Generalized Delay Integral Inequalities In Two Independent Variables On Time Scales

Qinghua Feng
Shandong University of Technology
School of Science
Zhangzhou Road 12, 255049, Zibo
China
fqhua@sina.com

Bosheng Fu
Wuhan University
School of Journalism and Communication
Wuhan, 430072
China
fengqinghua1978@sina.com

Abstract: In this paper, we discuss some new Gronwall-Bellman type delay integral inequalities in two independent variables on time scales, which contain arbitrary nonlinear items on unknown functions outside the integrals as well as inside the integrals. The inequalities established have different forms compared with the existing Gronwall-Bellman type inequalities so far in the literature. As applications, we research the upper bounds of one certain delay dynamic equation on time scales. New explicit bounds for the solutions are derived with the results established.

Key–Words: Delay integral inequality, Time scales, Delay dynamic equations, Qualitative analysis, Bounded

1 Introduction

As is known, various inequalities play important roles in the research of qualitative and quantitative properties such as boundedness, uniqueness, and continuous dependence on initial data of solutions of certain differential equations, integral equations as well as difference equations. Among the inequalities, the Gronwall-Bellman inequality [1,2] and its various generalizations are of particular importance as these inequalities provide explicit bounds for the unknown functions concerned. During the past decades, much effort has been done for developing such inequalities (for example, see [3-22] and the references therein). But we notice in the analysis of solutions of some certain delay dynamic equations on time scales, the bounds provided by the earlier inequalities are inadequate and it is necessary to generalize the existing inequalities to arbitrary time scales so as to obtain the desired results.

The aim of this paper is to establish some new Gronwall-Bellman type delay integral inequalities in two independent variables on time scales, which provide new bounds for the unknown functions concerned. Some of our results unify existing continuous and discrete analysis in the literature. For illustrating the validity of the established results, we will present some applications for them.

Throughout this paper, $\mathbf{R}$ denotes the set of real numbers and $\mathbf{R}_+ = [0, \infty)$, while $\mathbf{Z}$ denotes the set of integers. For $(x_1, y_1),\ (x_2, y_2) \in \mathbf{R}^2$, if $x_1 \leq x_2$ and $y_1 \leq y_2$, then we mean $(x_1, y_1) \leq (x_2, y_2)$. For a function f with two independent variables, if $(x_1, y_1) \leq (x_2, y_2)$ implies $f(x_1, y_1) \leq f(x_2, y_2)$, we mean f is nondecreasing. For the details about time scales, we refer the reader to [23,24].

In the rest of this paper, $\mathbf{T}$ denotes an arbitrary time scale, and $\mathbf{T}_0 = [x_0, \infty) \bigcap \mathbf{T}$, $\widetilde{\mathbf{T}}_0 = [y_0, \infty) \bigcap \mathbf{T}$, where $x_0,\ y_0 \in \mathbf{T}^\kappa$. Furthermore, assume $\mathbf{T}_0 \subseteq \mathbf{T}^\kappa$, $\widetilde{\mathbf{T}}_0 \subseteq \mathbf{T}^\kappa$.

2 Main Results

Theorem 1 *Suppose* $u,\ a,\ b,\ f \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \mathbf{R}_+)$*, and* $a(x, y),\ b(x, y)$ *are nondecreasing with* $a(0, 0) > 0$. $\varphi,\ \eta \in C(\mathbf{R}_+, \mathbf{R}_+)$*, and* φ *is nondecreasing with* $\varphi(r) > 0$ *for* $r > 0$*, while* η *is strictly increasing.* $\tau_1 \in (\mathbf{T}_0, \mathbf{T})$, $\tau_1(x) \leq x$, $-\infty < \alpha = \inf\{\tau_1(x), x \in \mathbf{T}_0\} \leq x_0$. $\tau_2 \in (\widetilde{\mathbf{T}}_0, \mathbf{T})$, $\tau_2(y) \leq y$, $-\infty < \beta = \inf\{\tau_2(y), y \in \widetilde{\mathbf{T}}_0\} \leq y_0$. $\phi \in C_{rd}(([\alpha, x_0] \times [\beta, y_0]) \bigcap \mathbf{T}^2, \mathbf{R}_+)$. *If for* $(x, y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0$, $u(x, y)$ *satisfies the following inequality*

$$\eta(u(x, y)) \leq a(x, y)$$
$$+b(x, y) \int_{y_0}^{y} \int_{x_0}^{x} f(s, t)\varphi(u(\tau_1(s), \tau_2(t)))\Delta s \Delta t, \tag{1}$$

with the initial condition:

$$\begin{cases} u(x, y) = \phi(x, y), \\ \quad if\ x \in [\alpha, x_0] \bigcap \mathbf{T}\ or\ y \in [\beta, y_0] \bigcap \mathbf{T}, \\ \phi(\tau_1(x), \tau_2(y)) \leq \eta^{-1}(a(x, y)),\ \forall (x, y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \\ \quad if\ \tau_1(x) \leq x_0\ or\ \tau_2(y) \leq y_0, \end{cases} \tag{2}$$

then

$$u(x,y) \le \eta^{-1}\{G^{-1}[G(a(x,y)) + b(x,y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t]\},\ (x,y)\in \mathbf{T}_0\times\widetilde{\mathbf{T}}_0, \tag{3}$$

where G is an increasing bijective function, and $G(v)=\int_1^v \frac{1}{\varphi(\eta^{-1}(r))}dr,\ v>0$ with $G(\infty)=\infty$.

