\alpha_2}{\widetilde{k}_2})\sin(xm_2)\\
&\qquad +i[(10\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})\cos(xm_2)-\frac{\beta_2}{k_2}]\Big]\\
&\cdot\Big[\frac{\alpha_2}{\widetilde{k}_2}\cos(xm_2)+\frac{\beta_2}{k_2}+i\widetilde{k}_2\sin(xm_2)\Big]+\cos(xm_1)\Big|\\
&= |\widehat{N}(x)|\Big|\cos(xm_1)+\frac{\sin(xm_1)}{|\widehat{N}(x)|^2}\cdot\Big[-\frac{\beta_2}{\widetilde{k}_2}\cdot\frac{10\alpha_2}{\widetilde{k}_2}\\
&\quad \cdot\sin(xm_2)+10(\widetilde{k}_2^2-\frac{\alpha_2^2}{\widetilde{k}_2^2})\sin(xm_2)\cos(xm_2)\Big]\\
&\quad +i\frac{\sin(xm_1)}{|\widehat{N}(x)|^2}\Big[(\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2)\cos^2(xm_2)-\frac{2\beta_2}{k_2}\\
&\ \cdot(5\widetilde{k}_2-\frac{\alpha_2}{\widetilde{k}_2})\cos(xm_2)+\frac{\beta_2^2}{k_2^2}-10\alpha_2+\widetilde{k}_2^2\Big]\Big|. \qquad (62)
\end{aligned}$$

Take $\{x_n\}\subset R$ such that

$$\lim_{n\to+\infty}\frac{|\det\Delta_0(ix_n)|}{x_n^4|N(x_n)|}=\inf_{x\in R\setminus\{0\}}\frac{|\det\Delta_0(ix)|}{x^4|N(x)|}.$$

If $\lim_{n\to+\infty}\sin(x_n m_1)=0$, we have, by (62),

$$\begin{aligned}&\inf_{x\in R\setminus\{0\}}\frac{3k_1k_2k_3|\det\Delta_0(ix)|}{x^4M|N(x)|}\\&=\lim_{n\to+\infty}|\widehat{N}(x_n)||\cos(x_nm_1)|\\&\geq\min\{\widetilde{k}_2,\frac{\alpha_2}{\widetilde{k}_2}\}+\frac{\beta_2}{k_2}(\frac{\beta_2}{k_2}-\frac{2\alpha_2}{\widetilde{k}_2})\\&>0.\end{aligned}\tag{63}$$

Otherwise, $\lim_{n\to+\infty}\sin(x_nm_1)\neq0$, then by (62) again, we have

$$\begin{aligned}&\frac{3k_1k_2k_3|\det\Delta_0(ix)|}{x^4M|N(x)|}\\&\geq\frac{|\sin(xm_1)|}{|\widehat{N}(x)|}\left|(\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2)\cos^2(xm_2)+\frac{2\beta_2}{k_2}\right.\\&\left.\cdot(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)\cos(xm_2)+\frac{\beta_2^2}{k_2^2}+\widetilde{k}_2^2-10\alpha_2\right|\\&=\frac{|\sin(xm_1)|}{|\widehat{N}(x)|}\cdot\left|(\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2)\left[\cos(xm_2)\right.\right.\\&\left.+\frac{\frac{\beta_2}{k_2}\cdot(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)}{\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2}\right]^2+\frac{\beta_2^2}{k_2^2}+\widetilde{k}_2^2-10\alpha_2\\&\left.-\frac{\frac{\beta_2^2}{k_2^2}\cdot(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2^2)^2}{\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2}\right|.\end{aligned}\tag{64}$$

Using (57) and (58), it is easy to see that

$$\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2>0,$$
$$\frac{\frac{\beta_2}{k_2}\cdot(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)}{\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2}>1,$$

and

$$\begin{aligned}&(\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2)-\frac{2\beta_2}{k_2}(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)+\frac{\beta_2^2}{k_2^2}+\widetilde{k}_2^2-10\alpha_2\\&=\left[\frac{\beta_2}{k_2}-(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)\right]^2-25\widetilde{k}_2^2>0.\end{aligned}$$

So, for any $x\in R\setminus\{0\}$, (64) implies that

$$\begin{aligned}&\frac{3k_1k_2k_3|\det\Delta_0(ix)|}{x^4M|N(x)|}\\&\geq\frac{|\sin(xm_1)|}{|\widehat{N}(x)|}\left[(\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2)-\frac{2\beta_2}{k_2}(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)\right.\\&\left.+\frac{\beta_2^2}{k_2^2}+\widetilde{k}_2^2-10\alpha_2\right]\\&>0.\end{aligned}\tag{65}$$

Now, from (62), (64) and (65), it has

$$\begin{aligned}&\inf_{x\in R\setminus\{0\}}\frac{3k_1k_2k_3|\det\Delta_0(ix)|}{x^4M|N(x)|}\\&\geq\lim_{x\to+\infty}\frac{|\sin(xm_1)|}{|\widehat{N}(x)|}\left[(\frac{\alpha_2^2}{\widetilde{k}_2^2}-\widetilde{k}_2^2)\right.\\&\left.-\frac{2\beta_2}{k_2}(\frac{\alpha_2}{\widetilde{k}_2}-5\widetilde{k}_2)+\frac{\beta_2^2}{k_2^2}+\widetilde{k}_2^2-10\alpha_2\right]\\&>0.\end{aligned}\tag{66}$$

Thus, (63) together with (66) indicates that

$$\inf_{x\in\mathbf{R}\setminus\{0\}}|\det\Delta_0(ix)|>0.$$

That is, system (4) is exponentially stable. □

5 Conclusion

This paper studies the exponential stabilization of a three-edge network system of strings with tip mass added on the common vertex. The tip mass attached would increase the flexibility of the system, and therefore modify the vibrating behavior of it. To stabilize the vibration of such system, the following two non-collocated controllers are designed

$$u_i(t)=\alpha_i\frac{\partial w_i}{\partial t}(0,t)+\beta_i\frac{\partial w_i}{\partial t}(\ell_i,t),\ i=2,3.$$

The detailed analysis shows that the exponential stability of the closed-loop system can be achieved while the feedback gain constants $\alpha_i,\ \beta_i$ satisfy some conditions. Moreover, The obtained results indicates that the non-collocated controllers can be used to deal with the exponential stabilization problem of the network system with tip mass, which will be applied to some more complex vibrating system of network in our next work.

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