Proof: Fix $X\in\mathbf{T}_0,\ Y\in\widetilde{\mathbf{T}}_0$, and let $x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}$. Denote $v(x,y)=a(X,Y)+b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\varphi(u(\tau_1(s),\tau_2(t)))\Delta s\Delta t$. Then

$$u(x,y)\le\eta^{-1}(v(x,y)),$$

$$x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}. \tag{4}$$

If $\tau_1(x)\ge x_0$ and $\tau_2(y)\ge y_0$, then $\tau_1(x)\in[x_0,X]\bigcap\mathbf{T},\ \tau_2(y)\in[y_0,Y]\bigcap\mathbf{T}$, and

$$u(\tau_1(x),\tau_2(y))\le\eta^{-1}(v(\tau_1(x),\tau_2(y)))\le\eta^{-1}(v(x,y)). \tag{5}$$

If $\tau_1(x)\le x_0$ or $\tau_2(y)\le y_0$, then from (2) we have

$$u(\tau_1(x),\tau_2(y))=\phi(\tau_1(x),\tau_2(y))$$

$$\le\eta^{-1}(a(x,y))\le\eta^{-1}(v(x,y)). \tag{6}$$

From (5) and (6) we always have

$$u(\tau_1(x),\tau_2(y))\le\eta^{-1}(v(x,y)),$$

$$x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}. \tag{7}$$

Moreover

$$v_y^{\Delta}(x,y)=b(X,Y)\int_{x_0}^{x} f(s,y)\varphi(u(\tau_1(s),\tau_2(y)))\Delta s$$

$$\le[b(X,Y)\int_{x_0}^{x} f(s,y)\Delta s]\varphi(\eta^{-1}(v(x,y))),$$

that is,

$$\frac{v_y^{\Delta}(x,y)}{\varphi(\eta^{-1}(v(x,y)))}\le b(X,Y)\int_{x_0}^{x} f(s,y)\Delta s, \tag{8}$$

On the other hand, from the definition of G, if $\sigma(y)>y$, then

$$[G(v(x,y))]_y^{\Delta}=\frac{G(v(x,\sigma(y)))-G(v(x,y))}{\sigma(y)-y}$$

$$=\frac{1}{\sigma(y)-y}\int_{v(x,y)}^{v(x,\sigma(y))}\frac{1}{\varphi(\eta^{-1}(r))}dr$$

$$\le\frac{v(x,\sigma(y))-v(x,y)}{\sigma(y)-y}\frac{1}{\varphi(\eta^{-1}(v(x,y)))}$$

$$=\frac{v_y^{\Delta}(x,y)}{\varphi(\eta^{-1}(v(x,y)))}. \tag{9}$$

If $\sigma(y)=y$, then

$$[G(v(x,y))]_y^{\Delta}=\lim_{t\to y}\frac{G(v(x,y))-G(v(x,t))}{y-t}$$

$$=\lim_{t\to y}\frac{1}{y-t}\int_{v(x,t)}^{v(x,y)}\frac{1}{\varphi(\eta^{-1}(r))}dr$$

$$=\lim_{s\to y}\frac{v(x,y)-v(x,t)}{y-t}\frac{1}{\varphi(\eta^{-1}(\xi))}=\frac{v_y^{\Delta}(x,y)}{\varphi(\eta^{-1}(v(x,y)))}, \tag{10}$$

where ξ lies between $v(x,t)$ and $v(x,y)$. So from (9) and (10) we always have

$$[G(v(x,y))]_y^{\Delta}\le\frac{v_y^{\Delta}(x,y)}{\varphi(\eta^{-1}(v(x,y)))}. \tag{11}$$

Combining (8) and (11) we obtain

$$[G(v(x,y))]_y^{\Delta}\le b(X,Y)\int_{x_0}^{x} f(s,y)\Delta s. \tag{12}$$

Setting $y=t$ in (12), an integration for (12) with respect to t from y_0 to y yields

$$G(v(x,y))-G(v(x,y_0))\le b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t.$$

Since $v(x,y_0)=a(X,Y)$, and G is increasing, then

$$v(x,y)\le G^{-1}[G(a(X,Y)) + b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t]. \tag{13}$$

Combining (4) and (13) we deduce

$$u(x,y)\le\eta^{-1}\{G^{-1}[G(a(X,Y))+b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t]\},$$

$$x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}. \tag{14}$$

Setting $x=X, y=Y$ in (14), since $X,\ Y$ are selected from $\mathbf{T}_0,\ \widetilde{\mathbf{T}}_0$ arbitrarily, then after replacing $X,\ Y$ by $x,\ y$ we get the desired result.

Theorem 2 *Suppose $g\in C_{rd}(\mathbf{T}_0\times\widetilde{\mathbf{T}}_0,\mathbf{R}_+)$, and $u,\ a,\ b,\ f,\ \eta,\ \varphi,\ \tau_1,\ \tau_2,\ \alpha,\ \beta,\ \phi$ are defined as in Theorem 1. Furthermore, define $\widetilde{H}(r)=\frac{\eta(r)}{r},\ r>0$, and assume $\widetilde{H}$ is increasing with $\widetilde{H}(\infty)=\infty$. If for $(x,y)\in\mathbf{T}_0\times\widetilde{\mathbf{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$\eta(u(x,y))\le a(x,y)+b(x,y)$$

$$\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)u(\tau_1(s),\tau_2(t))$$
$$+g(s,t)u(\tau_1(s),\tau_2(t))\varphi(u(\tau_1(s),\tau_2(t)))]\Delta s\Delta t, \tag{15}$$

with the initial condition (2), then for $(x,y)\in \mathbf{T}_0\times\widetilde{\mathbf{T}}_0$ *we have*

$$u(x,y)\le \widetilde{H}^{-1}\{\widetilde{G}^{-1}\{\widetilde{G}[\frac{a(x,y)}{\eta^{-1}(a(x,y))}+b(x,y)$$
$$\int_{y_0}^{y}\int_{x_0}^{x}f(s,t)\Delta s\Delta t]+b(x,y)\int_{y_0}^{y}\int_{x_0}^{x}g(s,t)\Delta s\Delta t\}\}. \tag{16}$$

where $\widetilde{G}$ *is an increasing bijective function, and* $\widetilde{G}(v)=\int_1^v\frac{1}{\varphi(\widetilde{H}^{-1}(r))}dr,\ v>0$ *with* $\widetilde{G}(\infty)=\infty$.

Proof: Fix $X\in\mathbf{T}_0,\ Y\in\widetilde{\mathbf{T}}_0$, and let $x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}$. Denote $v(x,y)=a(X,Y)+b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)u(\tau_1(s),\tau_2(t))+g(s,t)u(\tau_1(s),\tau_2(t))\varphi(u(\tau_1(s),\tau_2(t)))]\Delta s\Delta t$. Then

$$u(x,y)\le\eta^{-1}(v(x,y)),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}, \tag{17}$$

and similar to (5)-(7) we obtain

$$u(\tau_1(x),\tau_2(y))\le\eta^{-1}(v(x,y)),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ y\in[y_0,Y]\bigcap\mathbf{T}. \tag{18}$$

Furthermore,

$$v_y^{\Delta}(x,y)=b(X,Y)\int_{x_0}^{x}[f(s,y)u(\tau_1(s),\tau_2(y))$$
$$+g(s,y)u(\tau_1(s),\tau_2(y))\varphi(u(\tau_1(s),\tau_2(y)))]\Delta s$$
$$\le\eta^{-1}(v(x,y))\{b(X,Y)\int_{x_0}^{x}[f(s,y)$$
$$+g(s,y)\varphi(\eta^{-1}(v(s,y)))]\Delta s\},$$

that is,

$$\frac{v_y^{\Delta}(x,y)}{\eta^{-1}(v(x,y))}\le$$
$$b(X,Y)\{\int_{x_0}^{x}[f(s,y)+g(s,y)\varphi(\eta^{-1}(v(s,y)))]\Delta s\}, \tag{19}$$

Define $J(v)=\int_1^v\frac{1}{\eta^{-1}(r)}dr,\ v>0$. Then similar to the process of (9)-(11) we have

$$[J(v(x,y))]_y^{\Delta}\le\frac{v_y^{\Delta}(x,y)}{\eta^{-1}(v(x,y))}. \tag{20}$$

A combination of (19) and (20) yields

$$[J(v(x,y))]_y^{\Delta}\le b(X,Y)$$
$$\int_{x_0}^{x}[f(s,y)+g(s,y)\varphi(\eta^{-1}(v(s,y)))]\Delta s. \tag{21}$$

Setting $y=t$ in (21), an integration with respect to t from y_0 to y yields

$$J(v(x,y))-J(v(x,y_0))\le b(X,Y)$$
$$\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)+g(s,t)\varphi(\eta^{-1}(v(s,t)))]\Delta s\Delta t,$$

that is,

$$\int_{v(x,y_0)}^{v(x,y)}\frac{1}{\eta^{-1}(r)}dr\le b(X,Y)$$
$$\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)+g(s,t)\varphi(\eta^{-1}(v(s,t)))]\Delta s\Delta t. \tag{22}$$

On the other hand, according to the Mean-Value Theorem for integrals, there exists ξ such that $v(x,y_0)\le\xi\le v(x,y)$, and

$$\int_{v(x,y_0)}^{v(x,y)}\frac{1}{\eta^{-1}(r)}dr=\frac{v(x,y)-v(x,y_0)}{\eta^{-1}(\xi)}$$
$$\ge\frac{v(x,y)}{\eta^{-1}(v(x,y))}-\frac{v(x,y_0)}{\eta^{-1}(v(x,y_0))}$$
$$=\widetilde{H}(\eta^{-1}(v(x,y)))-\frac{a(X,Y)}{\eta^{-1}(a(X,Y))}.$$

So we deduce that

$$\widetilde{H}(\eta^{-1}(v(x,y)))\le\frac{a(X,Y)}{\eta^{-1}(a(X,Y))}+b(X,Y)$$
$$\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)+g(s,t)\varphi(\eta^{-1}(v(s,t)))]\Delta s\Delta t$$
$$\le[\frac{a(X,Y)}{\eta^{-1}(a(X,Y))}+b(X,Y)\int_{y_0}^{Y}\int_{x_0}^{X}f(s,t)\Delta s\Delta t]$$
$$+b(X,Y)[\int_{y_0}^{y}\int_{x_0}^{x}g(s,t)\varphi(\eta^{-1}(v(s,t)))\Delta s\Delta t]. \tag{23}$$

Now a suitable application of Theorem 1 to (23) yields

$$\eta^{-1}(v(x,y))\le\widetilde{H}^{-1}\{\widetilde{G}^{-1}\{\widetilde{G}[\frac{a(X,Y)}{\eta^{-1}(a(X,Y))}$$
$$+b(X,Y)\int_{y_0}^{Y}\int_{x_0}^{X}f(s,t)\Delta s\Delta t]$$
$$+b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x}g(s,t)\Delta s\Delta t\}\},$$

which implies

$$u(x,y) \le \widetilde{H}^{-1}\{\widetilde{G}^{-1}\{\widetilde{G}[\frac{a(X,Y)}{\eta^{-1}(a(X,Y))}$$

$$+b(X,Y)\int_{y_0}^{Y}\int_{x_0}^{X} f(s,t)\Delta s\Delta t]$$

$$+b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t\}\}. \tag{24}$$

Setting $x = X, y = Y$ in (24), since $X,\ Y$ are selected from $\mathbf{T}_0,\ \widetilde{\mathbf{T}}_0$ arbitrarily, then after replacing $X,\ Y$ by $x,\ y$ we get the desired result.

Theorem 3 *Under the conditions of Theorem 2, we have another estimate for $u(x,y)$ as follows:*

$$u(x,y) \le \eta^{-1}\{J^{-1}\{\overline{G}^{-1}\{\overline{G}[J(a(x,y)) + b(x,y)$$

$$\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t] + b(x,y)\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t\}\}\}, \tag{25}$$

where $\overline{G}$ is an increasing bijective function, and $\overline{G}(z) = \int_1^z \frac{1}{\varphi(\eta^{-1}(J^{-1}(r)))}dr,\ z > 0$ with $\overline{G}(\infty) = \infty$.

Proof: By the process of (17)-(22) we have

$$v(x,y) \le J^{-1}\{J(v(x,y_0)) + b(X,Y)$$

$$\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t) + g(s,t)\varphi(\eta^{-1}(v(s,t)))]\Delta s\Delta t\},$$

$$= J^{-1}\{J(a(X,Y)) + b(X,Y)$$

$$\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t) + g(s,t)\varphi(\eta^{-1}(v(s,t)))]\Delta s\Delta t\},$$

$$\le J^{-1}\{J(a(X,Y)) + b(X,Y)$$

$$\int_{y_0}^{Y}\int_{x_0}^{X} f(s,t)\Delta s\Delta t + b(X,Y)$$

$$[\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\varphi(\eta^{-1}(v(s,t)))\Delta s\Delta t]\}, \tag{26}$$

where $J,\ v(x,y),\ X,\ Y$ are defined as in Theorem 2.

Let

$$z(x,y) = J(a(X,Y)) + b(X,Y)\int_{y_0}^{Y}\int_{x_0}^{X} f(s,t)\Delta s\Delta t$$

$$+b(X,Y)[\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\varphi(\eta^{-1}(v(s,t)))\Delta s\Delta t].$$

Then

$$v(x,y) \le J^{-1}\{z(x,y)\},$$

$$x \in [x_0, X]\bigcap \mathbf{T},\ y \in [y_0, Y]\bigcap \mathbf{T},$$

and

$$z_y^{\Delta}(x,y) = b(X,Y)[\int_{x_0}^{x} g(s,y)\varphi(\eta^{-1}(v(s,y)))\Delta s]$$

$$\le [b(X,Y)\int_{x_0}^{x} g(s,y)\Delta s]\varphi(\eta^{-1}(J^{-1}(z(x,y)))),$$

that is,

$$\frac{z_y^{\Delta}(x,y)}{\varphi(\eta^{-1}(J^{-1}(z(x,y))))} \le b(X,Y)\int_{x_0}^{x} g(s,y)\Delta s. \tag{27}$$

On the other hand, similar to the process of (9)-(11) we have

$$[\overline{G}(z(x,y))]_y^{\Delta} \le \frac{z_y^{\Delta}(x,y)}{\varphi(\eta^{-1}(J^{-1}(z(s,y))))}. \tag{28}$$

A combination of (27) and (28) yields

$$[\overline{G}(z(x,y))]_y^{\Delta} \le b(X,Y)\int_{x_0}^{x} g(s,y)\Delta s. \tag{29}$$

Setting $y = t$ in (29), an integration with respect to t from y_0 to y yields

$$\overline{G}(z(x,y)) - \overline{G}(z(x,y_0)) \le b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t.$$

Since $\overline{G}$ is increasing, and $z(x,y_0) = J(a(X,Y)) + b(X,Y)\int_{y_0}^{Y}\int_{x_0}^{X} f(s,t)\Delta s\Delta t$, then furthermore we have

$$z(x,y) \le \overline{G}^{-1}\{\overline{G}[J(a(X,Y)) + b(X,Y)$$

$$\int_{y_0}^{Y}\int_{x_0}^{X} f(s,t)\Delta s\Delta t] + b(X,Y)\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t\}, \tag{30}$$

which confirms the desired result since $X,\ Y$ are selected from $\mathbf{T}_0,\ \widetilde{\mathbf{T}}_0$ arbitrarily.

Theorem 4 *Suppose $g_i,\ i = 1,2,...,l \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \mathbf{R}_+)$, where l is a positive integer. $u,\ a,\ b,\ f,\ \eta,\ \varphi,\ \tau_1,\ \tau_2,\ \alpha,\ \beta,\ \phi$ are defined as in Theorem 1. $\psi \in C(\mathbf{R}_+, \mathbf{R}_+)$, and ψ is strictly increasing. Furthermore, define $\widehat{H}(r) = \frac{\eta(\psi^{-1}(r))}{r},\ r > 0$, and assume $\widehat{H}$ is increasing with $\widehat{H}(\infty) = \infty$. If for $(x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$\eta(u(x,y)) \le a(x,y) + b(x,y)$$

$$\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\psi(u(\tau_1(s),\tau_2(t)))\Delta s\Delta t + b(x,y)$$

$$\int_{y_0}^{y}\int_{x_0}^{x}\sum_{i=1}^{l} g_i(s,t)\psi(u(\tau_1(s),\tau_2(t)))$$

$$\varphi(u(\tau_1(s),\tau_2(t)))\Delta s\Delta t, \tag{31}$$

with the initial condition (2), then for $(x,y)\in \mathbf{T}_0\times \widetilde{\mathbf{T}}_0$ *we have*

$$u(x,y)\le \widehat{H}^{-1}\{\widehat{G}^{-1}\{\widehat{G}[\frac{a(x,y)}{\eta^{-1}(a(x,y))}$$
$$+b(x,y)\int_{y_0}^{y}\int_{x_0}^{x}f(s,t)\Delta s\Delta t]$$
$$+b(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\sum_{i=1}^{l}g_i(s,t)\Delta s\Delta t\}\}. \tag{32}$$

where $\widehat{G}$ *is an increasing bijective function, and* $\widehat{G}(v)=\int_1^v\frac{1}{\varphi(\widehat{H}^{-1}(r))}dr,\ v>0$ *with* $\widetilde{G}(\infty)=\infty$.

Proof: Denote $v(x,y)=\psi(u(x,y))$. Then (31) can be rewritten as

$$\eta(\psi^{-1}(v(x,y)))\le a(x,y)+$$
$$b(x,y)\int_{y_0}^{y}\int_{x_0}^{x}f(s,t)v(\tau_1(s),\tau_2(t))\Delta s\Delta t+b(x,y)$$
$$\int_{y_0}^{y}\int_{x_0}^{x}\sum_{i=1}^{l}g_i(s,t)v(\tau_1(s),\tau_2(t))\varphi(u(\tau_1(s),\tau_2(t)))\Delta s\Delta t. \tag{33}$$

Then a suitable application of Theorem 2 to (33) yields the desired result.

Now we study the delay integral inequality on time scales with the following form:

$$u^p(x,y)\le a(x,y)+\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)u^q(\tau_1(s),\tau_2(t))$$
$$+g(s,t)\varphi(u(\tau_1(s),\tau_2(t)))+h(s,t)\varphi(u(s,t))]\Delta s\Delta t, \tag{34}$$

with the initial condition

$$\begin{cases} u(x,y)=\phi(x,y) \\ \quad if\ x\in[\alpha,x_0]\bigcap\mathbf{T}\ or\ y\in[\beta,y_0]\bigcap\mathbf{T}, \\ \phi(\tau_1(x),\tau_2(y))\le a^{\frac{1}{p}}(x,y),\ \forall(x,y)\in\mathbf{T}_0\times\widetilde{\mathbf{T}}_0 \\ \quad if\ \tau_1(x)\le x_0\ or\ \tau_2(y)\le y_0, \end{cases} \tag{35}$$

where $u,\ a,\ f,\ \varphi,\ \tau_1,\ \tau_2,\ \alpha,\ \beta,\ \phi$ are defined the same as in Theorem 1, and furthermore, φ is submultiplicative, that is, $\varphi(\alpha\beta)\le\varphi(\alpha)\varphi(\beta)$ for $\forall\alpha,\beta\in\mathbf{R}_+$. $g,\ h\in C_{rd}(\mathbf{T}_0\times\widetilde{\mathbf{T}}_0,\mathbf{R}_+)$. $p,\ q$ are constants with $p>q>0$.

Lemma 5 *Suppose* $u,\ q\in C_{rd}(\mathbf{T}^\kappa\times\mathbf{T}^\kappa,\mathbf{R})$, $p(x,y)\in\mathcal{R}^+$ *with respect to y, and* $u(x,y)$ *is partial delta differential at* $y\in\mathbf{T}^\kappa$, *then for* $y\in\mathbf{T}^\kappa$,

$$u_y^\Delta(x,y)\le p(x,y)u(x,y)+q(x,y)$$

implies

$$u(x,y)\le u(x,y_0)e_p(y,y_0)+\int_{y_0}^{y}q(x,t)e_p(y,\sigma(t))\Delta t.$$

Lemma 5 is a direct 2D extension of [25, Theorem 5.4].

Lemma 6 [26] *Assume that* $a\ge 0, p\ge q\ge 0$, *and* $p\neq 0$, *then for any* $K>0$

$$a^{\frac{q}{p}}\le\frac{q}{p}K^{\frac{q-p}{p}}a+\frac{p-q}{p}K^{\frac{q}{p}}.$$

Theorem 7 *If for* $(x,y)\in\mathbf{T}_0\times\widetilde{\mathbf{T}}_0$, $u(x,y)$ *satisfies the inequality (34) with the initial condition (35), then*

$$u(x,y)\le\{\widetilde{\widetilde{G}}^{-1}\{\widetilde{\widetilde{G}}(a(x,y)$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t)+\int_{y_0}^{y}[\int_{x_0}^{x}(g(s,t)$$
$$+h(s,t))\Delta s]\varphi((e_{F_2}(t,y_0))^{\frac{1}{p}})\Delta t\}e_{F_2}(y,y_0)\}^{\frac{1}{p}}, \tag{36}$$

where $\widetilde{\widetilde{G}}$ *is an increasing bijective function, and* $\widetilde{\widetilde{G}}(v)=\int_1^v\frac{1}{\varphi(r^{\frac{1}{p}})}dr,\ v>0$ *with* $\widetilde{\widetilde{G}}(\infty)=\infty$, *while*

$$F_2(x,y)=\int_{x_0}^{x}f(s,y)\frac{q}{p}K^{\frac{q-p}{p}}\Delta s, \tag{37}$$

Proof: Let $X\in\mathbf{T}_0,\ Y\in\widetilde{\mathbf{T}}_0$ be two fixed numbers, and $x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}$. Denote

$$v(x,y)=a(X,Y)+\int_{x_0}^{x}\int_{y_0}^{y}[f(s,t)u^q(\tau_1(s),\tau_2(t))$$
$$+g(s,t)\varphi(u(\tau_1(s),\tau_2(t)))+h(s,t)\varphi(u(s,t))]\Delta s\Delta t.$$

Then

$$u(x,y)\le v^{\frac{1}{p}}(x,y),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \tag{38}$$

Similar to the process of (5)-(7), we obtain

$$u(\tau_1(x),\tau_2(y))\le v^{\frac{1}{p}}(x,y),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \tag{39}$$

Furthermore, by (38), (39) and Lemma 6, we have

$$v(x,y)\le a(X,Y)+\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)v^{\frac{q}{p}}(s,t)$$
$$+g(s,t)\varphi(v^{\frac{1}{p}}(s,t))+h(s,t)\varphi(v^{\frac{1}{p}}(s,t))]\Delta s\Delta t$$

$$\leq a(X,Y)+\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)(\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}})$$
$$+(g(s,t)+h(s,t))\varphi(v^{\frac{1}{p}}(s,t))]\Delta s\Delta t$$
$$\leq F_1(X,Y)+\int_{y_0}^{y}F_2(x,t)v(x,t)\Delta t. \quad (40)$$

where

$$F_1(x,y)=a(x,y)+\int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}$$
$$+(g(s,t)+h(s,t))\varphi(v^{\frac{1}{p}}(s,t))]\Delta s\Delta t,$$

F_2 is defined in (37), and $K>0$ is an arbitrary constant.

Denote the right side of (40) by $c(x,y)$. Then

$$v(x,y)\leq c(x,y),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \quad (41)$$

Furthermore,

$$c_y^{\Delta}(x,y)=F_2(x,y)v(x,y)\leq F_2(x,y)c(x,y). \quad (42)$$

As one can see, $F_2(x,y)\in\mathcal{R}_+$ with respect to y. Then a suitable application of Lemma 5 to (42) yields

$$c(x,y)\leq c(x,y_0)e_{F_2}(y,y_0)=F_1(X,Y)e_{F_2}(y,y_0),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \quad (43)$$

Taking $x=X,\ y=Y$ in (43), since $X\in\mathbf{T}_0,\ Y\in\widetilde{\mathbf{T}}_0$ are selected arbitrarily, then in fact (43) holds for $\forall x\in\mathbf{T}_0,\ y\in\widetilde{\mathbf{T}}_0$, that is

$$c(x,y)\leq F_1(x,y)e_{F_2}(y,y_0),\ (x,y)\in\mathbf{T}_0\times\widetilde{\mathbf{T}}_0. \quad (44)$$

On the other hand, if we let

$$z(x,y)=a(X,Y)+\int_{x_0}^{X}\int_{y_0}^{Y}f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta t\Delta s$$
$$+\int_{x_0}^{x}\int_{y_0}^{y}(g(s,t)+h(s,t))\varphi(v^{\frac{1}{p}}(s,t))\Delta s\Delta t,$$

then

$$F_1(x,y)\leq z(x,y),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \quad (45)$$

Furthermore, we have

$$z_y^{\Delta}(x,y)=\int_{x_0}^{x}(g(s,y)+h(s,y))\varphi(v^{\frac{1}{p}}(s,y))\Delta s$$
$$\leq[\int_{x_0}^{x}(g(s,y)+h(s,y))\Delta s]\varphi(v^{\frac{1}{p}}(x,y))$$
$$\leq[\int_{x_0}^{x}(g(s,y)+h(s,y))\Delta s]\varphi(c^{\frac{1}{p}}(x,y))$$
$$\leq[\int_{y_0}^{y}(g(s,y)+h(s,y))\Delta s]\varphi((F_1(x,y)e_{F_2}(x,x_0))^{\frac{1}{p}})$$
$$\leq[\int_{x_0}^{x}(g(s,y)+h(s,y))\Delta s]$$
$$\varphi((F_1(x,y))^{\frac{1}{p}})\varphi((e_{F_2}(y,y_0))^{\frac{1}{p}})$$
$$\leq[\int_{x_0}^{x}(g(s,y)+h(s,y))\Delta s]$$
$$\varphi((z(x,y))^{\frac{1}{p}})\varphi((e_{F_2}(y,y_0))^{\frac{1}{p}}),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T},$$

that is,

$$\frac{z_y^{\Delta}(x,y)}{\varphi(z^{\frac{1}{p}}(x,y))}\leq$$
$$[\int_{x_0}^{x}(g(s,y)+h(s,y))\Delta s]\varphi((e_{F_2}(y,y_0))^{\frac{1}{p}}),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \quad (46)$$

Similar to the process of (9)-(11) we have

$$[\widetilde{\widetilde{G}}(z(x,y))]_y^{\Delta}\leq\frac{z_y^{\Delta}(x,y)}{\varphi(z^{\frac{1}{p}}(x,y))}.$$

So

$$[\widetilde{\widetilde{G}}(z(x,y))]_y^{\Delta}\leq$$
$$[\int_{x_0}^{x}(g(s,y)+h(s,y))\Delta s]\varphi((e_{F_2}(y,y_0))^{\frac{1}{p}}),$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \quad (47)$$

Setting $y=t$ in (47), an integration for (47) with respect to t from y_0 to y yields

$$\widetilde{\widetilde{G}}(z(x,y))-\widetilde{\widetilde{G}}(z(x,y_0))\leq$$
$$\int_{y_0}^{y}[\int_{x_0}^{x}(g(s,t)+h(s,t))\Delta s]\varphi((e_{F_2}(t,y_0))^{\frac{1}{p}})\Delta t,$$
$$x\in[x_0,X]\bigcap\mathbf{T},\ Y\in[y_0,Y]\bigcap\mathbf{T}. \quad (48)$$

Considering G is increasing, and $z(x,y_0)=a(X,Y)+\int_{x_0}^{X}\int_{y_0}^{Y}f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t$, we have

$$z(x,y)\leq$$
$$\widetilde{\widetilde{G}}^{-1}\{\widetilde{\widetilde{G}}(a(X,Y)+\int_{x_0}^{X}\int_{y_0}^{Y}f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t)$$
$$+\int_{y_0}^{y}[\int_{x_0}^{x}(g(s,t)+h(s,t))\Delta s]\varphi((e_{F_2}(t,y_0))^{\frac{1}{p}})\Delta t\}. \quad (49)$$

Combining (38), (41), (44), (45) and (49) we obtain for $x \in [x_0, X] \bigcap \mathbf{T},\ Y \in [y_0, Y] \bigcap \mathbf{T}$

$$u(x,y) \leq$$
$$\{\widetilde{\widetilde{G}}^{-1}\{\widetilde{\widetilde{G}}(a(X,Y)+\int_{x_0}^{X}\int_{y_0}^{Y} f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t)+$$
$$\int_{y_0}^{y}[\int_{x_0}^{x}(g(s,t)+h(s,t))\Delta s]$$
$$\varphi((e_{F_2}(t,y_0))^{\frac{1}{p}})\Delta t\}e_{F_2}(y,y_0)\}^{\frac{1}{p}}.$$

Setting $x = X,\ y = Y$ in the inequality above, since $X \in \mathbf{T}_0,\ Y \in \widetilde{\mathbf{T}}_0$ are selected arbitrarily, then after substituting $X,\ Y$ with $x,\ y$ we get the desired inequality (36).

Finally we study the delay integral inequality on time scales with the following form:

$$u^p(x,y) \leq a(x,y) + \int_{y_0}^{y} c(x,t)u^p(x,t)\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)u^q(\tau_1(s),\tau_2(t))\Delta s\Delta t+$$
$$\int_{y_0}^{y}\int_{x_0}^{x}[g(s,t)\varphi(u(\tau_1(s),\tau_2(t)))+h(s,t)\varphi(u(s,t))]\Delta s\Delta t, \tag{50}$$

with the initial condition (35), where $u,\ a,\ b,\ f,\ \varphi,\ \tau_1,\ \tau_2$ are defined the same as in Theorem 1, and furthermore, φ is submultiplicative, that is, $\varphi(\alpha\beta) \leq \varphi(\alpha)\varphi(\beta)$ for $\forall \alpha, \beta \in \mathbf{R}_+$. $g,\ h,\ c \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \mathbf{R}_+)$. $p,\ q$ are constants with $p > q > 0$.

Lemma 8 *Suppose $u,\ q \in C_{rd}(\mathbf{T}^\kappa \times \mathbf{T}^\kappa, \mathbf{R}_+)$, $p(x,y) \in \mathcal{R}^+$ with respect to y, $q(x,y)$ is nondecreasing in y, and $u(x,y)$ is partial delta differential at $y \in \mathbf{T}^\kappa$, then for $y \in \mathbf{T}^\kappa$,*

$$u(x,y) \leq q(x,y) + \int_{y_0}^{y} p(x,t)u(x,t)\Delta t$$

implies
$$u(x,y) \leq q(x,y)e_p(y,y_0).$$

Proof: Treat x as a constant, and According to [25, Theorem 5.6] we can deduce

$$u(x,y) \leq q(x,y) + \int_{y_0}^{y} q(x,t)p(x,t)e_p(y,\sigma(t))\Delta t. \tag{51}$$

Since $q(x,y)$ is nondecreasing in y, then

$$u(x,y) \leq q(x,y)[1 + \int_{y_0}^{y} p(x,t)e_p(y,\sigma(t))\Delta t]. \tag{52}$$

On the other hand, according to [24, Theorem 2.39 and 2.36 (i)] we have

$$1 + \int_{y_0}^{y} p(x,t)e_p(y,\sigma(t))\Delta t = e_p(y,y_0). \tag{53}$$

Then combining (52) and (53) we get the desired result.

Theorem 9 *If for $(x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0$, $u(x,y)$ satisfies the inequality (50) with the initial condition (35), then*

$$u(x,y) \leq$$
$$\{\widetilde{\widetilde{G}}^{-1}\{\widetilde{\widetilde{G}}(a(x,y)+\int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t)$$
$$+\int_{y_0}^{y}[\int_{x_0}^{x}(\widetilde{g}(s,t)+\widetilde{h}(s,t))\Delta s]$$
$$\varphi((e_{\widetilde{F}_2}(t,y_0))^{\frac{1}{p}})\Delta t\}e_{\widetilde{F}_2}(y,y_0)e_c(y,y_0)\}^{\frac{1}{p}}, \tag{54}$$

where $\widetilde{\widetilde{G}}$ is defined as in Theorem 7, and

$$\begin{cases} \widetilde{f}(x,y) = f(x,y)(e_c(y,y_0))^{\frac{q}{p}}, \\ \widetilde{g}(x,y) = g(x,y)\varphi[(e_c(y,y_0))^{\frac{1}{p}}], \\ \widetilde{h}(x,y) = h(x,y)\varphi[(e_c(y,y_0))^{\frac{1}{p}}], \\ \widetilde{F}_2(x,y) = \int_{x_0}^{x} \widetilde{f}(s,y)\frac{q}{p}K^{\frac{q-p}{p}}\Delta s, \end{cases} \tag{55}$$

Proof: Let the right side of (50) be $v^p(x,y)$. Then

$$u(x,y) \leq v(x,y),\ (x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \tag{56}$$

and similar to (5)-(7) we have

$$u(\tau_1(x),\tau_2(y)) \leq v(x,y),\ (x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0.$$

So we deduce

$$v^p(x,y) \leq a(x,y) + \int_{y_0}^{y} c(x,t)v^p(x,t)\Delta t$$
$$+b(x,y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)v^q(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}[g(s,t)\varphi(v(s,t))+h(s,t)\varphi(v(s,t))]\Delta s\Delta t, \tag{57}$$

Denote

$$z^p(x,y) = a(x,y) + b(x,y)\int_{y_0}^{y}\int_{x_0}^{x} f(s,t)v^q(s,t)\Delta s\Delta t$$
$$+\int_{y_0}^{y}\int_{x_0}^{x}[g(s,t)\varphi(v(s,t))+h(s,t)\varphi(v(s,t))]\Delta s\Delta t.$$

Then $v^p(x,y) \leq z^p(x,y) + \int_{y_0}^{y} c(x,t)v^p(x,t)\Delta t$. Obviously $z(x,y)$ is nondecreasing in y, and $c(x,y) \in$

$\mathcal{R}_+$ with respect to y. By an application of Lemma 8 we obtain

$$v^p(x,y) \le z^p(x,y)e_c(y,y_0). \tag{58}$$

Furthermore,

$$\begin{aligned} z^p(x,y) &\le a(x,y) \\ &+ \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)z^q(s,t)(e_c(t,y_0))^{\frac{q}{p}}\Delta s\Delta t \\ &+ \int_{y_0}^{y}\int_{x_0}^{x}\{g(s,t)\varphi[z(s,t)(e_c(t,y_0))^{\frac{1}{p}}] \\ &+h(s,t)\varphi[z(s,t)(e_c(t,y_0))^{\frac{1}{p}}]\}\Delta s\Delta t \\ &\le a(x,y) + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)z^q(s,t)(e_c(t,y_0))^{\frac{q}{p}}\Delta s\Delta t \\ &+ \int_{y_0}^{y}\int_{x_0}^{x}\{g(s,t)\varphi(z(s,t))\varphi[(e_c(t,y_0))^{\frac{1}{p}}] \\ &+h(s,t)\varphi(z(s,t))\varphi[(e_c(t,y_0))^{\frac{1}{p}}]\}\Delta s\Delta t \\ &\le a(x,y) + \int_{y_0}^{y}\int_{x_0}^{x} \widetilde{f}(s,t)z^q(s,t)\Delta s\Delta t \\ &+ \int_{y_0}^{y}\int_{x_0}^{x}[\widetilde{g}(s,t)\varphi(z(s,t)) + \widetilde{h}(s,t)\varphi(z(s,t))]\Delta s\Delta t, \end{aligned}$$

where $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$ are defined in (55). Then applying Theorem 7 we obtain

$$\begin{aligned} z(x,y) &\le \{\widetilde{\widetilde{G}}^{-1}\{\widetilde{\widetilde{G}}(a(x,y) \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} \widetilde{f}(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t) + \int_{y_0}^{y}[\int_{x_0}^{x}(\widetilde{g}(s,t) \\ &+\widetilde{h}(s,t))\Delta s]\varphi((e_{\widetilde{F}_2}(t,y_0))^{\frac{1}{p}})\Delta t\}e_{\widetilde{F}_2}(y,y_0)\}^{\frac{1}{p}}, \end{aligned} \tag{59}$$

where $\widetilde{F}_2(x,y)$ is defined in (55).

Combining (56), (58) and (59) we get the desired result.

Remark 10 *Our theorems generalize some known results including both continuous and discrete inequalities in the literature. For example, If we take* $\mathbf{T} = \mathbf{R}$, $x_0 = y_0 = 0$, $\eta(u) = u$, *then Theorem 1 reduces to [11, Theorem 2.4]. If we take* $\mathbf{T} = \mathbf{Z}$, $x_0 = y_0 = 0$, $\tau_1(x) = x$, $\tau_2(y) = y$, $a(x,y) \equiv C$, $b(x,y) \equiv 1$, $\eta(u) = u^p$, *then Theorem 1 reduces to [12, Theorem 2.1]. If we take* $\mathbf{T} = \mathbf{Z}$, $\tau_1(x) = x$, $\tau_2(y) = y$, $a(x,y) \equiv C$, $b(x,y) \equiv 1$, *then Theorem 1 reduces to [16, Theorem 2.1]. If we take* $\mathbf{T} = \mathbf{Z}$, $a(x,y) \equiv k$, $b(x,y) \equiv 1, \tau_1(x) = x$, $\tau_2(y) = y$, $x_0 = y_0 = 0$, *then Theorem 2 and Theorem 4 reduce to [16, Theorems 2.2, 2.3]. If we take* $\mathbf{T} = \mathbf{R}$, $x_0 = y_0 = 0$, $f(x,y) \equiv 0$, *then Theorem 7 reduces to [13, Theorem 2.4] with slight difference. If we take* $\mathbf{T} = \mathbf{R}$, $f(x,y) \equiv 0$, $\varphi(u) = u$, $a(x,y) \equiv k$, $b(x,y) \equiv 1$, $\eta(u) = u$, *then Theorem 7 reduces to [14, Theorem 3(c1)] with slight difference.*

Remark 11 *Theorem 3 is the extension on time scales of [17, Theorem 1] to 2D case.*

Remark 12 *Compared with the main results in [27-29], one can see the inequalities established in Theorems 1-4, 7 and 9 provide new Gronwall-Bellman type inequalities on time scales so far in the literature.*

3 Applications

In this section, we will present one application for the established results above, and new bounds are derived for the solutions of one given delay dynamic equation on time scales.

Example: Consider the following delay dynamic integral equation on time scales

$$u^2(x,y) = C + \int_{y_0}^{y}\int_{x_0}^{x} M[s,t,u(\tau_1(s),\tau_2(t))]\Delta s\Delta t,$$

$$(x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \tag{60}$$

with the initial condition

$$\begin{cases} u(x,y) = \phi(x,y), \\ \quad if\ x \in [\alpha, x_0]\bigcap \mathbf{T}\ or\ y \in [\beta, y_0]\bigcap \mathbf{T}, \\ \phi(\tau_1(x),\tau_2(y)) \le |C|^{\frac{1}{p}},\ \forall (x,y) \in (\mathbf{T}_0, \widetilde{\mathbf{T}}_0), \\ \quad if\ \tau_1(x) \le x_0\ or\ \tau_2(y) \le y_0, \end{cases} \tag{61}$$

where $u \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \mathbf{R})$, ϕ, α, β, τ_1, τ_2 are the same as in Theorem 1, C is a nonzero constant, $M \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0 \times \mathbf{R}, \mathbf{R})$.

Theorem 13 *Assume* $|M(s,t,u)| \le f(s,t)|u| + g(s,t)|u|^{\frac{3}{2}}$, *where* f, $g \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \mathbf{R}_+)$, *then we have the following estimate*

$$u(x,y) \le \{\sqrt{\sqrt{|C|} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t}$$

$$+\frac{1}{2}\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t\}^2,\ (x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0. \tag{62}$$

Proof: From (60) we have

$$|u(x,y)|^2 \le |C| + \int_{y_0}^{y}\int_{x_0}^{x} |M[s,t,u(\tau_1(s),\tau_2(t))|\Delta s\Delta t$$

$$\leq |C| + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)|u(\tau_1(s),\tau_2(t))| + g(s,t)|u(\tau_1(s),\tau_2(t))|^{\frac{3}{2}}]\Delta s\Delta t. \quad (63)$$

Then a suitable application of Theorem 2 yields

$$u(x,y) \leq \widetilde{H}^{-1}\{\widetilde{G}^{-1}\{\widetilde{G}[\frac{|C|}{\eta^{-1}(|C|)} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t] + \int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t\}\}. \quad (64)$$

where $\eta(r) = r^2,\ \widetilde{H}(r) = r,\ \varphi(r) = \sqrt{r},\ \widetilde{G}(v) = \int_1^v \frac{1}{\sqrt{r}}dr = 2\sqrt{v} - 2,\ v > 0$. Then from (64) and the expressions of $\widehat{H},\ \varphi,\ \widehat{G}$ we can deduce the desired result.

Remark 14 *Under the conditions of Theorem 13, if we apply Theorem 3 instead of Theorem 2 to (63), then we can obtain another estimate for $u(x,y)$ as follows:*

$$u(x,y) \leq \frac{1}{2}[\sqrt{2\sqrt{|C|} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t} + \frac{\sqrt{2}}{4}\int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t]^2,\ \ (x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0.$$

Theorem 15 *Assume $|M(s,t,u)| \leq f(s,t)\sqrt{|u|} + g(s,t)|u|^{\frac{7}{2}}$, where $f,\ g \in C_{rd}(\mathbf{T}_0 \times \widetilde{\mathbf{T}}_0, \mathbf{R}_+)$, then we have the following estimate*

$$u(x,y) \leq \sqrt[3]{\sqrt{|C|} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t} \sqrt[3]{exp\{\int_{y_0}^{y}\int_{x_0}^{x} \sum g(s,t)\Delta s\Delta t\}},\ (x,y) \in \mathbf{T}_0 \times \widetilde{\mathbf{T}}_0. \quad (65)$$

Proof: From (60) we have

$$|u(x,y)|^2 \leq |C| + \int_{y_0}^{y}\int_{x_0}^{x} |M[s,t,u(\tau_1(s),\tau_2(t))|\Delta s\Delta t$$
$$\leq |C| + \int_{y_0}^{y}\int_{x_0}^{x}[f(s,t)\sqrt{|u(\tau_1(s),\tau_2(t))|} + g(s,t)|u(\tau_1(s),\tau_2(t))|^{\frac{7}{2}}]\Delta s\Delta t. \quad (66)$$

Then a suitable application of Theorem 4 yields

$$u(x,y) \leq \widehat{H}^{-1}\{\widehat{G}^{-1}\{\widehat{G}[\frac{|c|}{\eta^{-1}(|C|)} + \int_{y_0}^{y}\int_{x_0}^{x} f(s,t)\Delta s\Delta t] + \int_{y_0}^{y}\int_{x_0}^{x} g(s,t)\Delta s\Delta t\}\}. \quad (67)$$

where $\eta(r) = r^2,\ \widehat{H}(r) = \varphi(r) = r^3,\ \widehat{G}(v) = \int_1^v \frac{1}{r}dr = lnv,\ v > 0$. Then from (67) and the expressions of $\widehat{H},\ \varphi,\ \widehat{G}$ we can deduce the desired result.

Remark 16 *In Theorem 15, if we assume $|M(s,t,u)| \leq f(s,t)\sqrt{|u|} + g(s,t)|u|^{\frac{3}{2}}$, then from Theorem 7 we can obtain another estimate for $u(x,y)$ as follows:*

$$u(x,y) \leq \{\sqrt[4]{|C| + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\frac{p-q}{p}K^{\frac{q}{p}}\Delta s\Delta t} + \frac{1}{4}\int_{y_0}^{y}[\int_{x_0}^{x}(g(s,t)+h(s,t))\Delta s]\varphi((e_{F_2}(t,y_0))^{\frac{1}{p}})\Delta t\}^2 \times\sqrt{e_{F_2}(y,y_0)},$$

where $F_2(x,y) = \frac{1}{4}K^{-\frac{3}{4}}\int_{x_0}^{x} f(s,y)\Delta s$, and $K > 0$ is an arbitrary constant.

4 Conclusion

Some new generalized Gronwall-Bellman type delay integral inequalities in two independent variables on time scales have been presented. We also present one example as applications, in which new explicit bounds for the solutions of certain delay dynamic equations on time scales are derived using the results established. The results established in this paper are generalization of some existing Gronwall-Bellman type continuous and discrete inequalities in the literature. Finally, we note the established results can be extended to n dimensional cases, which is supposed to further research.

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